Non-commutative Algebra 3, SS 2020

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Exercises 5

We work over an algebraically-closed field K.

1. Recall the fibre dimension theorem (3.3(E) Main Lemma).

   If $\pi\colon X \to Y$ is a dominant morphism of irreducible varieties, then any irreducible component of a fibre $\pi^{-1}(y)$ has dimension at least $\dim X - \dim Y$. Moreover, there is a non-empty open $U \subset Y$ with $\dim \pi^{-1}(y) = \dim X - \dim Y$ for all $u \in U$.

   (a) Assume this holds when X and Y are both affine. Show how to deduce the result when just Y is affine. Show how this implies the general case.

   (b) Translate the affine case into a result about commutative algebras.

2. Recall that a constructible subset is a finite union of locally closed subsets.

   (a) Prove that the class of constructible subsets is closed under finite unions, finite intersections, complements, and inverse images. (Chevalley's Theorem says it is also closed under images.)

   (b) Prove that every constructible set V contains an open dense subset of its closure $\bar{V}$, so there exists $U \subset V$ with U open and dense in $\bar{V}$.

   (c) Suppose a connected algebraic group G acts on a variety X, and let $V \subset X$ be a G-stable constructible subset, so $G \cdot V = V$. Show that we can decompose V into a finite disjoint union of G-stable, irreducible and locally closed subsets.

3. We know that if $f\colon X \to Y$ is a dominant morphism of varieties and X is irreducible, then Y is irreducible. In general the converse fails, so Y irreducible does not imply X irreducible. We prove that the converse holds in three situations.

   Let $f\colon X \to Y$ be a dominant morphism of varieties with Y irreducible. Assume further that each non-empty fibre is irreducible of the same dimension d.

   (a) Suppose f is an open map. Prove that X is irreducible. (In fact, all we need for this is that there is a dense set of points $y \in Y$ such that $f^{-1}(y)$ is nonempty and irreducible.)

   Hint. Suppose U, V are nonempty disjoint open subsets of X. Show that there exists a point $y \in f(U) \cap f(V)$ such that $f^{-1}(y)$ is irreducible. Obtain a contradiction.

   (b) In general, decompose $X = X_1 \cup \cdots \cup X_n$ into its irreducible components. Show that some $f(X_i)$ is dense in Y. Apply the Main Lemma to get that there is some i such that $f(X_i)$ is dense, and $f^{-1}(y) \subset X_i$ for all $y \in f(X_i)$. Deduce that if $f(X_i) = Y$, then $X = X_i$ is irreducible.

   Hint. If $y \in f(X)$, then $f^{-1}(y) = \bigcup_i (f^{-1}(y) \cap X_i)$, and $f^{-1}(y) \cap X_i$ is the fibre over y of the restriction $f|_{X_i}$.

(c) Suppose f is a closed map. Prove that X is irreducible.

(d) Suppose instead that f admist a section s, so a morphism $s\colon Y \to X$ with $fs = \mathrm{id}_Y$. Prove that X is irreducible.

Hint. Taking i as in (b), consider the intersection $X_i \cap s(Y)$.

Suppose an algebraic group G acts on a variety X. In general there is no geometric quotient X/G, but we can always form the variety $Z := \{(g,x) \in G \times X : gx = x\}$. We can then define $\dim_G X := \dim Z - \dim G$, and also $\mathrm{top}_G X := \mathrm{top}\, Z$, which is the number of irreducible components of Z having maximal dimension. By the result above we know that, if there is a geometric quotient, and all points in X have irreducible stabiliser $\mathrm{Stab}_G(x) := \{g \in G : gx = x\}$, then $\dim X/G = \dim_G X$ and $\mathrm{top}\, X/G = \mathrm{top}_G X$.

4. Let $\mathbb{M}_d(K)$ denote the variety of $d \times d$ matrices, and $C_d := \{(M,N) \in \mathbb{M}_d(K)^2 : MN = NM\}$ the commuting variety. We give a different proof of Theorem 3.3 (G).

(a) Given $M \in \mathbb{M}_d(K)$, show that $Z_M := \{N \in \mathbb{M}_d(K) : MN = NM\}$ is a subspace, so in particular an irreducible cone. Deduce that the map $M \mapsto \dim Z_M$ is upper semicontinuous. Show further that its minimal value is d, and $\dim Z_M = d$ if and only if $M \in U$, the set of regular matrices (that is, those matrices whose Jordan Normal Form has Jordan blocks with pairwise distinct eigenvalues). In particular, U is open dense in $\mathbb{M}_d(K)$.

(b) Set $C'_d := \{(M,N) \in C_d : N \in U\}$ and let $p\colon C'_d \to \mathbb{M}_d(K)$ be the projection onto the second co-ordinate. Use the previous exercise to show that C'_d is irreducible of dimension $d^2 + d$.

Hint: The map p has an obvious section.

(c) As in the lectures, given $(M,N) \in C_d$ there exists $R \in U$ commuting with M. Consider the morphism $f\colon \mathbb{A}^1 \to C_d$, $\lambda \mapsto (M, N + \lambda R)$. Then $f^{-1}(C'_d)$ is non-empty open, so $\mathrm{Im}(f) \subset \bar{C}'_d$. Hence C'_d is dense in C_d, so C_d is irreducible of dimension $d^2 + d$.

To be handed in by 8th June.