Non-commutative Algebra 3, SS 2020

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Exercises 7

We work over an algebraically-closed field K. Recall the degeneration order $M \leq N$ provided $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$.

1. Let $A = K[x,y]/(x^2,y^2)$, and let M_t be the two-dimensional module where the x and y actions are given by

$$M_t(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_t(y) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

   (a) Let $V \subset \mathrm{Mod}(A,4)$ be the subset consisting of the modules $M_s \oplus M_t$, so pairs of matrices

$$\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & & \\ s & 0 & & \\ & & 0 & 0 \\ & & t & 0 \end{pmatrix}, \qquad s,t \in K.$$

   Consider the morphism

$$\theta \colon \mathrm{GL}_4(K) \times V \to \mathrm{Mod}(A,4),$$

   given by the restriction of the usual $\mathrm{GL}_4(K)$ action on $\mathrm{Mod}(A,4)$.
   Show that $\mathrm{GL}_4(K) \times V$ is irreducible and has dimension 18.
   Show further the generic fibre of θ over the image has dimension 6 (it is the Zariski dimension of $\mathrm{Aut}(M_s \oplus M_t)$, equivalently the K-dimension of $\mathrm{End}(M_s \oplus M_t)$, for $s \neq t$).
   Deduce that $\overline{\mathrm{Im}(\theta)}$ is irreducible of dimension 12.

   (b) Show that the orbit closure $\overline{\mathcal{O}}_A$ is irreducible of dimension 12.

   (c) Deduce that there is some s,t such that A does not degenerate to $M_s \oplus M_t$.

2. Let A be a finite dimensional algebra, and $\binom{a}{b} \colon X \to M \oplus X$ a monomorphism.

   (a) Show that $f_t := b + t \cdot \mathrm{id}_X$ is an automorphism of X for all but finitely many $t \in K$.

   (b) Set $M_t := \mathrm{Coker}\binom{a}{f_t}$. Deduce that $M_t \cong M$ for all but finitely many $t \in K$.

   (c) Deduce that $M \leq M_0$.

> This result, due to Riedtmann, shows that if we have a short exact sequence
> $$0 \to X \to M \oplus X \to N \to 0$$
> then $M \leq N$. Zwara proved the (harder) converse implication.

3. Let $k = \mathbb{F}_q$ be a finite field with q elements. Set $\mathcal{N}_d$ to be the set of all nilpotent matrices in $\mathbb{M}_d(k)$. In this question we want to prove $|\mathcal{N}_d| = q^{d(d-1)}$.

   Recall that every matrix $M \in \mathbb{M}_d(K)$ determines a $K[T]$-module of dimension d, by letting T act as M. Then, given $M, M' \in \mathbb{M}_d(K)$, we have

   $$\mathrm{Hom}_{K[T]}(M, M') = \{\theta \in \mathbb{M}_d(K) : \theta M = M'\theta\}.$$

   Set $N = J_d(0)$ to be the Jordan block

   $$N := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

   and consider the set $\mathcal{S}_d$ of pairs (A, θ) such that $A \in \mathcal{N}_d$ and $\theta \in \mathrm{Hom}_{K[T]}(N, A)$.

   (a) Show that for every nilpotent matrix A we have $\dim \mathrm{Hom}_{K[T]}(N, A) = d$. Thus the projection $\mathcal{S}_d \to \mathcal{N}_d$ on to the first co-ordinate is surjective and every fibre has size q^d. In other words, $|\mathcal{S}_d| = q^d|\mathcal{N}_d|$.

   (b) Now consider the projection $\mathcal{S}_d \to \mathbb{M}_d(K)$ on to the second co-ordinate. Show that the fibre over θ has the same size as the fibre over $g\theta$ for every $g \in \mathrm{GL}_d(K)$. Thus we may assume that θ is in row-reduced form.

   (c) Take $(A, \theta) \in \mathcal{S}_d$. Since $\theta N = A\theta$, we know that $N(\mathrm{Ker}(\theta)) \subset \mathrm{Ker}(\theta)$. Assuming θ is in row reduced form, show that we must have $\theta = E_r := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $I_r \in \mathbb{M}_r(K)$ is the identity matrix and $r = \mathrm{rank}\,\theta$.

   (d) Show that for $\theta = E_r$, the number of nilpotent matrices A for which $\theta N = A\theta$ is $q^{r(d-r)}|\mathcal{N}_{d-r}|$. By induction this equals $q^{(d-1)(d-r)}$ for $r > 0$.

   (e) Show that the number of θ which have row reduced form E_r is $(q^d-1)(q^d-q)\cdots(q^d-q^{r-1})$.

   (f) It follows that

   $$|\mathcal{S}_d| - |\mathcal{N}_d| = \sum_{r>0}(q^d-1)(q^d-q)\cdots(q^d-q^{r-1})q^{(d-1)(d-r)}.$$

   Prove that this equals $(q^d-1)q^{d(d-1)}$, and hence that $|\mathcal{N}_d| = q^{d(d-1)}$.

4. Fix a finite dimensional A-module X. We want to show that for each i, the map $Y \mapsto \dim \operatorname{Ext}^i(X,Y)$ is upper semi-continuous on $\operatorname{Mod}(A,d)$.

   We fix a free resolution of X,

$$\cdots \to A^{r_2} \xrightarrow{f_2} A^{r_1} \xrightarrow{f_1} A^{r_0} \to X \to 0.$$

   We also set

$$C(U,V,W) := \{(\theta,\phi) : \phi\theta = 0\} \subset \operatorname{Hom}(U,V) \times \operatorname{Hom}(V,W).$$

   In the lectures we saw that the function

$$C(U,V,W) \to \mathbb{Z}, \quad (\theta,\phi) \mapsto \dim(\operatorname{Ker}(\phi)/\operatorname{Im}(\theta))$$

   is upper semi-continuous.

   (a) We fix a surjective algebra homomorphism $K\langle p_1,\dots,p_k\rangle \twoheadrightarrow A$. Then $\operatorname{Mod}(A,d) \subset \mathbb{M}_d(K)^k$. Given $a \in A$, lift it to a non-commutative polynomial $\alpha \in K\langle p_1,\dots,p_k\rangle$. Right multiplication by a gives a map $A \to A$, and thus a map $\rho_a \colon A \to A$.

   Now let $(y_1,\dots,y_k) \in \operatorname{Mod}(A,d)$, corresponding to a d-dimensional A-module Y. Show that, under the standard identification $\operatorname{Hom}(A,Y) \cong Y$, the induced homomorphism $\rho_a^* \colon \operatorname{Hom}(A,Y) \to \operatorname{Hom}(A,Y)$ corresponds to the linear map $\alpha(y_1,\dots,y_k) \colon K^d \to K^d$.

   (b) A homomorphism $f \colon A^s \to A^r$ corresponds to a matrix $(f_{ij}) \in \mathbb{M}_{r\times s}(A)$. We lift each f_{ij} to a non-commutative polynomial $f_{ij} \in K\langle p_1,\dots,p_k\rangle$. Show that the induced homomorphism $f_Y^* \colon \operatorname{Hom}(A^r,Y) \to \operatorname{Hom}(A^s,Y)$ corresponds to the block matrix $(f_{ij}(y_1,\dots,y_k)) \colon (K^d)^r \to (K^d)^s$.

   (c) Deduce that for each i there is a morphism of varieties

$$\operatorname{Mod}(A,d) \to C(K^{dr_{i-1}}, K^{dr_i}, K^{dr_{i+1}}), \quad Y \mapsto ((f_i)_Y^*, (f_{i+1})_Y^*).$$

   Conclude that the function $Y \mapsto \dim \operatorname{Ext}^i(X,Y)$ is upper semi-continuous on $\operatorname{Mod}(A,d)$.

To be handed in by 29th June.