Non-commutative Algebra 3, SS 2020

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Exercises 8

We work over an algebraically-closed field K. Recall that we have three partial orders on $\mathrm{Mod}(A,d)$.

Ext $M \leq_{\mathrm{ext}} N$ if there exists a sequence $M = M_0, M_1, \ldots, M_n = N$ and short exact sequences

$$0 \to L_i' \to M_{i-1} \to L_i'' \to 0, \quad M_i \cong L_i' \oplus L_i''.$$

Deg $M \leq_{\mathrm{deg}} N$ if $N \in \overline{\mathcal{O}}_M$.

Hom $M \leq_{\mathrm{hom}} N$ if $\dim \mathrm{Hom}(X,M) \leq \dim \mathrm{Hom}(X,N)$ for all finite dimensional A-modules X.

We saw in the lectures that $M \leq_{\mathrm{ext}} N$ implies $M \leq_{\mathrm{deg}} N$ implies $M \leq_{\mathrm{hom}} N$.

1. Let A be a (finite dimensional) K-algebra, and $M, N \in \mathrm{Mod}(A,d)$.

   (a) Show that for finite dimensional A-modules M, N we have $M \cong N$ if and only if $\dim \mathrm{Hom}(X,M) = \dim \mathrm{Hom}(X,N)$ for all finite dimensional A-modules X.
   Hint. We may proceed as follows. Take a basis $f_1, \ldots, f_d$ for $\mathrm{Hom}(M,N)$. Use these to construct a morphism $f \colon M \to N^d$, and show that the induced map $\mathrm{Hom}(N^d, N) \to \mathrm{Hom}(M,N)$ is onto. Deduce that the map $\mathrm{Hom}(N^d, M) \to \mathrm{End}(M)$ is onto, and hence that f is a split monomorphism. By Krull-Remak-Schmidt, M and N have a common direct summand. Finish by induction on $\dim M$.

   (b) Given a minimal projective presentation $Q \to P \to X \to 0$, show that

   $$\dim \mathrm{Hom}(X,M) - \dim \mathrm{Hom}(M,\tau X) = \dim \mathrm{Hom}(P,M) - \dim \mathrm{Hom}(Q,M).$$

   Deduce that $M \leq_{\mathrm{hom}} N$ if and only if $\dim \mathrm{Hom}(M,X) \leq \dim \mathrm{Hom}(N,X)$ for all finite dimensional A-modules X.
   Hint. We have the (minimal) projective presentation $P^\vee \to Q^\vee \to \mathrm{Tr}X \to 0$ as right A-modules, where $P^\vee := \mathrm{Hom}(P,A)$. Now tensor with M to obtain an exact sequence

   $$0 \to \mathrm{Hom}(X,M) \to \mathrm{Hom}(P,M) \to \mathrm{Hom}(Q,M) \to D\,\mathrm{Hom}(M,\tau X) \to 0.$$

The first result is due to Auslander, and generalised by Bongartz. It shows that the hom order, so $M \leq_{\mathrm{hom}} N$ provided $\dim \mathrm{Hom}(X,M) \leq \dim \mathrm{Hom}(X,N)$ for all X, is antisymmetric, and hence a partial order on the set of isomorphism classes.
The second result is due to Auslander and Reiten, and shows that the partial orders determined by $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$ for all X agree.

2. Consider the path algebra KQ, where Q is the quiver $1 \longrightarrow 3 \longleftarrow 2$

   (a) Show that this has Auslander-Reiten quiver

$$\begin{tikzcd}
 & P_1 \arrow[dr] & & S_2 \\
S_3 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & \\
 & P_2 \arrow[ur] & & S_1
\end{tikzcd}$$

   (b) Consider a finite dimensional module M. By the Krull-Remak-Schmidt Theorem, we can write

$$M \cong S_3^a \oplus P_1^b \oplus P_2^c \oplus I_3^d \oplus S_2^e \oplus S_1^f,$$

   and so we can use the shorthand $M \leftrightarrow (a, b, c, d, e, f)$.

   Compute the dimension vector $\underline{\dim}\, M$.

   Compute $\dim \operatorname{Hom}(X, M)$ as X runs through all six indecomposable modules.

   (c) Suppose $\underline{\dim}\, M = \underline{\dim}\, N$ with $N \leftrightarrow (a', b', c', d', e', f')$. Write out the conditions that $M \leq_{\text{hom}} N$, that is, $\dim \operatorname{Hom}(X, M) \leq \dim \operatorname{Hom}(X, N)$ for all six indecomposable modules X.

   (d) Using the three Auslander-Reiten sequences

$$0 \to S_3 \to P_1 \oplus P_2 \to I_3 \to 0$$

   and

$$0 \to P_1 \to I_3 \to S_2 \to 0 \quad 0 \to P_2 \to I_3 \to S_1 \to 0$$

   show that $M \leq_{\text{hom}} N$ implies $M \leq_{\text{ext}} N$.

It is known that $\leq_{\text{hom}}$ implies $\leq_{\text{ext}}$ when the algebra A is representation directed. This implies that A is representation finite, and includes all path algebras of Dynkin quivers.

3. Consider the algebra $A = KQ/I$ given by the quiver

$$\begin{tikzcd} 2 \arrow[r, "b"] & 1 \arrow[loop right, "a"] \end{tikzcd}$$

and I is the ideal generated by a^2 (c.f. Exercise 6.3 last semester). We know that this algebra is representation finite, having precisely seven indecomposables up to isomorphism. Moreover, there are two non-isomorphic indecomposables of dimension vector $2e_1 + e_2$, namely

$$X\colon\ K \xrightarrow{\binom{1}{0}} K^2 \circlearrowleft \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \qquad \text{and} \qquad Y\colon\ K \xrightarrow{\binom{0}{1}} K^2 \circlearrowleft \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$$

Compute $\mathrm{End}(X)$ and $\mathrm{End}(Y)$. This again shows that $X \ncong Y$.

Set

$$X_t\colon\ K \xrightarrow{\binom{t}{1}} K^2 \circlearrowleft \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$$

Show that $X_t \cong X$ for all $t \neq 0$.

This proves that $X \leq_{\mathrm{deg}} Y$. Explain why $X \nleq_{\mathrm{ext}} Y$.

> Thus $\leq_{\mathrm{deg}}$ does not imply $\leq_{\mathrm{ext}}$ for representation finite algebras.

4. We introduce a new partial order by saying $M \leq_{\mathrm{v.ext}} N$ (virtual extension) provided $M \oplus X \leq_{\mathrm{ext}} N \oplus X$ for some finite dimensional module X.

   (a) Show that $M \leq_{\mathrm{v.ext}} N$ implies $M \leq_{\mathrm{hom}} N$.

   As in Exercise 7.1, let $A = K[x,y]/(x^2,y^2)$, and let M_t be the two-dimensional module where the x and y actions are given by

$$M_t(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_t(y) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

   We have shown that $A \nleq_{\mathrm{deg}} M_s \oplus M_t$ for some s, t.

   (b) Show that we have a short exact sequence (in fact an Auslander-Reiten sequence)

$$0 \to \mathrm{rad}\, A \to A \oplus \mathrm{rad}\, A/\mathrm{soc}\, A \to A/\mathrm{soc}\, A \to 0.$$

   (c) Show that for all s, t we have short exact sequences

$$0 \to M_s \to \mathrm{rad}\, A \to K \to 0 \quad \text{and} \quad 0 \to K \to A/\mathrm{soc}\, A \to M_t \to 0.$$

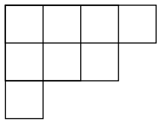
   (d) Using that $\mathrm{rad}\, A/\mathrm{soc}\, A \cong K^2$ deduce that

$$A \oplus K^2 \leq_{\mathrm{ext}} M_s \oplus M_t \oplus K^2 \quad \text{for all } s, t \in K,$$

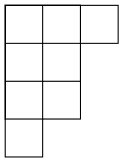
   and hence that $A \leq_{\mathrm{v.ext}} M_s \oplus M_t$.

> This example, due to Carlsson, shows that $\leq_{\text{v.ext}}$ does not imply $\leq_{\text{deg}}$. On the other hand, Zwara showed that if $M \leq_{\text{deg}} N$, then there is a Riedtamnn sequence $0 \to X \to M \oplus X \to N \to 0$, and so $M \leq_{\text{v.ext}} N$.

5. A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative integers, arranged in decreasing order. The Young diagram of λ has rows of size λ_i. For example

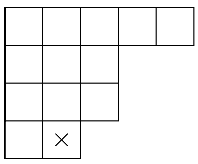
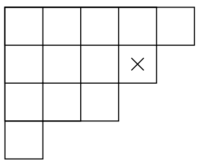
$\lambda = (4, 3, 1)$ has Young diagram 

The dual partition λ' is given by reflection the Young diagram in the diagonal. For example, the dual of $(4, 3, 1)$ above is

$\lambda' = (3, 2, 2, 1)$ having Young diagram 

The dominance order on partitions of n is given by $\lambda \lhd \mu$ provided $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i.

(a) Show that $\lambda \lhd \mu$ if and only if $\mu' \lhd \lambda'$.
Hint. Suppose that μ is obtained from λ by moving a 'bottom right corner' block to the next available space to the upper left. For example

$\lambda =$  $\rightsquigarrow \quad \mu =$ 

Show that, taking dual partitions, the inverse move is of the same type, but now from μ' to λ'. As in the lectures, the covers in the dominance order are all of this type.

Let λ and μ be two partitions. Their sum $\lambda + \mu$ is the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$; their cup product $\lambda \cup \mu$ is the partition given by rearranging the parts of λ and μ into decreasing order.

(b) Show that $\lambda' + \mu' = (\lambda \cup \mu)'$.

Recall that for each partition λ we have a nilpotent $K[t]$-module $M(\lambda)$, having Jordan blocks of sizes λ'_i. In particular, $M(d)$ is semisimple, and $M(1^d)$ is indecomposable.

(c) Show that $M(\lambda) \oplus M(\mu) \cong M(\lambda + \mu)$.

(d) Let $U \leq M(\lambda)$ be a submodule such that $\dim \operatorname{soc} U \leq r$. Show that $\dim U \leq \lambda'_1 + \cdots + \lambda'_r$.
Hint. Show that $\dim \operatorname{soc}(U/\operatorname{soc} U) \leq r$. Show that $M(\lambda)/\operatorname{soc} M(\lambda) \cong M(\lambda_{\geq 2})$, where $\lambda_{\geq 2} = (\lambda_2, \lambda_3, \ldots)$. Now use $U/\operatorname{soc} U \leq M(\lambda_{\geq 2})$ and induction.

(e) Deduce that $\lambda'_1 + \cdots + \lambda'_r$ is the maximum dimension of a submodule $U \leq M(\lambda)$ having $\dim \operatorname{soc} U \leq r$.

(f) Suppose now that we have a short exact sequence

$$0 \to M(\lambda) \to M(\xi) \to M(\mu) \to 0.$$

Use the previous part to deduce that $\xi' \lhd \lambda' + \mu'$, and hence that $\lambda \cup \mu \lhd \xi$.

Putting this together with the result from the lectures we see that if we have a short exact sequence

$$0 \to M(\lambda) \to M(\xi) \to M(\mu) \to 0,$$

then $\lambda \cup \mu \lhd \xi \lhd \lambda + \mu$. On the other hand, starting from the pair λ, μ, there is no simple criterion for determining which ξ arise as the middle term of such a short exact sequence; it is equivalent to saying the the Littlewood–Richardson coefficient $c^{\xi}_{\lambda\mu}$ is non-zero, which is a well-known hard problem.

To be handed in by 6th July.