Non-commutative Algebra 3, SS 2020

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Exercises 9

We work over an algebraically-closed field K.

1. (a) Consider the family of algebras $A_t := K[x]/(x^2 - tx)$ for $t \in K$. Compute $\operatorname{gl.dim} A_t$.

   (b) Consider the family of algebras $B_t := K[x, y]/(x^2, y^2 - ty)$ for $t \in K$. For which t is B_t representation finite?

2. Let G be a connected algebraic group acting on a variety X. For a G-stable constructible subset $Y \subset X$, set

$$Y_{(d)} := \{y \in Y : \dim Gy = d\} \quad \text{and} \quad Y_{(\leq d)} := \{y \in Y : \dim Gy \leq d\}.$$

We define

$$\dim_G Y := \max\{\dim Y_{(d)} - d : d \geq 0\} \quad \text{and} \quad \operatorname{top}_G Y := \sum_{\dim Y_{(d)} = \dim_G Y + d} \operatorname{top} Y_{(d)}.$$

Let $Y, Y_i \subset X$ be G-stable constructible subsets.

   (a) Show that $\dim_G(Y_1 \cup Y_2) = \max\{\dim_G Y_1, \dim_G Y_2\}$.

   (b) Show that $\dim_G Y = \max\{\dim Y_{(\leq d)} - d : d \geq 0\}$.

   (c) Suppose $Z \subset Y$ is constructible and meets every orbit in Y. Show that $\dim_G Y \leq \dim Z$.

   (d) Show that $\dim_G Y = 0$ if and only if Y is a finite union of G-orbits, in which case $\operatorname{top}_G Y$ equals the number of orbits.

   (e) Define $Z := \{(g, x) : gx = x\} \subset G \times X\}$, and let $\pi\colon Z \to X$ be the projection. Show that $\dim_G Y = \dim \pi^{-1}(Y) - \dim G$. Show further that if $\operatorname{Stab}_G(y)$ is connected for all $y \in Y$, then $\operatorname{top}_G Y = \operatorname{top} Z$.

3. Consider the path algebra KQ of the Kronecker quiver. The indecomposable modules of dimension vector smaller than $\alpha = (1,2)$ are, up to isomorphism, given by the following list

$$S_1\colon K \rightrightarrows 0 \qquad S_2\colon 0 \rightrightarrows K \qquad P_1\colon K \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} K^2$$

together with

$$R_a\colon K \underset{a}{\overset{1}{\rightrightarrows}} K \quad \text{for } a \in K, \qquad R_\infty\colon K \underset{1}{\overset{0}{\rightrightarrows}} K$$

Set $X := \mathrm{Mod}(KQ, \alpha)$. Describe $X_{(d)}$ for all d, giving representatives for the orbits it contains and computing its dimension. Hence compute the number of parameters $\dim_{\mathrm{GL}(\alpha)} X$.

4. Set $A := K[x,y]/(x,y)^2$ and KQ the path algebra of the Kronecker quiver. Consider the functor $F\colon A\text{−mod} \to KQ\text{−mod}$ defined as follows. Given an A-module M, we can regard this as a K-vector space M equipped with two endomorphisms x_M and y_M. The Jacobson radical of A is $J = A(x,y) = Kx + Ky$, so $x_M M, y_M M \subset JM$ and $x_M JM = 0 = y_M JM$. We thus have induced maps $\bar{x}_M, \bar{y}_M\colon M/JM \to JM$. The functor F then sends M to the Kronecker representation

$$F(M)\colon M/JM \underset{\bar{y}_M}{\overset{\bar{x}_M}{\rightrightarrows}} JM$$

(a) How does F act on morphisms?

(b) Deduce that F sends indecomposable A-modules to indecomposable KQ-modules. Moreover, $M \cong M'$ if and only if $FM \cong FM'$.

(c) Show further that if N is an indecomposable KQ-module other than the simple S_2, then $N \cong FM$ for some indecomposable A-module.

(d) Using that KQ is tame, deduce that A is tame.

(e) Is the functor F a representation embedding?

To be handed in by 13th July.