Non-commutative Algebra 3, SS 2020

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Solutions 3

Throughout K will be an algebraically closed field. For simplicity we define an affine K-algebra to be one which is finitely generated, commutative and reduced.

1. Let A and B be affine K-algebras, and consider the K-algebra $A \otimes_K B$.

(a) Let $\mathfrak{m} \lhd A$ be a maximal ideal. Show that $(A \otimes B)/(\mathfrak{m} \otimes B) \cong B$.

(b) Show that $A \otimes B$ is reduced.

(c) Prove that $A \otimes_K B$ is again an affine K-algebra.

(d) Deduce that if X, Y are affine varieties, then $K[X \times Y] \cong K[X] \otimes K[Y]$.

Proof. (a) By the Nullstellensatz we know that $A/\mathfrak{m} = K$. We therefore have an algebra homomorphism $A \otimes B \to B$, $a \otimes b \mapsto (a + \mathfrak{m})b$. This is clearly surjective, with kernel containing $\mathfrak{m} \otimes B$. For the converse, take a K-basis $\{b_i\}$ for B and suppose $\sum_i a_i \otimes b_i$ (finite sum) lies in the kernel. Thus $\sum_i (a_i + \mathfrak{m}) b_i = 0$, so $a_i + \mathfrak{m} = 0$ for all i, whence $a_i \in \mathfrak{m}$ and $\sum_i a_i \otimes b_i \in \mathfrak{m} \otimes B$.

(b) Again, $\{b_i\}$ is a K-basis of B. Take a maximal ideal $\mathfrak{m} \lhd A$. The image of $c = \sum_i a_i \otimes b_i$ in $(A \otimes B)/(\mathfrak{m} \otimes B) \cong B$ is nilpotent, hence zero since B is reduced. Thus $a_i \in \mathfrak{m}$ for all i. This holds for all maximal ideals, so $a_i \in \bigcap \mathfrak{m} = 0$ since A is reduced. Thus $c = 0$.

(c) There exist surjections $K[x_, \ldots, x_m] \twoheadrightarrow A$ and $K[y_1, \ldots, y_n] \twoheadrightarrow B$, so there exists a surjection $K[x_i, y_j] \twoheadrightarrow A \otimes B$. Thus $A \otimes B$ is finitely generated, it is clearly commutative, and we saw in (b) that it is reduced.

(d) A duality swaps products and coproducts. Thus $K[X \times Y]$ is the coproduct of $K[X]$ and $K[Y]$ in the category of affine K-algebras. As K-algebras, the coproduct is $K[X] \otimes K[Y]$, and since this is affine, this is automatically the coproduct in the category of affine K-algebras. Thus $K[X \times Y] \cong K[X] \otimes K[Y]$. $\square$

2. Define a map of sets $\phi\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$, $([x_i],[y_p]) \mapsto ([x_i y_p])$.

(a) Show that this map is well-defined and injective.

(b) Let $Z \subset \mathbb{P}^{mn+m+n}$ be the closed subset $V'(\{Z_{ip}Z_{jq} - Z_{iq}Z_{jp}\})$. Show that ϕ has image Z.

(c) Show further that ϕ restricts to an isomorphism of spaces with functions

$$\mathbb{A}^m \times \mathbb{A}^n \cong D'(X_i) \times D'(Y_p) \xrightarrow{\sim} Z \cap D'(Z_{ip}).$$

(d) Deduce that $\phi\colon \mathbb{P}^m \times \mathbb{P}^n \to Z$ is an isomorphism of spaces with functions.

Proof. (a) If $\lambda, \mu \in K^\times$, then $([\lambda x_i],[\mu y_p]) \mapsto [\lambda\mu x_i y_p] = [x_i y_p]$. Thus ϕ doesn't depend on the choice of representatives.

Now suppose $[x_i y_p] = [x'_i y'_p]$. Then there exists $\lambda \in K^\times$ with $x'_i y'_p = \lambda x_i y_p$. There exists some (i,p) such that this is nonzero. Thus $y_p, y'_p \neq 0$ and we have $x'_i = (\lambda y_p / y'_p) x_i$ for all i. Thus $[x'_i] = [x_i]$. Swapping the roles of x and y gives $[y'_p] = [y_p]$, proving that the map is injective.

Alternatively, though this wasn't explicitly stated in the online notes, a morphism between projective varieties is a map which is given locally by homogeneous polynomials of the same degree. Here we actually have a global description using the homogeneous polynomials $X_i Y_p$ of degree two. In fact, every map from projective space has such a global definition; this holds more generally for a projective variety $V'(I) \subset \mathbb{P}^n$ such that the homogeneous ring $K[X_0,\dots,X_n]/I$ is a unique factorisation domain. There are easy examples when this is not the case, for example the map $V'(X^2 - YZ) \to \mathbb{P}^1$ given by $[X,Y]$ on $D'(Y)$ and $[Z,X]$ on $D'(Z)$.

(b) An element of the image is of the form $[x_i y_p]$, which is sent by the function $Z_{ip}Z_{jq}$ to $x_i x_j y_p y_q$. It follows that $\mathrm{Im}(\phi) \subset Z$. Conversely, take $z \in Z$, say with $z_{ip} \neq 0$. Set $x_j := z_{jp}/z_{ip}$ and $y_q := z_{iq}/z_{ip}$. Then $x_i = 1 = y_p$, so we have a point $([x_i],[y_p]) \in \mathbb{P}^m \times \mathbb{P}^n$. Under ϕ this is sent to $[x_j y_q]$. Using that $z \in Z$ we compute

$$x_j y_q = z_{jp} z_{iq} / z_{ip}^2 = z_{jq}/z_{ip}$$

and so $[x_j y_q] = [z_{jq}]$. Thus ϕ is surjective.

(c) For convenience we take $i = 0 = p$. We then have the ismorphism $\mathbb{A}^m \cong D'(X_0)$, $(x_1,\dots,x_m) \mapsto [1,x_1,\dots,x_m]$; similarly $\mathbb{A}^n \cong D'(Y_0)$ and $\mathbb{A}^{mn+m+n} \cong D'(Z_{00})$. Let $W \subset \mathbb{A}^{mn+m+n}$ be the closed subset corresponding to $Z \cap D'(Z_{00})$.

With these identifications, ϕ retricts to $\mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^{mn+m+n}$, $(x_i, y_p) \mapsto (x_i y_p)$. This is given by polynomials, hence is a morphism, and its image lies in W. Conversely, using (b) we see that the inverse to ϕ is given by the map $\mathbb{A}^{mn+m+n} \to \mathbb{A}^m \times \mathbb{A}^n$, $(z_{ip}) \mapsto (z_{i0}, z_{0p})$, restricted to W. Again, this is given by polynomials, so is a morphism.

(d) We now have the bijective map $\phi\colon \mathbb{P}^m \times \mathbb{P}^n \to Z$ which is locally an isomorphism of spaces with functions. Hence ϕ is a morphism of spaces with functions, as is its set-theoretic inverse. Thus ϕ is an isomorphism. □

3. Let $\theta\colon X \to Y$ be a continous map of topological spaces.

(a) Show that X connected implies $\overline{\theta(X)}$ is connected.

(b) Show that X irreducible implies $\overline{\theta(X)}$ is irreducible.

(c) Show that X is irreducible if and only if every non-empty open is dense.

Proof. (a) Write $\overline{\theta(X)} = Y_1 \sqcup Y_2$ as a disjoint union of two closeds. Set $X_i := \theta^{-1}(Y_i)$. Then $X = X_1 \cup X_2$ is a union of two closeds, and $X_1 \cap X_2 = \emptyset$. As X is connected, we may assume $X = X_1$. Thus $\theta(X)$ is contained in the closed set Y_1, whence $Y_1 = \overline{\theta(X)}$.

(b) Write $\overline{\theta(X)} = Y_1 \cup Y_2$ as a union of two closeds. Set $X_i := \theta^{-1}(Y_i)$. Then $X = X_1 \cup X_2$ is a union of two closeds, so by irreducibility we may assume $X = X_1$. Thus $\theta(X)$ is contained in the closed set Y_1, whence $Y_1 = \overline{\theta(X)}$.

(c) Suppose X is irreducible, and let $U \subset X$ be open, with complement C. Then $X = \overline{U} \cup C$, so either $\overline{U} = \emptyset$ or else $C = \emptyset$, equivalently U is either empty or dense. If X is not irreducible, then we can write $X = X_1 \cup X_2$ as a union of proper closeds. Now $U = X_1^c$ is nonempty open and contained in X_2, so its closure is contained in the proper subset X_2. □

4. We have the surjective continuous map $\pi\colon \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$. Show that π is a morphism of spaces with functions.

Proof. Let $U \subset \mathbb{P}^n$ be open, and $f \in \mathcal{O}_{\mathbb{P}}(U)$ regular. Take $(x_i) \in \pi^{-1}(U)$. Since f is regular, there exists an open neighbourhood W of $[x_i]$ with $W \subset U$, homogeneous polynomials P, Q of the same degree with $Q(x_i) \neq 0$ and $f = P/Q$ on W. Now $\pi^{-1}(W)$ is an open neighbourhood of (x_i) with $\pi^{-1}(W)$ contained in $\pi^{-1}(U)$, we have polynomials $P\pi = P, Q\pi = Q$ with $Q(x_i) \neq 0$, and $f\pi = P/Q$ on $\pi^{-1}(W)$. This shows that $f\pi$ is locally a regular function on $\pi^{-1}(U)$, whence $f\pi \in \mathcal{O}_{\mathbb{A}}(\pi^{-1}(U))$ is regular. □

5. Let $X = K \cup \{0'\}$ with the cofinite topology. Set $K' := X - \{0\}$, which we can again identify with $\mathbb{A}^1$.
   For an open subset $U \subset X$ we define $\mathcal{O}(U)$ as follows:
   If $0' \notin U$, then $U \subset K \cong \mathbb{A}^1$ and we can define $\mathcal{O}(U)$.
   If $0 \notin U$, then $U \subset K' \cong \mathbb{A}^1$ and we can define $\mathcal{O}(U)$.
   If $U = X$ then we set $\mathcal{O}(X) = K[t]$, where t is the usual co-ordinate function on K and sends $0'$ to zero.

(a) Show that this construction determines a space with functions.

(b) Show that X is not separated.

Proof. (a) We apply Exercise Sheet 1, Question 4. Take $\mathcal{B} = \{\emptyset, W, K, K', X\}$, where $W = K \cap K' = K - \{0\}$. Then $\mathcal{B}$ is closed under finite intersections. We have $\mathcal{O}'(U)$ for $U \in \mathcal{B}$, where $\mathcal{O}'(W) := K[t^{\pm}]$. These satisfy the necessary restriction condition. Thus we have a space with functions.

We check that this construction yields the cofinite topology on X. Suppose $X - U$ is finite. There exists $g \in K[t^{\pm}]$ such that $D(g, W) = U \cap W \subset \mathbb{A}^1$. Multiplying by a suitable power of t we may assume that $g \in K[t]$ has nonzero constant term.

If $0, 0' \in U$, then $U = D(g, X)$. If $0 \in U$, $0' \notin U$, then $U = D(g, K)$. Similarly if $0 \notin U$, $0' \in U$, then $U = D(g, K')$. Finally, if $0, 0' \notin U$, then $U = U \cap W = D(g, W)$.

Thus every cofinite subset is open. Conversely, since a nonzero (Laurent) polynomial has only finitely many zeros, each distinguished $D(g, U)$ must be cofinite. So every open is cofinite.

Now we check that there are no more regular functions on K, K', X. Let $f \in \mathcal{O}_X(K)$ be regular. Then locally it is given by a quotient of polynomials, so actually lies in $\mathcal{O}_{\mathbb{A}}(K) = K[t] = \mathcal{O}'_X(K)$. Similarly $\mathcal{O}_X(K') = K[t]$.

Suppose instead that $f \in \mathcal{O}_X(X)$. Then $f|_K$ is regular, so is given by a polynomial $g[t]$. Similarly $f|_{K'} = h[t]$. These polynomials must agree on W, so $g = h \in K[t^{\pm}]$, so actually $g = h \in K[t]$. Thus $f = g \in K[t]$ is a polynomial, and $\mathcal{O}_X(X) = K[t]$ (and $f(0) = f(0')$).

(b) The complement of the diagonal $\Delta_X \subset X \times X$ clearly contains $K \times \{0'\}$, since $0' \notin K$. Now K, K' are both isomorphic to $\mathbb{A}^1$, so $K \times K' \subset X \times X$ is isomorphic to $\mathbb{A}^2$. Let $U \subset \mathbb{A}^2$ be an open containing $K \times \{0\}$. Then U intersects the diagonal $\Delta_{\mathbb{A}}$ in $\mathbb{A}^2$ in the point $(0,0)$, and $\Delta_{\mathbb{A}} \cong \mathbb{A}^1$, so the intersection is cofinite, and hence contains some (a, a) with $a \neq 0$. It follows that every open in $K \times K'$ containing $K \times \{0'\}$ must contain some (a, a) with $a \neq 0$, so intersects Δ_X. Thus Δ_X is not closed, and X is not separated. □