Non-commutative Algebra 3, SS 2020

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Solutions 4

Throughout K will be an algebraically closed field. For simplicity we define an affine K-algebra to be one which is finitely generated, commutative and reduced.

1. G is an affine algebraic group, and $A = K[G]$.

(a) $A \otimes_K A$ acts on $G \times G$ via $(a \otimes b)(g, h) = a(g)b(h)$. Multiplication $G \times G \to G$, $(g, h) \mapsto gh$, induces an algebra homomorphism $\Delta\colon A \to A \otimes_K A$, $\Delta(a)(g, h) := a(gh)$. Show that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

(b) Have constant map $\operatorname{Spec} K \to G$ with image 1, giving an algebra homomorphism $\varepsilon\colon A \to K$, $\varepsilon(a) = a(1)$. Show that $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$.

(c) Multiplication μ on A corresponds to diagonal $G \to G \times G$, $g \mapsto (g, g)$.

(d) Inversion on G corresponds to an algebra endomorphism S of A. Show that S is invertible and satisfies $\mu(S \otimes \mathrm{id})\Delta = \varepsilon = \mu(\mathrm{id} \otimes S)\Delta$.

(e) Have $\tau\colon G \times G \to G \times G$, $(g, h) \mapsto (h, g)$, corresponding to an algebra automorphism τ of $A \otimes_K A$. Show that $\tau(a \otimes b) = b \otimes a$.

Know A is commutative if and only if $\mu\tau = \mu$. We say A is cocommutative if $\tau\Delta = \Delta$. Show that A is cocommutative if and only if G is commutative.

Proof. (a) We have $(\Delta \otimes \mathrm{id})(a \otimes b)(f, g, h) = \Delta(a)(f, g)b(h) = a(fg)b(h) = (a \otimes b)(fg, h)$. Thus $(\Delta \otimes \mathrm{id})\Delta(a)(f, g, h) = \Delta(a)(fg, h) = a(fgh)$.

(b) We have $(\varepsilon \otimes \mathrm{id})(a \otimes b)(g) = a(1)b(g) = (a \otimes b)(1, g)$. Thus $(\varepsilon \otimes \mathrm{id})\Delta(a)(g) = \Delta(a)(1, g) = a(g)$.

(c) We have $\mu(a \otimes b)(g) = (ab)(g) = a(g)b(g) = (a \otimes b)(g, g)$.

(d) We have $S(a)(g) = a(g^{-1})$, so $S^2(a) = a$ and S is invertible. Also, $\mu(S \otimes \mathrm{id})(a \otimes b)(g) = S(a)(g)b(g) = a(g^{-1})b(g) = (a \otimes b)(g^{-1}, g)$. Thus $\mu(S \otimes \mathrm{id}\Delta(a)(g) = \Delta(a)(g^{-1}, g) = a(1) = \varepsilon(a)$.

(e) We have $\tau(a \otimes b)(g, h) = (a \otimes b)(h, g) = a(h)b(g) = (b \otimes a)(g, h)$. Now $\tau\Delta(a)(g, h) = a(hg)$, so $\tau\Delta = \Delta$ if and only if $a(hg) = a(gh)$ for all a, g, h.

Given an affine variety X and distinct points $x, y \in X$, there always exists some $f \in K[X]$ with $f(x) = 0$, $f(y) \neq 0$. (In terms of affine algebras, this says that given two distinct maximal ideals, there is an element in one of them but not in the other.)

So, knowing that $a(hg) = a(gh)$ for all a implies that $hg = gh$. Knowing this for all g, h says that G is commutative. □

2. Consider the affine variety $G = K^\times \times K$, together with the action

$$G \times G \to G, \quad (a,b)\cdot(c,d) := (ac, bc+d).$$

(a) Show that G is an affine algebraic group.

(b) Compute $A = K[G]$.

(c) Compute the comultiplication $\Delta\colon A \to A \otimes_K A$.

(d) Compute the antipode $S\colon A \to A$.

Proof. (a) The multiplication is a morphism of varieties. Inversion is $(a,b) \mapsto (a^{-1}, -ba^{-1})$, again a morphism of varieties. Unit is $(1,0)$. Hence affine algebraic group.

(b) Product of varieties corresponds to coproduct of algebras, so

$$K[G] \cong K[X^{\pm 1}] \otimes_K K[Y] \cong K[X^{\pm 1}, Y].$$

(c) $\Delta(X)((a,b),(c,d)) = X(ac, bc+d) = ac = (X \otimes X)((a,b),(c,d))$, so $\Delta(X) = X \otimes X$.

$\Delta(Y)((a,b),(c,d)) = Y(ac, bc+d) = bc+d = (Y \otimes X + 1 \otimes Y)((a,b),(c,d))$, so $\Delta(Y) = Y \otimes X + 1 \otimes Y$.

(d) $S(X)(a,b) = X(a^{-1}, -ba^{-1}) = a^{-1} = X^{-1}(a,b)$, so $S(X) = X^{-1}$.

$S(Y)(a,b) = Y(a^{-1}, -ba^{-1}) = -ba^{-1} = (-X^{-1}Y)(a,b)$, so $S(Y) = -X^{-1}Y$. $\square$

We check that $\mu(S \otimes \mathrm{id})\Delta = \varepsilon = \mu(\mathrm{id} \otimes S)\Delta$.

$$\mu(S \otimes \mathrm{id})\Delta(X) = \mu(S(X) \otimes X) = X^{-1}X = 1$$

and

$$\mu(S \otimes \mathrm{id})\Delta(Y) = \mu(S(Y) \otimes X + S(1) \otimes Y) = -X^{-1}YX + Y = 0.$$

Similarly

$$\mu(\mathrm{id} \otimes S)\Delta(X) = \mu(X \otimes S(X)) = XX^{-1} = 1$$

and

$$\mu(\mathrm{id} \otimes S)\Delta(Y) = \mu(Y \otimes S(X) + 1 \otimes S(Y)) = YX^{-1} - X^{-1}Y = 0.$$

Now $X(1,0) = 1$ and $Y(1,0) = 0$, so $\varepsilon(X) = 1$ and $\varepsilon(Y) = 0$.

3. Affine variety $G = \{M \in \mathrm{GL}_2(K) : MM^t = 1\}$.

(a) Show that G is an affine algebraic group.

(b) Compute $A = K[G]$.

(c) Compute the comultiplication $\Delta\colon A \to A \otimes_K A$.

(d) Compute the antipode $S\colon A \to A$.

Proof. (a) Know that $\mathrm{GL}_2(K)$ is an affine algebraic group. Now G is a subgroup, and is closed (see (b) below), so is also an affine algebraic group.

(b) $K[\mathbb{M}_2(K)] = K[W, X, Y, Z]$, using the co-ordinate functions $\left(\begin{smallmatrix} W & X \\ Y & Z \end{smallmatrix}\right)$.

If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $MM^t = 1$ if and only if

$$a^2 + b^2 = 1 = c^2 + d^2, \quad ac + bd = 0.$$

So we definitely have $W^2 + X^2 = 1$, $Y^2 + Z^2 = 1$, $WY + XZ = 0$. It is not obvious that this is reduced, however.

Assume characteristic not 2. Taking just the first two relations, we obtain the reduced ring

$$K[W, X]/(W^2 + X^2 - 1) \otimes K[Y, Z]/(Y^2 \otimes Z^2 - 1).$$

This is free as a $K[X, Z]$-module, with basis $1, W, Y, WY$. The quotient by $(XZ + WY)$ is therefore free over $K[X, Z]$ with basis $1, W, Y$. Suppose $(p + qW + rY)^2 = 0$ in this quotient ring, where $p, q, r \in K[X, Y]$. Then

$$(p^2 + (1 - X^2)q^2 + (1 - Z^2)r^2 - 2XZqr) + 2Wpq + 2Ypr = 0.$$

Looking at the coefficients of W and Y we see that either $p = 0$, or else $q = r = 0$.

If $q = r = 0$, then $p^2 = 0$, so $p = 0$ as well. Otherwise $p = 0$, and then $(1 - X^2)q^2 + (1 - Z^2)r^2 = 2XZqr$ in $K[X, Z]$. Suppose q, r have greatest common divisor f. Then q/f and r/f also satisfy this equation, so we may assume that q, r have no common divisors. Now q divides $(1 - Z^2)r^2$, so must divide $1 - Z^2$. Similarly r divides $1 - X^2$. If $q \neq 0$ for some $Z = \pm 1$, then $q = 2$ and we get $1 - X^2 = \pm Xr$, a contradiction. Thus $q = \lambda(1 - Z^2)$, and similarly $r = \mu(1 - X^2)$, and then $\lambda + \mu = 2XZ\lambda\mu$. We conclude that $\lambda = \mu = 0$, so again $p = q = r = 0$.

This proves that, in characteristic not 2,

$$K[G] = K[W, X, Y, Z]/(W^2 + X^2 - 1, Y^2 + Z^2 - 1, WY + XZ).$$

In characteristic 2 we have $a + b = 1$, $c + d = 1$, $ac + bd = 0$. So $b = 1 + a$ and $c = 1 + d$, and then $0 = ac + bd = a + d$. Thus $G \cong K$ via $\left(\begin{smallmatrix} a & a+1 \\ 1+a & a \end{smallmatrix}\right)$, and $K[G] \cong K$.

(c) The comultiplication comes from the group multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

Thus

$$\Delta(W) = W \otimes W + X \otimes Y, \quad \Delta(X) = W \otimes X + X \otimes Z$$
$$\Delta(Y) = Y \otimes W + Z \otimes Y, \quad \Delta(Z) = Y \otimes X + Z \otimes Z.$$

(d) The antipode comes from inversion, which in G is given by the transpose. Thus

$$S(W) = W, \quad S(X) = Y, \quad S(Y) = X, \quad S(Z) = Z.$$ □

Note that $\mu(S \otimes \mathrm{id})\Delta(W) = W^2 + Y^2$. This equals 1 in $K[G]$, since

$$W^2Y^2 = X^2Z^2 = (1 - W^2)(1 - Y^2) = 1 - (W^2 + Y^2) + W^2Y^2.$$

4. Consider the action of $G = K^\times$ on K^2 given by $g \cdot (x, y) := (gx, g^{-1}y)$.

(a) Compute the comodule structure $K[X, Y] \to K[T, T^{-1}, X, Y]$.

(b) Show that every morphism $K^2 \to \operatorname{Spec} A$ which is constant on orbits factors through uniquely through $\pi\colon K^2 \to K$, $(x, y) \mapsto xy$.

(c) Show that π is not a geometric quotient.

Proof. (a) The image of X is the function $G \times K^2 \to K$, $(g, x, y) \mapsto gx$. Thus $X \mapsto TX$.

The image of Y is the function $G \times K^2 \to K$, $(g, x, y) \mapsto g^{-1}y$. Thus $Y \mapsto T^{-1}Y$.

(b) Morphism $\theta\colon K^2 \to \operatorname{Spec} A$ constant on G-orbits. Then $\theta^{-1}\theta(t, 0)$ is closed, so contains $(0, 0)$. Deduce that $\theta(t, 0) = \theta(0, t)$ for all $t \in K$. Now have morphism $\bar{\theta}\colon K \to \operatorname{Spec} A$, $t \mapsto \phi(t, 1)$, and $\theta = \bar{\theta}\pi$.

Alternatively, have algebra homomorphism $\phi\colon A \to K[X, Y]$. Constant on orbits if and only if the image in $K[T^{\pm 1}, X, Y]$ does not involve T. The monomial X^aY^b is sent to $T^{a-b}X^aY^b$, so does not involve T if and only if $a = b$. So only polynomials in XY are allowed in ϕ. Hence ϕ factors through the subalgebra $K[XY]$. Equivalently, the map $K^2 \to \operatorname{Spec} A$ factors through $\pi\colon K^2 \to K$, $(x, y) \mapsto xy$.

(c) There are four types of orbits. $(0, 0)$, $K^\times \times \{0\}$, $\{0\} \times K^\times$, and $C_\lambda := \{(t, \lambda/t) : t \in K^\times\}$ for each $\lambda \in K^\times$.

The map π sends the first three of these to 0, and $C_\lambda \mapsto \lambda$. Hence π is not a geometric quotient. □

5. Affine algebraic group G acts on variety X, with $\pi\colon X \to X/G$ a geometric quotient.

(a) If $U \subset X$ open, then so is $gU = \{g \cdot u : u \in U\}$ for each $g \in G$.

(b) Show that π is an open map.

Proof. (a) Multiplication by g is continuous as a map $X \to X$, so if $U \subset X$ is open, then so is its preimage under multiplication by g, so $g^{-1}U \subset X$ is open. Doing this for g^{-1} instead of g, we get that $gU \subset X$ is open.

(b) Let U be open. Then $\pi^{-1}\pi(U) = \bigcup_{g \in G} gU$ is open, so that $\pi(U)$ is open. □

6. G an algebraic group. Here G-bundle means Zariski-locally trivial principal G-bundle.

(a) G acts on X, $\pi\colon X \to Y$ a G-bundle. Open cover $Y = \bigcup_i V_i$ and local trivialisations $\phi_i\colon G \times V_i \xrightarrow{\sim} \pi^{-1}(V_i)$, $\phi_i(gh, v) = g\phi_i(h, v)$.

For each i, j set $V_{ij} := V_i \cap V_j$. Transition functions $\phi_j^{-1}\phi_i$ on $G \times V_{ij}$. Show that $\phi_j^{-1}\phi_i$ is of the form $(g, v) \mapsto (g\gamma(v), v)$ for some $\gamma\colon V_{ij} \to G$.

(b) $\pi'\colon X' \to Y$ another G-bundle. Morphism of G-bundles over Y is morphism $\theta\colon X \to X'$ such that $\theta(g \cdot x) = g\theta(x)$ and $\pi'\theta = \pi$. Show that every such morphism is an isomorphism.

It follows that we have a category of G-bundles over Y, and this category is a groupoid.

(c) Show that a G-bundle $\pi\colon X \to Y$ is trivial, so isomorphic to $G \times Y \to Y$, $(g, y) \mapsto y$, if and only if π admits a section, so a morphism $\sigma\colon Y \to X$ such that $\pi\sigma = \mathrm{id}_Y$.

(d) Let $\pi\colon X \to Y$ be a G-bundle. Given a morphism $\psi\colon Y' \to Y$, we can form the pullback

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

Show that the map $\pi'\colon X' \to Y'$ is again a G-bundle.

(e) Let $\pi\colon X \to Y$ be a G-bundle. Show that the pullback $X \times_Y X \to X$ is a trivial G-bundle.

Proof. (a) Have morphism $\gamma\colon V_{ij} \to G$, given by $v \mapsto (1, v) \mapsto \phi_j^{-1}\phi_i(1, v)$ followed by projection onto G. Now $\phi_i(g, v) = g\phi_i(1, v) = g\phi_j(\gamma(v), v) = \phi_j(g\gamma(v), v)$, so $\phi_j^{-1}\phi_i(g, v) = (g\gamma(v), v)$.

(b) Open cover V'_j and local trivialisations $\psi_j\colon G \times V'_j \xrightarrow{\sim} \pi'^{-1}(V'_j)$. Intersecting gives common local trivialisation, so we may assume that $V'_i = V_i$. Now $\pi'\theta(x) = \pi(x)$, so θ restricts to a morphism $\pi^{-1}(V_i) \to \pi'^{-1}(V_i)$. Composing with the local trivialisations, get morphism $\psi_i^{-1}\theta\phi_i\colon G \times V_i \to G \times V_i$.

As in (a), have morphism $\theta_i\colon V_i \to G$, given by $v \mapsto (1, v) \mapsto \psi_i^{-1}\theta\phi_i(1, v)$ followed by projection onto G. Now $\theta\phi_i(g, v) = g\theta\phi_i(1, v) = g\psi_i(\theta_i(v), v) = \psi_i(g\theta_i(v), v)$, so $\psi_i^{-1}\theta\phi_i(g, v) = (g\theta_i(v), v)$.

In particular, this map is an isomorphism, so the restriction $\pi^{-1}(V_i) \to \pi'^{-1}(V_i)$ is an isomorphism for all i, so θ is an isomorphism.

(c) The trivial bundle clearly admits the section $Y \to G \times Y$, $y \mapsto (1, y)$. Conversely, suppose π admits a section σ. We then have a morphism $\theta \colon G \times Y \to X$ of G-bundles over Y, $\theta(g, y) = g\sigma(y)$. Now θ is an isomorphism by (b).

(d) We have a morphism $G \times X \times Y' \to X$, $(g, x, y') \mapsto gx$, and a morphism $G \times X \times Y' \to Y'$, $(g, x, y') \mapsto y'$. The induced morphisms to Y agree, so we get a morphism $G \times X \times Y' \to X \times_Y Y'$, $(g, x, y') \mapsto (gx, y')$, and hence by restriction a morphism $G \times (X \times_Y Y') \to X \times_Y Y'$. Thus G acts on $X \times_Y Y'$.

We have an open cover V_i of Y and local trivialisations $\phi_i \colon G \times V_i \xrightarrow{\sim} \pi^{-1}(V_i)$. Set $V_i' := \psi^{-1}(V_i)$, yielding an open cover of Y'. We have the projection map $G \times V_i' \to V_i'$, $(g, v') \mapsto v'$, and the morphism $G \times V_i' \to \pi^{-1}(V_i)$, $(g, v') \mapsto \phi_i(g, \psi(v'))$. These induce a morphism $\phi_i' \colon G \times V_i' \to {\pi'}^{-1}(V_i')$, necessarily satisfying $\phi_i'(gh, v') = g\phi_i'(h, v')$.

We also have a morphism ${\pi'}^{-1}(V_i') \to \pi^{-1}(V_i) \xrightarrow{\sim} G \times V_i \to G$ and a morphism ${\pi'}^{-1}(V_i') \to V_i'$, which together yield a morphism ${\pi'}^{-1}(V_i') \to G \times V'$, which is an inverse to ψ_i'.

This shows that π' is locally trivial, so is a G-bundle.

(e) We know that the pullback $X \times_Y X \to X$ is a G-bundle, and it has a section $x \mapsto (x, x)$, so it is trivial by (c).

□

Q7 (Question below)

Proof. (a) To be a submodule we need that $M\binom{a}{b} \in K\binom{a'}{b'}$, so $\binom{a}{0} \in K\binom{a'}{b'}$, equivalently $ab' = 0$.

In this case there exists a unique $\lambda \in K$ such that $\binom{a}{0} = \lambda\binom{a'}{b'}$, and so fitting into a commutative diagram

$$\begin{array}{ccccc} M & \qquad & K^2 & \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)} & K^2 \\ \uparrow & & \binom{a}{b}\uparrow & & \uparrow\binom{a'}{b'} \\ U & & K & \xrightarrow{\lambda} & K \end{array}$$

(b) Under the Segre embedding, the polynomial ab' corresponds to x, so the quiver Grassmannian $\mathrm{Gr}_Q(M, (1, 1))$ is isomorphic to the subvariety $V'(wz - xy, x) = V'(x, wz)$.

(c) Under the isomorphism $\mathbb{P}^2 \cong V'(x)$, the polynomial wz corresponds to su, so the Grassmannian $\mathrm{Gr}_Q(M, (1, 1))$ is isomorphic to $V'(x, wz) \cong V'(su)$.

(d) Now $\mathrm{Gr}_Q(M, (1, 1)) \cong V'(su) \subset \mathbb{P}^2$ is the union of the two lines $V'(s)$ and $V'(u)$. Now $s = 0$ corresponds first to $w = 0$, and then to $aa' = 0$. So $V'(s)$ corresponds to $([a, b], [a', b'])$ such that $ab' = 0 = aa'$. Since $[a', b'] \in \mathbb{P}^1$, this implies $a = 0$.

We compute the submodule as in (a). Then $\lambda\binom{a'}{b'} = 0$, so $\lambda = 0$.

(e) The complement is given by $u = 0$, $s \neq 0$. This corresponds first to $z = 0$, $w \neq 0$, and then to $bb' = 0$ and $aa' \neq 0$ (and $ab' = 0$). So $b' = 0$, $a \neq 0$, so we may assume $a = a' = 1$. Computing the submodule, we see that $\lambda = 1$. □

7. Let Q be the quiver $1 \to 2$. We have $\mathrm{Mod}(Q,(d,e)) \cong \mathbb{M}_{e\times d}(K)$, the variety of matrices of size $e \times d$.

(a) Let $M = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \in \mathrm{Mod}(Q,(2,2))$ and consider the quiver Grassmannian $\mathrm{Gr}_Q(M,(1,1)) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Show that the pair of lines $([a,b],[a',b']) \in \mathbb{P}^1 \times \mathbb{P}^1$ corresponds to a submodule of M, so a point of $\mathrm{Gr}_Q(M,(1,1))$, if and only if the point $(a,0) \in K^2$ lies on the line $[a',b']$, which is if and only if $ab' = 0$.

Note that the corresponding submodule is the image of the injective module homomorphism

$$\begin{array}{ccc} M & \qquad & \begin{array}{ccc} K^2 & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} & K^2 \\ {\scriptstyle \binom{a}{b}}\uparrow & & \uparrow{\scriptstyle \binom{a'}{b'}} \\ K & \xrightarrow{\ \lambda\ } & K \end{array} \\ \uparrow & & \\ U & & \end{array}$$

Here λ is the unique map making the diagram commute.

(b) Recall the Segre embedding, $\mathbb{P}^1 \times \mathbb{P}^1 \cong V'(wz - xy) \subset \mathbb{P}^3$, sending the point $([a,b],[a',b'])$ to $[aa',ab',ba',bb']$. Show that this induces an isomorphism between $\mathrm{Gr}_Q(M,(1,1))$ and $V'(x,wz) \subset \mathbb{P}^3$.

(c) Show that the isomorphism $\mathbb{P}^2 \cong V'(x) \subset \mathbb{P}^3$, $[s,t,u] \mapsto [s,0,t,u]$, induces an isomorphism $\mathrm{Gr}_Q(M,(1,1)) \cong V'(su) \subset \mathbb{P}^2$.

In other words, we can regard $\mathrm{Gr}_Q(M,(1,1))$ as the union of the two projective lines $V'(s)$ and $V'(u)$ inside the projective plane $\mathbb{P}^2$.

(d) Show that the submodules corresponding to the projective line $V'(s)$ are those of the form

$$\begin{array}{ccc} K^2 & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} & K^2 \\ {\scriptstyle \binom{0}{1}}\uparrow & & \uparrow{\scriptstyle \binom{a'}{b'}} \\ K & \xrightarrow{\ 0\ } & K \end{array}$$

(e) Show that the complement, so the submodules corresponding to the open affine $V(u) \cap D(s) = \{[1,t,0] : t \in K\} \cong \mathbb{A}^1$, are those of the form

$$\begin{array}{ccc} K^2 & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} & K^2 \\ {\scriptstyle \binom{1}{t}}\uparrow & & \uparrow{\scriptstyle \binom{1}{0}} \\ K & \xrightarrow{\ 1\ } & K \end{array}$$