Non-commutative Algebra 3, SS 2020

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Exercises 6

We work over an algebraically-closed field K.

1. Let an algebraic group G act on a variety X, and assume that a geometric quotient $\pi\colon X \to X/G$ exists.

   (a) Given a point $x \in X$, we have the action $\rho_x\colon G \to X$, $g \mapsto gx$. We therefore get maps on tangent spaces

   $$\mathfrak{g} \xrightarrow{d(\rho_x)} T_x X \xrightarrow{(d\pi)_x} T_{Gx} X/G.$$

   Show that the composition is zero.

   (b) Now suppose further that π is Zariski locally trivial. Show that

   $$T_{(u,v)}(U \times V) \cong T_u U \times T_v V$$

   for any varieties U, V.

   Deduce that $(d\pi)_x$ is a cokernel for $d(\rho_x)$, so we have a natural isomorphism

   $$T_x X / \operatorname{Im}(d(\rho_x)) \xrightarrow{\sim} T_{Gx} X/G.$$

2. Consider the Grassmannian $\operatorname{Gr}(M,d) = \operatorname{Inj}(K^d, M)/\operatorname{GL}_d(K)$. Let $\theta \in \operatorname{Inj}(K^d, M)$ have image U and set $\rho_\theta\colon \operatorname{GL}_d(K) \to \operatorname{Inj}(K^d, M)$, $g \mapsto \theta g^{-1}$.

   (a) Show that $T_\theta \operatorname{Inj}(K^d, M) \cong \operatorname{Hom}(U, M)$.

   (b) Choose a cokernel $\phi\colon M \to M/U$ for θ. Show that the map

   $$T_\theta \operatorname{Inj}(K^d, M) \to \operatorname{Hom}(U, M/U), \quad \theta' \mapsto \phi\theta',$$

   is a cokernel for the map $d(\rho_\theta)\colon \mathbb{M}_d(K) \to T_\theta \operatorname{Inj}(K^d, M)$.

   (c) Deduce that $T_U \operatorname{Gr}(M,d) \cong \operatorname{Hom}(U, M/U)$.

3. More generally, consider a quiver Grassmannian $\operatorname{Gr}_A(M,d)$, where A is a K-algebra and M is a finite dimensional A-module.

   Set $\operatorname{Inj}_A(K^d, M)$ to be those $\theta \in \operatorname{Inj}(K^d, M)$ such that $\phi a_M \theta = 0$ for all $a \in A$, where ϕ is a cokernel for θ, and $a_M \in \operatorname{End}_K(M)$ is the map $m \mapsto a \cdot m$.

   (a) Show that if $\theta \in \operatorname{Inj}_A(K^d, M)$ has image U, then a_M restricts to an endomorphism of U, so that $U \leq M$ is an A-submodule.

   (b) Show that $\operatorname{Gr}_A(M,d) = \operatorname{Inj}_A(M,d)/\operatorname{GL}_d(K)$ is a geometric quotient, and that $\operatorname{Inj}_A(M,d) \to \operatorname{Gr}_A(M,d)$ is Zariski locally trivial.

   (c) Deduce that $T_U \operatorname{Gr}_A(M,d) \cong \operatorname{Hom}_A(U, M/U)$.

4. Let $A = KQ$ be the path algebra of a quiver, and M a finite dimensional A-module.

   (a) Using the long exact sequences for hom, show that if $\mathrm{Ext}^1(M,M) = 0$ and $U \leq M$ is a submodule, then $\mathrm{Ext}^1(U,M)$, $\mathrm{Ext}^1(M,M/U)$ and $\mathrm{Ext}^1(U,M/U)$ all vanish.

   (b) Use the Ringel form to deduce that $\dim \mathrm{Hom}_A(U,M)$ depends only on the dimension vectors of U and M.

   (c) Deduce that $\mathrm{Gr}_A(M,\underline{d})$ is smooth.

5. (a) Consider the surjective morphism $\phi\colon V(t^2-xt+y) \to \mathbb{A}^2$, $(x,y,t) \mapsto (x,y)$. For which $q \in V(t^2-xt+y)$ is the differential $(d\phi)_q$ surjective?

   (b) Suppose $\mathrm{char}\, K = p > 0$ and consider the surjective morphism $\theta\colon \mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^p$. Compute the differential $(d\theta)_q$.

To be handed in by 22nd June.