Non-commutative Algebra 3, SS 2020

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Solutions 2

Throughout K will be an algebraically closed field. For simplicity we define an affine K-algebra to be one which is finitely generated, commutative and reduced.

Nullstellensatz. *Let A be a finitely generated (commutative) K-algebra.*

- *If $\mathfrak{m} \lhd A$ is a maximal ideal, then the natural map $K \to A/\mathfrak{m}$ is an isomorphism.*
- *If $I \lhd A$ is any ideal, then its radical $\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n\}$ is the intersection of all maximal ideals containing I, so $\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$.*

Sketch of proof. We know that $\bar{A} := A/\mathfrak{m}$ is a field which is finitely generated as a K-algebra. By Noether's Normalisation Lemma, $\bar{A}$ is finite over a polynomial subalgebra $R = K[x_1, \dots, x_r]$. Since finite extensions are integral, we know that R is a field. If $r \geq 1$, then (x_1) is a proper ideal, a contradiction. Thus $R = K$ and $\bar{A}$ is finite dimensional over K, so algebraic. Since K is algebraically closed, we conclude that $\bar{A} = K$.
By passing to the quotient $A/\sqrt{I}$, which is reduced, it is enough to show that $\bigcap \mathfrak{m} = 0$ in every affine algebra. Suppose $a \neq 0$. Then a is not nilpotent, so the localisation $A_a \cong A[t]/(1 - at)$ is non-zero and hence contains a maximal ideal $\mathfrak{n}$. Its preimage $\mathfrak{m}$ is a prime ideal of A, and we have the inclusions $K \subset A/\mathfrak{m} \subset A_a/\mathfrak{n}$. The composite is an isomorphism by the first part, so $A/\mathfrak{m} \cong K$ and $\mathfrak{m}$ is a maximal ideal of A. Clearly a is a unit in A_a, so does not lie in $\mathfrak{m}$. Hence $\bigcap \mathfrak{m} \subset 0$ as required. $\square$

1. Let A be a affine K-algebra, and set $\operatorname{Spec} A$ to be the set of maximal ideals of A.

(a) There is an injective algebra homomorphism from A to the algebra of functions $\operatorname{Spec} A \to K$, sending a to the map $\mathfrak{m} \mapsto a + \mathfrak{m} \in A/\mathfrak{m} \cong K$.

Observe: $\mathfrak{m}$ is the kernel of the map $A \to K$, $a \mapsto a(\mathfrak{m})$.
Turn $\operatorname{Spec} A$ into a space with functions by taking $\mathcal{B} := \{\emptyset, \operatorname{Spec} A\}$ and $\mathcal{O}'(\operatorname{Spec} A) := A$. If X is any space with functions, then a map $\theta \colon X \to \operatorname{Spec} A$ is a morphism of spaces with functions if and only if $a\theta \in \mathcal{O}(X)$ for all $a \in A$.

(b) Let $\theta \colon X \to \operatorname{Spec} A$ be a morphism of spaces with functions. Show that $a \mapsto a\theta$ is a K-algebra homomorphism $\phi \colon A \to \mathcal{O}(X)$. Moreover, for each $x \in X$, the maximal ideal $\theta(x) \in \operatorname{Spec} A$ is precisely the kernel of the map $A \to K$, $a \mapsto \phi(a)(x)$.

(c) Conversely, given a K-algebra homomorphism $\phi \colon A \to \mathcal{O}(X)$, show that there is a map $\theta \colon X \to \operatorname{Spec} A$ sending $x \in X$ to the kernel of $A \to K$, $a \mapsto \phi(a)(x)$. Show that $a\theta = \phi(a) \in \mathcal{O}(X)$. Deduce that θ is a morphism of spaces with functions.

(d) Show that these two constructions are mutually inverse, so yield a bijection between morphisms $X \to \operatorname{Spec} A$ and K-algebra homomorphisms $A \to \mathcal{O}(X)$.

Proof. (a) Note first that the composite $K \to A \to A/\mathfrak{m}$ is an isomorphism, so we are not making any choices in the isomorphism $A/\mathfrak{m} \cong K$. Also, the map $A \to A/\mathfrak{m}$ is an algebra homomorphism, and since the algebra structure on functions $\operatorname{Spec} A \to K$ is given pointwise, it follows that the map from A to functions $\operatorname{Spec} A \to K$ is an algebra homomorphism.

That it is injective follows from the NSS, since the kernel is $\bigcap \mathfrak{m} = 0$.

(b) The morphism θ defines a map $\phi \colon A \to \mathcal{O}(X)$, $a \mapsto a\theta$. Again, the algebra structure on $\mathcal{O}(X)$ is given pointwise, so ϕ will be an algebra homomorphism if and only if the map $A \to K$, $a \mapsto \phi(a)(x)$, is an algebra homomorphism for all $x \in X$. This clearly holds, since $\phi(a)(x) = a(\theta(x))$.

In particular, $\theta(x)$ is the kernel of the map $a \mapsto \phi(a)(x)$.

(c) Given ϕ, we define $\theta(x)$ to be the kernel of the K-algebra homomorphism $A \to K$, $a \mapsto \phi(a)(x)$. This map must be onto, so $\theta(x)$ is maximal and $A/\theta(x) = K$. Moreover, the induced map $A/\theta(x) \to K$, $a + \theta(x) \mapsto \phi(a)(x)$ is now an isomorphism of K-algebras, so is the identity on K. Thus $(a\theta)(x) = \phi(a)(x)$ for all $x \in X$, so $a\theta = \phi \in \mathcal{O}(X)$, and hence θ is a morphism of spaces with functions.

(d) Starting from $\theta \colon X \to \operatorname{Spec} A$, we obtain $\phi \colon A \to \mathcal{O}(X)$, $a \mapsto a\theta$. The kernel of $a \mapsto \phi(a)(x)$ is $\theta(x)$, so from ϕ we can reconstruct θ.

Conversely, starting from $\phi \colon A \to \mathcal{O}(X)$ we obtain $\theta \colon X \to \operatorname{Spec} A$, $x \mapsto \operatorname{Ker}(a \mapsto \phi(a)(x))$. Then $a\theta = \phi$, so from θ we can reconstruct ϕ.

This proves that the two constructions are mutually inverse. $\square$

2. Let $\phi\colon A \to B$ be a homomorphism between two affine K-algebras.

(a) If $\mathfrak{n} \lhd B$ is maximal, then $\phi^{-1}(\mathfrak{n}) \lhd A$ is maximal.

(b) The morphism of spaces with functions $\theta\colon \operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to ϕ is given by $\theta(\mathfrak{n}) := \phi^{-1}(\mathfrak{n})$.

(c) $\operatorname{Im}(\theta) \subset V(I)$ iff $I \subset \operatorname{Ker}(\phi)$. Deduce that $\overline{\operatorname{Im}(\theta)} = V(\operatorname{Ker}(\phi))$.

(d) Show that θ is dense if and only if ϕ is injective.

Proof. (a) We have the inclusions $K \subset A/\phi^{-1}(\mathfrak{n}) \subset B/\mathfrak{n}$. The composite is an isomorphism by the NSS, so $A/\phi^{-1}(\mathfrak{n}) \cong K$ and $\phi^{-1}(\mathfrak{n})$ is maximal.

(b) We have $B = \mathcal{O}'(\operatorname{Spec} B) \rightarrowtail \mathcal{O}(B)$, so ϕ determines an algebra homomorphism $A \to \mathcal{O}(B)$, and hence a morphism $\theta\colon \operatorname{Spec} B \to \operatorname{Spec} A$. By construction, $\theta(\mathfrak{n})$ is the kernel of the map $a \mapsto \phi(a)(\mathfrak{n})$. Using that $K \to A/\phi^{-1}(\mathfrak{n}) \to B/\mathfrak{n}$ are both isomorphisms, this is the same as the the map $a \mapsto a(\phi^{-1}(\mathfrak{n}))$, so $\theta(\mathfrak{n}) = \phi^{-1}(\mathfrak{n})$.

(c) $\operatorname{Im}(\theta) \subset V(I)$ iff for all $\mathfrak{n} \in \operatorname{Spec} B$ we have $\phi^{-1}(\mathfrak{n}) \supset I$, equivalently $\phi(I) \subset \mathfrak{n}$. Since $\bigcap \mathfrak{n} = 0$, this is the same as saying $\phi(I) = 0$, or $I \subset \operatorname{Ker}(\phi)$.

Using this we see that $\overline{\operatorname{Im}(\theta)} = \bigcap_{I \subset \operatorname{Ker}\phi} V(I)$. If $I \subset \operatorname{Ker}\phi$, then $V(\operatorname{Ker}\phi) \subset V(I)$, and so the right hand side equals $V(\operatorname{Ker}\phi)$ as required.

(d) θ is dense if and only if $V(\operatorname{Ker}\phi) = \operatorname{Spec} A$, so $\operatorname{Ker}\phi$ is contained in $\bigcap \mathfrak{m} = 0$, equivalently ϕ is injective. $\square$

3. Let A be a affine K-algebra, and $I \lhd A$ a radical ideal.

(a) Show that A/I is a affine K-algebra.

(b) The canonical algebra homomorphism $A \to A/I$ yields a morphism $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$. This has image contained in $V(I)$, so have morphism $\theta\colon \operatorname{Spec}(A/I) \to V(I)$.

(c) The restriction map $A \to \mathcal{O}(V(I))$ induces an algebra homomorphism $A/I \to \mathcal{O}(V(I))$, and hence corresponds to a morphism $\phi\colon V(I) \to \operatorname{Spec}(A/I)$. Show further that $\phi(\mathfrak{m}) = \mathfrak{m}/I$ for all maximal ideals $\mathfrak{m} \supset I$, and hence that ϕ and θ are mutually inverse.

(d) Give an example of a continuous map $\theta\colon X \to Y$ between two topological spaces such that θ is bijective but not a homeomorphism.

Proof. (a) It is clear that A/I is finitely generated and commutative; it is reduced since I is radical.

(b) Using Question 2, the map $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ sends a maximal ideal of A/I to its preimage in A. The preimage necesaarily contains I, so lies in $V(I)$. By Sheet 1 Question 3 we have the induced morphism $\theta\colon \operatorname{Spec}(A/I) \to V(I)$.

(c) We have the algebra homomorphisms $A = \mathcal{O}'(A) \rightarrowtail \mathcal{O}(\operatorname{Spec} A) \to \mathcal{O}(V(I))$, the latter map being restriction. We know that I lies in the kernel of the composite, so we have the induced algebra map $A/I \to \mathcal{O}(V(I))$. This determines a morphism $\phi\colon V(I) \to \operatorname{Spec}(A/I)$, sending $\mathfrak{m}$ containing I to the kernel of the map $A/I \to A/\mathfrak{m}$, $a + I \mapsto a(\mathfrak{m})$, which is precisely the ideal $\mathfrak{m}/I$.

Now ϕ and θ are mutually inverse by the Third Isomorphism Theorem.

(d) The easiest example is maybe the following. Given a set X, let X_{disc} be X equipped with the discrete topology, where every set is open, and X_{indisc} be X equipped with the indiscrete topology, where the only opens are $\emptyset$ and X. Thus there is a continuous bijection $X_{\mathrm{disc}} \to X_{\mathrm{indisc}}$, which is a homeomorphism precisely when $|X| \leq 1$.

Perhaps a more interesting example is $[0, 1) \to S^1$, $t \mapsto \exp(2\pi i t)$, where both sets have their usual topologies. This is not a homeomorphism, since $[0, 0.5)$ is open but its image is not open. □

4. Let A be a affine K-algebra. Given $0 \neq a \in A$, we can form the algebra $A_a := A[t]/(1 - at)$.

(a) Show that A_a is a affine K-algebra.

(b) The canonical algebra map $A \to A_a$ yields a morphism $\operatorname{Spec} A_a \to \operatorname{Spec} A$. This has image contained in $D(a)$, so have morphism $\theta\colon \operatorname{Spec} A_a \to D(a)$.

(c) The restriction map $A \to \mathcal{O}(D(a))$ induces an algebra homomorphism $A_a \to \mathcal{O}(D(a))$, and hence corresponds to a morphism $\phi\colon D(a) \to \operatorname{Spec} A_a$.

(d) Show that ϕ and θ are mutually inverse.

Proof. (a) Every element of A_a is represented by a polynomial in $A[t]$. Using that $at = 1$ we have $bt^r = ba^s t^{r+s}$, and hence we can represent every element by a polynomial of the form bt^r. In particular, A finitely generated and commutative implies the same for A_a.

Next, the natural map $A \to A_a$ is injective. For, if $b \mapsto 0$, then $b = (1 - at)p(t)$ in $A[t]$, and computing coefficients of t yields $p = 0$, whence $b = 0$.

Finally, A_a is reduced. For, if bt^r is nilpotent, then so too is $b = b(at)^r$, whence $b \in A$ is nilpotent. Now use that A is reduced.

(b) The map $\operatorname{Spec} A_a \to \operatorname{Spec} A$ is given by taking the preimage of an ideal, equivalently by sending $\mathfrak{n}$ to the kernel $\mathfrak{m}$ of the map $A \to A_a \to A_a/\mathfrak{n}$. Since a is sent to a unit in A_a, it cannot lie in the kernel, so $\mathfrak{m} \in D(a)$. By Sheet 1, Question 3 we get a morphism $\operatorname{Spec} A_a \to D(a)$.

(c) We have the algebra homomorphisms $A = \mathcal{O}'(\operatorname{Spec} A) \rightarrowtail \mathcal{O}(\operatorname{Spec} A) \to \mathcal{O}(D(a))$, the latter map being restriction. We know that $1/a \in \mathcal{O}(D(a))$, so we have the induced algebra map $A_a \to \mathcal{O}(D(a))$, $t \mapsto 1/a$. This determines a morphism $\phi\colon D(a) \to \operatorname{Spec} A_a$ sending $\mathfrak{m}$ not containing a to the kernel of the map $A_a \to K$, $bt^r \mapsto b(\mathfrak{m})/a(\mathfrak{m})^r$, which is precisely the ideal $\{bt^r : b \in \mathfrak{m}\}$ (equivalently $\mathfrak{m}A_a$).

(d) We have $\theta(\mathfrak{n}) = \{b \in A : b \in \mathfrak{n}\}$ and $\phi(\mathfrak{m}) = \{bt^r : b \in \mathfrak{m}\}$. Thus $\phi\theta(\mathfrak{n}) \subset \mathfrak{n}$ and $\mathfrak{m} \subset \theta\phi(\mathfrak{m})$. Since all the ideals are maximal, we must have equalities. Thus θ and ϕ are mutually inverse. □

5. Let A be a affine K-algebra. We show that $\mathcal{O}(\operatorname{Spec} A) = A$. Combining with the previous two exercises yields $\mathcal{O}(D(a)) = A_a$ and $\mathcal{O}(V(I)) = A/I$.

(a) If $f \in A$ is zero on $D(g)$, then fg is zero on $\operatorname{Spec} A$, and hence $fg = 0$ in A.

(b) Take $\theta \in \mathcal{O}(\operatorname{Spec} A)$. We can write $\operatorname{Spec} A = \bigcup_\alpha D(g'_\alpha)$ such that $\theta = f'_\alpha/g'_\alpha$ on $D(g'_\alpha)$. Set $f_\alpha := f'_\alpha g'_\alpha$ and $g_\alpha = (g'_\alpha)^2$. Show:

   (i) $D(g_\alpha) = D(g'_\alpha)$.

   (ii) $\theta = f_\alpha/g_\alpha$ on $D(g_\alpha)$.

   (iii) $f_\alpha g_\beta = f_\beta g_\alpha$ as elements of A.

(c) Set $I := (g_\alpha)$. Show that $V(I) = \emptyset$, so $I = A$. Deduce that $1 = \sum_\alpha s_\alpha g_\alpha$ (finite sum).

(d) Set $h := \sum_\alpha s_\alpha f_\alpha$ in A. Show $hg_\beta = f_\beta$, so $h = f_\beta/g_\beta = \theta$ on $D(g_\beta)$. Deduce that $\theta = h$ as functions $\operatorname{Spec} A \to K$.

Proof. (a) We have $\operatorname{Spec} A = D(g) \sqcup V(g)$, and g is zero as a function on $V(g)$. Thus fg is zero as a function on $\operatorname{Spec} A$, so is contained in $\bigcap \mathfrak{m} = 0$, whence $fg = 0$ in A.

(b) θ is regular, so by definition we can find such f'_α, g'_α.

(i) Maximal ideals are prime, so $g_\alpha \in \mathfrak{m}$ iff $g'_\alpha \in \mathfrak{m}$. Hence $V(g_\alpha) = V(g'_\alpha)$, and so their complements also agree, $D(g_\alpha) = D(g'_\alpha)$.

(ii) For all $x \in D(g_\alpha)$ we have $f_\alpha(x)/g_\alpha(x) = f'_\alpha(x)g'_\alpha(x)/g'_\alpha(x)^2 = f'_\alpha(x)/g'_\alpha(x)$.

(iii) Set $f'' := f'_\alpha g'_\beta - f'_\beta g'_\alpha$. Then $f''(x) = 0$ for all $x \in D(g'_\alpha g'_\beta)$, so $f'' g'_\alpha g'_\beta = 0$ in A by Part (a). In other words, $f_\alpha g_\beta = f_\beta g_\alpha$ in A.

(c) We have $V(I) = \bigcap_\alpha V(g_\alpha)$, so its complement is $\bigcup_\alpha D(g_\alpha) = \operatorname{Spec} A$. Hence $V(I) = \emptyset$, so I is not contained in any maximal ideal, and hence $I = A$. In particular, $1 \in I$, so we can write $1 = \sum_\alpha s_\alpha g_\alpha$ as a finite sum, so almost all s_α are zero.

(d) Using (b)(iii) we have $hg_\beta = \sum_\alpha s_\alpha f_\alpha g_\beta = \sum_\alpha s_\alpha f_\beta g_\alpha = f_\beta$. Thus $h(x) = f_\beta(x)/g_\beta(x) = \theta(x)$ for all $x \in D(g_\beta)$ (Here was a small typo on the exercise sheet.) Hence $h(x) = \theta(x)$ on all of $\operatorname{Spec} A$, so $\theta = h \in A$. □

It now follows that the map sending affine A to $\operatorname{Spec} A$ yields a contravariant equivalence between the category of affine K-algebras and the category of affine varieties, the inverse map being $X \mapsto \mathcal{O}(X)$.