Non-commutative Algebra 3, SS 2020

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Solutions 1

1. Let X be a tolopological space. Recall that a subset is closed provided its complement is open.

(a) Show that an arbitrary intersection of closed sets is again closed.

(b) We define the closure of a subset $Y \subset X$ to be the intersection over all closed subsets of X containing Y.

Prove that the following conditions are equivalent for a subset $Y \subset X$.

(i) Y is an open subset of a closed subset of X.

(ii) Y is an open subset of its closure.

(iii) Y is the intersection of an open and a closed subset of X.

Such a subset Y is said to be locally closed.

(c) Now let $Y \subset X$ be a locally closed subset. Show that a subset $Z \subset Y$ is locally closed in Y if and only if it is locally closed in X.

Proof. A topological space is given by a set X together with the collection of open subsets of X, such that $\emptyset$ and X are both open, a finite intersection of open sets is again open, and an arbitrary union of open sets is again open.

(a) A set is closed if its complement is open. Since the complement of an arbitrary intersection equals the union of the complements, $\left(\bigcap_i C_i\right)^c = \bigcup_i C_i^c$, it follows that an arbitrary intersection of closed sets is again closed.

(b) (ii) $\Rightarrow$ (i). Trivial, since the closure of Y is a closed set.

(iii) $\Rightarrow$ (ii). Write $Y = U \cap C$ with U open and C closed. Then $Y \subset C$ and C closed implies $\bar{Y} \subset C$. Since $Y \subset \bar{Y}$, we have $Y \subset U \cap \bar{Y} \subset U \cap C = Y$. Thus $Y = U \cap \bar{Y}$.

(i) $\Rightarrow$ (iii). Let Y be open inside the closed subset C. The set C is understood to have the subspace topology, so its open subsets are $U \cap C$ with $U \subset X$ open. So, $Y = U \cap C$ for some open U of X.

(c) Write $Y = U \cap C$ with U open in X, and C closed. Equip Y with the subspace topology.

If $Z \subset Y$ is locally closed in X, then $Z = U' \cap C'$ with U' open and C' closed, and $Z = Z \cap Y = (U \cap Y) \cap (C \cap Y)$. So Z is locally closed in Y.

Conversely, if $Z \subset Y$ is locally closed, then $Z = (U' \cap Y) \cap (C' \cap Y)$ for some U' open in X and C' closed in X. Thus $Z = (U \cap U') \cap (C \cap C')$ is locally closed in X. $\square$

2. Let $\theta\colon Z \to Y$ and $\phi\colon Y \to X$ be morphisms of spaces with functions. Show that the composition $\phi\theta\colon Z \to X$ is a morphism of spaces with functions.

Proof. A space with functions is a topological space X together with a K-subalgebra $\mathcal{O}(U)$ of the algebra of all functions $U \to K$ for each open subset U of X, satisfying the axioms

- if $U = \bigcup_i U_i$ and $f\colon U \to K$, then $f \in \mathcal{O}(U)$ if and only if $f|_{U_i} \in \mathcal{O}(U_i)$ for all i.
- if $f \in \mathcal{O}(U)$, then $D(f) := \{u \in U : f(u) \neq 0\}$ is open and $1/f \in \mathcal{O}(D(f))$.

Aside. The first axiom says that we have a sheaf of rings on X, so that X is a ringed space. Ringed spaces are more general than this, though, since we don't require that we have an algebraically closed field K, and we don't require that $\mathcal{O}(U)$ is a subalgebra of functions $U \to K$.

A morphism $\phi\colon Y \to X$ of spaces with functions consists of a continuous map $\phi\colon Y \to X$ such that $f\phi \in \mathcal{O}_Y(\phi^{-1}(U))$ for all open $U \subset X$ and $f \in \mathcal{O}_X(U)$.

Let $\theta\colon Z \to Y$ be another morphism of spaces with functions, and consider the composite $\phi\theta\colon Z \to X$. Let $U \subset X$ be open and $f \in \mathcal{O}_X(U)$. Then $\phi^{-1}(U)$ is open in Y and $f\phi \in \mathcal{O}_Y(\phi^{-1}(U)$. Hence $\theta^{-1}\phi^{-1}(U)$ is open in Z and $f\phi\theta \in \mathcal{O}_Z(\theta^{-1}\phi^{-1}(U))$. The result follows, noting that $(\phi\theta)^{-1}(U) = \theta^{-1}\phi^{-1}(U)$. $\square$

3. Let X be a space with functions, and Y a subset of X with its induced structure as a space with functions.

(a) Show that the inclusion $\iota\colon Y \hookrightarrow X$ is a morphism of spaces with functions.

(b) Let Z be any space with functions and $\theta\colon Z \to Y$ any map. Show that θ is a morphism of spaces if and only if $\iota\theta\colon Z \to X$ is a morphism of spaces.

In other words, morphisms $Z \to Y$ can be thought of as morphisms $Z \to X$ with image contained in Y.

Proof. Let X be a space with functions, and Y a subspace. Then we can give Y the subspace topology, so the open sets are $U' := U \cap Y$ for $U \subset X$ open. We then define $\mathcal{O}_Y(U')$ to be those functions $f\colon U' \to K$ such that, for each $y \in U'$, there exists an open $V \subset X$ with $y \in V$ and a $g \in \mathcal{O}_X(V)$ such that f and g agree on $V \cap U'$.

Check that this is indeed a space with functions. Let $U' = \bigcup_i U_i'$ and $f\colon U' \to K$, and write $f_i := f|_{U_i'}$.

Suppose $f \in \mathcal{O}_Y(U')$. Then for each $y \in U_i'$ there exists an open $V \subset X$ with $y \in V$ and $g \in \mathcal{O}_X(V)$ such that f, g agree on $V \cap U'$. Clearly f_i and g agree on $V \cap U_i'$, so $f_i \in \mathcal{O}_Y(U_i')$.

Suppose instead that $f_i \in \mathcal{O}_Y(U_i')$ for all i. Take $y \in U'$. Then $y \in U_i'$ for some i and there exists $\hat{V} \subset X$ open with $y \in \hat{V}$, and $\hat{g} \in \mathcal{O}_X(\hat{V})$ such that $f_i, \hat{g}$ agree on $U_i' \cap \hat{V}$. Since U_i' is open in Y we can write $U_i' = U_i \cap Y$ for some $U_i \subset X$ open. Set $V := U_i \cap \hat{V}$ and $g := \hat{g}|_V \in \mathcal{O}_X(V)$. Then $U' \cap V = U_i' \cap \hat{V}$, so f, g agree on $U' \cap V$, and hence $f \in \mathcal{O}_Y(U')$.

(a) Let $\iota\colon Y \hookrightarrow X$ be the inclusion, so that $\iota^{-1}(U) = U \cap Y$ and ι is continuous. Also, if $f \in \mathcal{O}_X(U)$, then $f\iota = f|_{Y\cap U}$, so clearly $f\iota \in \mathcal{O}_Y(\iota^{-1}(U))$. This shows that ι is a morphism of spaces with functions.

(b) If θ is a morphism, then so too is the composition $\iota\theta$. Suppose therefore that $\iota\theta$ is a morphism. Let $U' \subset Y$ be open, and write $U' = U \cap Y$ with $U \subset X$ open. Then $U' = \iota^{-1}(U)$, so $\theta^{-1}(U') = (\iota\theta)^{-1}(U)$ is open in Z, whence $\iota\theta$ is continuous.

Next take $f \in \mathcal{O}_Y(U')$. Given $y \in U'$, take an open $V \subset X$ containing y and $g \in \mathcal{O}_X(V)$ such that f, g agree on $V \cap U'$. Then the restriction of $f\theta$ lies in $\mathcal{O}_Z(\theta^{-1}(V \cap U')$. We can do this for all points of U', obtaining an open covering $U' = \bigcup_i U_i'$, whose preimages under θ form an open covering $\theta^{-1}(U') = \bigcup_i \theta^{-1}(U_i')$. Thus each restriction of $f\theta$ is regular, so $f\theta$ is itself regular. This proves that θ is a morphism of spaces with functions. $\square$

4. Let X be a set, and $\mathcal{B}$ a collection of subsets of X which is closed under finite intersections and contains $\emptyset$ and X. Suppose further that we have algebras $\mathcal{O}_X'(U)$ for each $U \in \mathcal{B}$ with the property that, if $f \in \mathcal{O}_X'(U)$ and $V \subset U$ in $\mathcal{B}$, then $f|_V \in \mathcal{O}_X'(V)$.

(a) Define the distinguished open sets as

$$D(g, U) := \{u \in U : g(u) \neq 0\} \quad \text{for} \quad U \in \mathcal{B},\ g \in \mathcal{O}_X'(U).$$

Show that this collection of subsets is again closed under finite intersections, and contains $\mathcal{B}$.

Show that there is a topology on X whose open sets are precisely the arbitrary unions of distinguished opens.

(b) Let $W \subset X$ be open. We define $\mathcal{O}_X(W)$ to be the set of those functions $h\colon W \to K$ for which there exists, for each point $w \in W$, an open $U \in \mathcal{B}$ and $f, g \in \mathcal{O}_X'(U)$ such that

- $w \in D(g, U)$.
- $h = f/g$ on $W \cap D(g, U)$.

Show that this construction gives X the structure of a space with functions.

(c) Let Z be any space with functions, and $\theta\colon Z \to X$ any map. Show that θ is a morphism of spaces with functions if and only if

- $\theta^{-1}(U)$ is open in Z for all $U \in \mathcal{B}$.
- $f\theta \in \mathcal{O}_Z(\theta^{-1}(U))$ for all $U \in \mathcal{B}$ and $f \in \mathcal{O}_X'(U)$.

Proof. I should have said that each $\mathcal{O}'(U)$ is a K-subalgebra of functions $U \to K$. This was implicit in Exercise (b), but should have been made explicit. Sorry if this lead to any confusion.

(a) Suppose $f \in \mathcal{O}'(U)$ and $g \in \mathcal{O}'(V)$. Then $U \cap V$ is in $\mathcal{B}$, and the restrictions of f, g lie in $\mathcal{O}'(U \cap V)$. Thus $D(f, U) \cap D(g, V) = D(fg, U \cap V)$. Thus the collection of distinguished opens is closed under finite intersections. If $U \in \mathcal{B}$, then the identity lies in $\mathcal{O}'(U)$ (this is the function $u \mapsto 1$ for all $u \in U$), and $D(1, U) = U$. Thus each element of $\mathcal{B}$ is distinguished.

Now, given any collection $\mathcal{U}$ of subsets of X, closed under finite intersections and containing $\emptyset, X$, there exists a topology on X whose opens are precisely the arbitrary unions of elements of $\mathcal{U}$. For, it is clear that this larger collection contains $\emptyset$ and X, and is closed under arbitrary unions. It is therefore enough to show that the intersection of two such elements is again in the set, since then by induction it will be closed under all finite intersections.

Let $U = \bigcup_i U_i$ and $V = \bigcup_j V_j$ with $U_i, V_j \in \mathcal{U}$. Then $U \cap V = \bigcup_{i,j}(U_i \cap V_j)$, and $U_i \cap V_j \in \mathcal{U}$, so we are done.

(b) Let $W = \bigcup_i W_i$, and $h\colon W \to K$. Write $h_i := h|_{W_i}$.

Suppose $h \in \mathcal{O}(W)$. Take $w \in W_i$. Then there exists $U \in \mathcal{B}$, and $f, g \in \mathcal{O}'(U)$, such that $w \in D(g, U)$ and $h = f/g$ on $W \cap D(g, U)$. Then $h_i = f/g$ on $W_i \cap D(g, U)$, so $h_i \in \mathcal{O}(W_i)$ for all i.

Suppose instead that $h_i \in \mathcal{O}(W_i)$ for all i. Take $w \in W$, say with $w \in W_i$. Since W_i is open, it is a union of distinguished opens, so we can write $w \in D(p, V) \subset W_i$ for some $V \in \mathcal{B}$ and $p \in \mathcal{O}'(V)$. Next, as above, we know that $h|_{D(p,V)} = h_i|_{D(p,V)}$ is regular, so lies in $\mathcal{O}(D(p, V))$. Thus there exists $\hat{U} \in \mathcal{B}$ and $\hat{f}, \hat{g} \in \mathcal{O}'(\hat{U})$ such that $w \in D(\hat{g}, \hat{U})$ and $h = \hat{f}/\hat{g}$ on $D(p, V) \cap D(\hat{g}, \hat{U})$. Set $U := \hat{U} \cap V \in \mathcal{B}$, as well as $f := \hat{f}p$ and $g := \hat{g}p$ in $\mathcal{O}'(U)$. Then $D(g, U) \subset D(p, V) \subset W$, it contains w, and $h = f/g$ on $D(g, U)$. We conclude that $h \in \mathcal{O}(W)$ is regular.

Next, take $h \in \mathcal{O}(W)$. We need to show that $W' := \{w \in W : h(w) \neq 0\}$ is open, and $1/h \in \mathcal{O}(W')$. Given $w \in W'$, take $U \in \mathcal{B}$ and $f, g \in \mathcal{O}'(W')$ such that $w \in D(g, U)$ and $h = f/g$ on $W \cap D(g, U)$. Then $v \in W' \cap D(g, U)$ if and only if $f(v) \neq 0$, so $W' \cap D(g, U) = D(fg, U)$ is open. Moreover, $1/h = g/f = g^2/fg$ on $D(fg, U)$, so is regular. It follows that W' is a union of distinguished opens, so is itself open, and $1/h$ is regular on each of these distinguished opens, so $1/h$ is itself regular. Thus W' is open and $1/h \in \mathcal{O}(W')$.

This proves that X is a space with functions. Furthermore, it is clear from the construction that $\mathcal{O}'(U)$ is a subalgebra of $\mathcal{O}(U)$ for all $U \in \mathcal{B}$.

(c) If θ is a morphism of spaces with functions, then the two conditions necessarily hold. Suppose therefore that $\theta\colon Z \to X$ satisfies the two conditions. Consider a distinguished open $D(g, U)$. Then $\theta^{-1}(U)$ is open and $g\theta \in \mathcal{O}_Z(\theta^{-1}(U))$. Thus $D(g\theta, \theta^{-1}(U))$ is open in Z, and this is precisely the preimage of $D(g, U)$. In general, every open of X is a union of distinguished opens, so its preimage is the union of the preimages, and hence is open. This proves that θ is continuous.

Now let $W \subset X$ be open, and $h \in \mathcal{O}_X(W)$. Given $z \in \theta^{-1}(W)$, we can find $U \in \mathcal{B}$, and $f, g \in \mathcal{O}'_X(U)$, such that $\theta(z) \in D(g, U)$ and $h = f/g$ on $W \cap D(g, U)$. Then $V := \theta^{-1}(W \cap D(g, U))$ is an open neighbourhood of z, and $h\theta = (f\theta)/(g\theta)$

on V. Since $f\theta, g\theta$ are regular on $\theta^{-1}(U)$, it follows that $h\theta$ is regular on V. It follows that $\theta^{-1}(W)$ has an open covering such that the restriction of $h\theta$ to any of the open pieces is regular, and hence $h\theta \in \mathcal{O}_Z(\theta^{-1}(W))$.

This proves that θ is a morphism of spaces with functions. $\square$

5. Apply the previous result to the set K^n, using $\mathcal{B} = \{K^n, \emptyset\}$ together with $\mathcal{O}'(K^n) = K[X_1, \ldots, X_n]$ and $\mathcal{O}'(\emptyset) = 0$, where the X_i are the co-ordinate functions on K^n.

(a) Show that the resulting space with functions is $\mathbb{A}^n$.

(b) Deduce moreover that if Z is a space with functions and $\theta\colon Z \to \mathbb{A}^n$ is any map, then θ is a morphism of spaces with functions if and only if $X_i\theta \in \mathcal{O}(Z)$ for all i.

Proof. Recall that we have the co-ordinate functions $X_i\colon K^n \to K$, $x \mapsto x_i$. Thus $K[X_1, \ldots, X_n]$ is a subalgebra of functions $K^n \to K$.

(a) The distinguished opens are just the $D(g) = \{x \in K^n : g(x) \neq 0\}$ for $g \in K[X_1, \ldots, X_n]$, so the opens in K^n are the arbitrary unions of such distinguished opens. Hence K^n has the Zariski topology.

Let $U \subset K^n$ be open and $h\colon U \to K$. Then $h \in \mathcal{O}(U)$ if and only if, for each $u \in U$, there exist $f, g \in K[X_1, \ldots, X_n]$ such that $g(u) \neq 0$ and $h = f/g$ on $U \cap D(g)$. Thus the resulting space with functions is precisely affine space $\mathbb{A}^n$, as defined in the lectures.

(b) Now let Z be a space with functions, and $\theta\colon Z \to \mathbb{A}^n$ any map. By (c) of the previous exercise, we see that θ is a morphism of spaces with functions if and only if $f\theta \in \mathcal{O}(Z)$ for all $f \in K[X_1, \ldots, X_n]$. Since the regular functions form a K-algebra, it is sufficient to ask that $X_i\theta$ is regular for all i; equivalently, writing $\theta(z) = (\theta_1(z), \ldots, \theta_n(z)) \in K^n$ as in the lectures, we see that $X_i\theta = \theta_i$, so θ is a morphism if and only if θ_i is regular for all i. $\square$

Here is another example. Take the set $K \cup \{\infty\}$, the collection $\mathcal{B} = \{\emptyset, K - \{0\}, K, U, K \cup \infty\}$, where U is the complement of $\{0\}$. Now take $\mathcal{O}'(\emptyset) = 0, \mathcal{O}'(K \cup \{\infty\}) = K$, $\mathcal{O}'(K) = K[X]$, $\mathcal{O}'(U) = K[X^{-1}]$, and $\mathcal{O}'(K - \{0\}) = K[X, X^{-1}]$, where X is the usual co-ordinate function on K, and X^{-1} acts on U by sending $x \mapsto 1/x$ for $0 \neq x \in K$, and $\infty \mapsto 0$.

This satisfies the conditions, and so we obtain a space with functions, which is precisely the projective line $\mathbb{P}^1$. The topology is the cofinite topology. If Z is a space with functions, and $\theta\colon Z \to \mathbb{P}^1$ and map, then θ is a morphism provided $\theta^{-1}(K)$ is open and $X\theta$ is a regular function on it, and also $\theta^{-1}(U)$ is open and $X^{-1}\theta$ is a regular function on it.

6. We can also apply this result to describe the product. Let X and Y be spaces with functions.

(a) Using Exercise (4) show that we can endow the set $X \times Y$ with the structure of a space with functions as follows.

We take $\mathcal{B}$ to be the collection of $U \times V$ such that $U \subset X$ and $V \subset Y$ are both open. We take $\mathcal{O}'_{X\times Y}(U \times V)$ to be those functions h such that

$$h(u,v) := \sum_{\text{finite}} f_i(u)g_i(v), \quad \text{with } f_i \in \mathcal{O}_X(U) \text{ and } g_i \in \mathcal{O}_Y(V).$$

(b) Show that the projection maps $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ are both morphisms of spaces with functions.

(c) Let $p_X \colon Z \to X$ and $p_Y \colon Z \to Y$ be morphisms of spaces with functions. Show that there is a unique morphism of spaces with functions $p \colon Z \to X \times Y$ such that $p_X = \pi_X p$ and $p_Y = \pi_Y p$.

This proves that $X \times Y$ with the above structure as a space with functions is a categorical product. Namely, we have morphisms $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ such that the induced map

$$\mathrm{Hom}(Z, X\times Y) \to \mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,Y), \quad p \mapsto (\pi_X p, \pi_Y p),$$

is bijective.

Proof. Just to be clear, $h \in \mathcal{O}'(U \times V)$ provided there exists n and $f_i \in \mathcal{O}_X(U)$ and $g_i \in \mathcal{O}_Y(V)$ for $i = 1, \dots, n$ such that $h(u,v) = \sum_{i=1}^n f_i(u)g_i(v)$ for all $(u,v) \in U \times V$.

(a) The collection $\mathcal{B}$ contains $\emptyset$ and $X \times Y$, and is closed under taking finite intersections. If $h \in \mathcal{O}'(U \times V)$ and $U' \times V' \subset U \times V$, then clearly the restriction of h lies in $\mathcal{O}'(U \times V')$. Thus the conditions are satisfied, and we have endowed $X \times Y$ with the structure of a space with functions.

(b) Let $U \subset X$ be open, and $f \in \mathcal{O}(X)$. Then $\pi_X^{-1}(U) = U \times Y$ is open, and $f\pi_X(u,y) = f(u)$ clearly lies in $\mathcal{O}'(U \times Y)$, so is regular. Thus π_X is a morphism. Similarly π_Y is a morphism.

(c) Sorry, there was a typo here: clearly we should have $p_Y \colon Z \to Y$.

We can define the map $p = (p_X, p_y) \colon Z \to X \times Y$, $z \mapsto (p_X(z), p_Y(z))$. This satisfies $\pi_X p = p_X$ and $\pi_Y p = p_Y$, and is the unique map (of sets) satisfying this. We therefore just need to check that p is a morphism of spaces with functions.

Consider the open set $U \times V$. Then $p^{-1}(U \times V) = p_X^{-1}(U) \cap p_Y^{-1}(V)$, so is open in Z. Now take $h \in \mathcal{O}'(U \times V)$, say with $h(u,v) = \sum_{\text{finite}} f_i(u)g_i(v)$ where $f_i \in \mathcal{O}_X(U)$ and $g_i \in \mathcal{O}_Y(V)$. Then $hp(z) = \sum_{\text{finite}} f_i p_X(z) g_i p_Y(z)$. We know that $f_i p_X \in \mathcal{O}_Z(p_X^{-1}(U))$ and $g_i p_Y \in \mathcal{O}_Z(p_Y^{-1}(V))$, so their restrictions all lie in $\mathcal{O}_Z(p^{-1}(U \times V))$, and hence $hp \in \mathcal{O}_Z(p^{-1}(U \times V))$. By Exercise 4 (c) we conclude that p is indeed a morphism of spaces with functions. $\square$

Here is some context/motivation for the category of spaces with functions. I don't know what was said in the lectures, but there was not much discussion about this in the online notes.

We are interested in the affine and projective varieties, so more generally quasi-projective varieties (locally closed subspaces of some projective space). The resulting category is easily seen to contain all finite products (as in Exercise 6) and coproducts (disjoint unions, which we can embed in suitable larger variety). It also contains all equalisers, so contains all finite limits (standard exercise in basic category theory). The basic problem is that it does not contain coequalisers, so does not contain all colimits. We also say that the category is complete, but not cocomplete.

Examples of coequalisers include quotients by group actions. If a group G acts on a set X, then we have the group action $\rho\colon G \times X \to X$, and the second projection $\pi\colon G \times X \to X$, and their coequaliser is the set of G-orbits on X, denoted X/G.

One way around this problem is thus to embed the category of (quasi-projective) varieties into a larger category which is both complete and cocomplete, and such that colimits and coproducts agree. We can then form a coequaliser/colimit in the larger category, and ask whether it lies in the smaller category. One subtlety then arises that a colimit may exist in the category of varieties, but it is not the colimit in the larger category. This can be seen as the motivation for the definition of a geometric quotient: it is a quotient (colimit) in the category of varieties which satisfies the stronger property of remaining a quotient (colimit) in the larger category.

The usual approach in algebraic geometry is to define the category of locally ringed spaces. This is very general and covers all affine schemes, so $\operatorname{Spec} R$ for an arbitrary commutative ring R. Kempf instead restricted to algebraically closed fields, and introduced the category of spaces with functions. Both of these categories are complete and cocomplete, and so can be used as described above. They are distinct, however; for example, if X is a space with functions, then each ring $\mathcal{O}(U)$ is necessarily a domain, whereas a locally ringed space can have nilpotents.

As a consequence, spaces with functions are more elementary, but the price one pays is in doing some of the more sophisticated constuctions such as considering tangent spaces as morphisms from the ring $K[t]/(t^2)$, or the more unified approach between the 'closed' points and the 'generic' points of a scheme.