Non-commutative Algebra 3, SS 2020

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Exercises 4

We work over an algebraically-closed field K. We define an affine K-algebra to be a finitely generated, commutative and reduced K-algebra.

1. Let G be an affine algebraic group (affine group scheme), and set $A = \mathcal{O}(G) = K[G]$.

   (a) Recall that $\mathcal{O}(G\times G) \cong A \otimes_K A$, where $a\otimes b$ is the map $(g,h)\mapsto a(g)b(h)$. The multiplication $G\times G\to G$, $(g,h)\mapsto gh$, induces an algebra homomorphism $\Delta\colon A\to A\otimes_K A$, $\Delta(a)(g,h) := a(gh)$.

   Show that, as maps $A\to A\otimes_K A\otimes_K A$, we have $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$.

   (b) Recall that $\operatorname{Spec} K = \{*\}$ is a single point. Thus the unit $1\in G$ can be regarded as a morphism $\operatorname{Spec} K\to G$, and as such corresponds to an algebra homomorphism $\varepsilon\colon A\to K$, $\varepsilon(a) = a(1)$.

   Using the canonical identifications $K\otimes_K A\cong A\cong A\otimes_K K$, show that, as maps $A\to A$, we have $(\varepsilon\otimes\mathrm{id})\Delta = \mathrm{id} = (\mathrm{id}\otimes\varepsilon)\Delta$.

   (c) Show that the multiplication $\mu\colon A\otimes_K A\to A$, $a\otimes b\mapsto ab$, corresponds to the diagonal embedding $G\to G\times G$, $g\mapsto (g,g)$.

   (d) Taking inverses gives a map $G\to G$, $g\mapsto g^{-1}$, so corresponds to an algebra homomorphism $S\colon A\to A$. Show that S is an algebra automorphism satisfying $\mu(S\otimes\mathrm{id})\Delta = \varepsilon = \mu(\mathrm{id}\otimes S)\Delta$.

   (e) Consider the map $\tau\colon G\times G\to G\times G$, $(g,h)\mapsto (h,g)$. This corresponds to an algebra automorphism τ of $A\otimes_K A$. Show that $\tau(a\otimes b) = b\otimes a$.

   We know that A is a commutative algebra, so $\mu\tau = \mu$. We say that A is cocommutative provided $\tau\Delta = \Delta$. Show that this is equivalent to the group G being commutative.

> (a) says that Δ is a coassociative comultiplication.
> (b) says that ε is a counit for Δ.
> (d) says that S is an (invertible) antipode.
> This proves that if G is a (commutative) affine algebraic group, then $\mathcal{O}(G)$ is a commutative (and cocommutative) Hopf algebra. In fact, the duality between affine varieties and affine algebras restricts to a duality between affine algebraic groups and affine Hopf algebras, and restricts further to a duality between commutative affine algebraic groups and cocommutative affine Hopf algebras.

2. Consider the affine variety $G = K^\times \times K$, together with the action

$$G \times G \to G, \quad (a,b)\cdot(c,d) := (ac, bc+d).$$

   (a) Show that G is an affine algebraic group.
   (b) Compute $A := \mathcal{O}(G)$.
   (c) Compute the comultiplication $\Delta\colon A \to A \otimes_K A$, corresponding to the group multiplication.
   (d) Compute the antipode $S\colon A \to A$, corresponding to the inverse map $g \mapsto g^{-1}$ on G.

3. Repeat Question 2 for the affine variety $G = \{M \in \mathrm{GL}_2(K) : MM^t = 1\}$.

4. Consider the action of $G = K^\times$ on K^2 given by $g \cdot (x,y) := (gx, g^{-1}y)$.

   (a) Compute the comodule structure $K[X,Y] \to K[T,T^{-1},X,Y]$.
   (b) Show that every morphism $K^2 \to \operatorname{Spec} A$ which is constant on orbits factors uniquely through $\pi\colon K^2 \to K$, $(x,y) \mapsto xy$.
   (c) Show that π is not a geometric quotient.

5. Let an affine algebraic group G act on a variety X in such a way that $\pi\colon X \to X/G$ is a geometric quotient.

   (a) Show that if $U \subset X$ is open, then so too is $gU = \{g \cdot u : u \in U\}$ for each $g \in G$.
   (b) Show that π is an open map, so $U \subset X$ open implies $\pi(U)$ open.

6. Let G be an algebraic group. In this question we will simply write G-bundle instead of Zariski-locally trivial principal G-bundle.

   (a) Let G act on a variety X, and let $\pi\colon X \to Y$ be a G-bundle. Thus there exists an open cover $Y = \bigcup_i V_i$ and local trivialisations $\phi_i\colon G \times V_i \xrightarrow{\sim} \pi^{-1}(V_i)$, $\phi_i(gh,v) = g\phi_i(h,v)$.
   For each i,j set $V_{ij} := V_i \cap V_j$. We then have the automorphism $\phi_j^{-1}\phi_i$ on $G \times V_{ij}$, called a transition function. Show that $\phi_j^{-1}\phi_i$ is of the form $(g,v) \mapsto (g\gamma(v), v)$ for some morphism $\gamma\colon V_{ij} \to G$.
   (b) Let $\pi'\colon X' \to Y$ be another G-bundle. A morphism of G-bundles over Y is a morphism $\theta\colon X \to X'$ such that $\theta(g \cdot x) = g\theta(x)$ and $\pi'\theta = \pi$. Show that every such morphism is an isomorphism.
   It follows that we have a category of G-bundles over Y, and this category is a groupoid.
   (c) Show that a G-bundle $\pi\colon X \to Y$ is trivial, so isomorphic to $G \times Y \to Y$, $(g,y) \mapsto y$, if and only if π admits a section, so a morphism $\sigma\colon Y \to X$ such that $\pi\sigma = \mathrm{id}_Y$.
   (d) Let $\pi\colon X \to Y$ be a G-bundle. Given a morphism $\psi\colon Y' \to Y$, we can form the pullback

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

   Show that the map $\pi'\colon X' \to Y'$ is again a G-bundle.

(e) Let $\pi\colon X \to Y$ be a G-bundle. Show that the pullback $X \times_Y X \to X$ is a trivial G-bundle.

7. Let Q be the quiver $1 \to 2$. We have $\mathrm{Mod}(Q,(d,e)) \cong \mathbb{M}_{e\times d}(K)$, the variety of matrices of size $e \times d$.

(a) Let $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mod}(Q,(2,2))$ and consider the quiver Grassmannian $\mathrm{Gr}_Q(M,(1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Show that the pair of lines $([a,b],[a',b']) \in \mathbb{P}^1 \times \mathbb{P}^1$ corresponds to a submodule of M, so a point of $\mathrm{Gr}_Q(M,(1,1))$, if and only if the point $(a,0) \in K^2$ lies on the line $[a',b']$, which is if and only if $ab' = 0$.

Note that the corresponding submodule is the image of the injective module homomorphism

$$\begin{tikzcd} M & K^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & K^2 \\ U \arrow[u, hook] & K \arrow[u, "\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)"] \arrow[r, "\lambda"] & K \arrow[u, "\left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right)"'] \end{tikzcd}$$

Here λ is the unique map making the diagram commute.

(b) Recall the Segre embedding, $\mathbb{P}^1 \times \mathbb{P}^1 \cong V'(wz - xy) \subset \mathbb{P}^3$, sending the point $([a,b],[a',b'])$ to $[aa', ab', ba', bb']$. Show that this induces an isomorphism between $\mathrm{Gr}_Q(M,(1,1))$ and $V'(x,wz) \subset \mathbb{P}^3$.

(c) Show that the isomorphism $\mathbb{P}^2 \cong V'(x) \subset \mathbb{P}^3$, $[s,t,u] \mapsto [s,0,t,u]$, induces an isomorphism $\mathrm{Gr}_Q(M,(1,1)) \cong V'(su) \subset \mathbb{P}^2$.

In other words, we can regard $\mathrm{Gr}_Q(M,(1,1))$ as the union of the two projective lines $V'(s)$ and $V'(u)$ inside the projective plane $\mathbb{P}^2$.

(d) Show that the submodules corresponding to the projective line $V'(s)$ are those of the form

$$\begin{tikzcd} K^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & K^2 \\ K \arrow[u, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[r, "0"] & K \arrow[u, "\left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right)"'] \end{tikzcd}$$

(e) Show that the complement, so the submodules corresponding to the open affine $V(u) \cap D(s) = \{[1,t,0] : t \in K\} \cong \mathbb{A}^1$, are those of the form

$$\begin{tikzcd} K^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & K^2 \\ K \arrow[u, "\left(\begin{smallmatrix} 1 \\ t \end{smallmatrix}\right)"] \arrow[r, "1"] & K \arrow[u, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] \end{tikzcd}$$

To be handed in by 1st June.