Non-commutative Algebra 3, SS 2020

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Solutions 5

Throughout K will be an algebraically closed field.

1. Recall the fibre dimension theorem (3.3(E) Main Lemma).
   If $\pi\colon X \to Y$ is a dominant morphism of irreducible varieties, then any irreducible component of a fibre $\pi^{-1}(y)$ has dimension at least $\dim X - \dim Y$. Moreover, there is a non-empty open $U \subset Y$ with $\dim \pi^{-1}(y) = \dim X - \dim Y$ for all $u \in U$.

(a) Assume this holds when X and Y are both affine. Show how to deduce the result when just Y is affine. Show how this implies the general case.

(b) Translate the affine case into a result about commutative algebras.

Proof. (a) Cover X by finitely many open affines X_i, so each X_i is irreducible of dimension $\dim X$. Also, $\pi^{-1}(\overline{\pi(X_i)})$ is closed in the irreducible X and contains the non-empty open X_i, so equals X. Hence $\pi(X) \subset \overline{\pi(X_i)}$, and each $\pi_i\colon X_i \to Y$ is a dominant morphism between irreducible affines.

If $C \subset \pi^{-1}(y)$ is an irreducible component, then $C = \bigcup_i C_i$, where $C_i = C \cap X_i$. Each C_i is open in C, so is either empty or else irreducible with $\dim C_i = \dim C$. By the Main Lemma for affines, if C_i is non-empty, then $\dim C_i \geq \dim X - \dim Y$. Hence $\dim C \geq \dim X - \dim Y$.

Next take $U_i \subset Y$ open dense such that $\dim \pi_i^{-1}(y) = \dim X - \dim Y$ for all $y \in U_i$. Set $U = \bigcap_i U_i$ to be their intersection. Since Y is irreducible, U is open dense, and $\dim \pi^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.

This proves the Main Lemma when just Y is affine.

Now for the general case. Take a finite open affine cover Y_i of Y, and set $X_i := \pi^{-1}(Y_i)$. Then $\pi_i\colon X_i \to Y_i$ is a dominant morphism between irreducibles with Y_i affine, so the Main Lemma applies. Moreover, $\dim X_i = \dim X$ and $\dim Y_i = \dim Y$ for all i. If $C \subset \pi^{-1}(y)$ is an irreducible component, then $C = \bigcup_i C_i$, where $C_i = C \cap \pi_i^{-1}(y)$. Again, each C_i is open in C, so is either empty or irreducible with $\dim C = \dim C_i \geq \dim X - \dim Y$.

For each i take $U_i \subset Y_i$ open dense such that $\dim \pi_i^{-1}(y) = \dim X - \dim Y$ for all $y \in U_i$. Then $U = \bigcap_i U_i$ is open dense and $\dim \pi^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.

This proves the Main Lemma in general.

(b) A dominant morphism between irreducible affines corresponds to an injective algebra homomorphism $\phi\colon A \rightarrowtail B$ between affine domains (=finitely generated domains). A point in $\operatorname{Spec} A$ corresponds to a maximal ideal $\mathfrak{m} \lhd A$, equivalently to an algebra homomorphism $A \to K$, in which case the fibre over the point corresponds to the algebra $B \otimes_A K$. The irreducible components of the fibre correspond to the minimal primes of this algebra, equivalently to the primes $\mathfrak{q} \lhd B$, minimal with respect to lying over $\mathfrak{m}$ (that is, $\mathfrak{q} \cap A = \mathfrak{m}$). Since $\mathfrak{m}$ is maximal, every prime of B containing $\mathfrak{q}$ lies over $\mathfrak{m}$, so the dimension of this component of the fibre is the Krull dimension of $B/\mathfrak{q}$, which equals $\operatorname{tr.deg} \kappa(\mathfrak{q})$, where $\kappa(\mathfrak{q})$ is the field of fractions of $B/\mathfrak{q}$.

So the Main Lemma becomes the statement $\operatorname{tr.deg} \kappa(\mathfrak{q}) \geq \dim B - \dim A$ for all primes $\mathfrak{q} \lhd B$, minimal over some $\mathfrak{m} \lhd A$ (and with equality for $\mathfrak{m}$ in a dense open subset). □

More generally, one can prove the following. Again $A \subset B$ are affine domains (hence universally catenary), $\mathfrak{q} \lhd B$ a prime of B, and $\mathfrak{p} := A \cap \mathfrak{q}$ a prime of A. Then

$$\operatorname{tr.deg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = (\dim B - \dim A) + (\dim A_{\mathfrak{p}} - \dim B_{\mathfrak{q}}),$$

where the right hand side involves the Krull dimensions of the localisations.

2. Recall that a constructible subset is a finite union of locally closed subsets.

(a) Prove that the class of constructible subsets is closed under finite unions, finite intersections, complements, and inverse images. (Chevalley's Theorem says it is also closed under images.)

(b) Prove that every constructible set V contains an open dense subset of its closure $\bar{V}$, so there exists $U \subset V$ with U open and dense in $\bar{V}$.

(c) Suppose a connected algebraic group G acts on a variety X, and let $V \subset X$ be a G-stable constructible subset, so $G \cdot V = V$. Show that we can decompose V into a finite disjoint union of G-stable, irreducible and locally closed subsets.

Proof. (a) Unions is clear. Intersection of two locally closeds is again locally closed, so get the result for intersections. Complement of locally closed is locally closed, so get complements. Preimage of locally closed is locally closed, so get preimages.

(b) Assume first that $\bar{V}$ is irreducible. Write $V = V_1 \cup \cdots \cup V_r$ with V_i locally closed. Since $\bar{V} = \bar{V}_1 \cup \cdots \cup \bar{V}_r$ is irreducible, we must have $\bar{V} = \bar{V}_i$ for some i. Now $V_i = U_i \cap \bar{V}$ for some open U_i, so U_i is open dense in $\bar{V}$ and contained in V.

In general write $\bar{V} = C_1 \cup \cdots \cup C_n$ as the union of its irreducible components, and set $V_i := V \cap C_i$, so constructible with $\bar{V}_i = C_i$. By the first part, each V_i contains some U_i', open dense in $\bar{V}_i$. Set $U_i := U_i' - \bigcup_{j \neq i} C_j$. Then U_i is open in $\bar{V}$, contained in V, with closure C_i. Thus $U := \bigcup_i U_i$ is open dense in $\bar{V}$ and contained in V.

(c) Obviously for this to work we need the group G to be connected (so irreducible as a variety).

Write $\bar{V} = C_1 \cup \cdots \cup C_n$ as the union of its irreducible components. As in (b) we can find disjoint $U_i \subset V$ with U_i open in $\bar{V}$ and closure C_i. Now set $V_i := \bigcup_{g \in G} gU_i$. This is still contained in V and open in $\bar{V}$. Moreover, as it is the image of $G \times U_i \to X$, it is irreducible, so $\bar{V}_i = C_i$. Thus $V_1 \cup \cdots \cup V_n$ is a disjoint union of G-stable, irreducible locally closed subsets.

Now repeat the argument with $V' := V - \bigcup_i V_i$. By construction, $\bar{V}' \cap C_i$ is a proper subset for each i, so we are done by induction on dimension. □

3. We know that if $f\colon X \to Y$ is a dominant morphism of varieties and X is irreducible, then Y is irreducible. In general the converse fails, so Y irreducible does not imply X irreducible. We prove that the converse holds in three situations.
   Let $f\colon X \to Y$ be a dominant morphism of varieties with Y irreducible. Assume further that each non-empty fibre is irreducible of the same dimension d.

(a) Suppose f is an open map. Prove that X is irreducible. (In fact, all we need for this is that there is a dense set of points $y \in Y$ such that $f^{-1}(y)$ is nonempty and irreducible .)

(b) In general, decompose $X = X_1 \cup \cdots \cup X_n$ into its irreducible components. Show that some $f(X_i)$ is dense in Y. Apply the Main Lemma to get that there is some i such that $f(X_i)$ is dense, and $f^{-1}(y) \subset X_i$ for all $y \in f(X_i)$. Deduce that if $f(X_i) = Y$, then $X = X_i$ is irreducible.

(c) Suppose f is a closed map. Prove that X is irreducible.

(d) Suppose instead that f admist a section s, so a morphism $s\colon Y \to X$ with $fs = \mathrm{id}_Y$. Prove that X is irreducible.

Proof. (a) Let $Y' \subset Y$ be an open dense subset such that $f^{-1}(y)$ is non-empty irreducible for all $y \in Y'$.

Suppose we have non-empty open disjoint $U, V \subset X$. Since $f(U), f(V), Y'$ are non-empty open and Y is irreducible, they must intersect. Take $y \in f(U) \cap f(V) \cap Y'$. Then $f^{-1}(y) = (f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap V)$ is a disjoint union of non-empty opens, contradicting the fact that the fibre is irreducible.

We deduce that X is irreducible.

(b) Each $f(X_i)$ is irreducible, and $Y = \bigcup_i \overline{f(X_i)}$, so Y irreducible implies some $f(X_i)$ is dense. Set $I := \{i : f(X_i) \text{ dense}\}$.

Set Y' to be the complement of $\bigcup_{j \notin I} \overline{f(X_j)}$, so that Y' is open dense in Y. Then for all $y \in Y'$ we have $f^{-1}(y) = \bigcup_{i \in I} f^{-1}(y) \cap X_i$.

By the Main Lemma, for each $i \in I$ there exists an open dense $U_i \subset Y'$ such that $f^{-1}(y) \cap X_i$ has dimension $\dim X_i - \dim Y$. Since $f^{-1}(y)$ is irreducible of dimension d, there exists some i for which $f^{-1}(y) \cap X_i$ has dimension d. Now for any other $y \in f(X_i)$ we have $f^{-1}(y) \cap X_i$ has dimension at least d, by the Main Lemma, so has dimension precisely d. Finally, since $f^{-1}(y) \cap X_i$ is closed in the irreducible $f^{-1}(y)$, it follows that $f^{-1}(y) \subset X_i$ for all $y \in f(X_i)$.

If $f(X_i) = Y$, then it follows that $f^{-1}(Y) \subset X_i$, so that $X = X_i$ is irreducible.

(c) If f is closed, then for i as in (b) we have that $f(X_i)$ is closed and dense, so equals Y, and we are done.

(d) Again, take i as in (b). We know that $f(X_i)$ is dense, and if $y \in f(X_i)$, then $f^{-1}(y) \subset X_i$. Since $s(y) \in f^{-1}(y)$, we see that $s(Y) \cap X_i = sf(X_i)$ is closed and dense in the irreducible $s(Y)$, so equals $s(Y)$. Thus $f(X_i) = Y$ and we are done. □

4. Let $\mathbb{M}_d(K)$ denote the variety of $d \times d$ matrices, and $C_d := \{(M, N) \in \mathbb{M}_d(K)^2 : MN = NM\}$ the commuting variety.

(a) Given $M \in \mathbb{M}_d(K)$, show that $Z_M := \{N \in \mathbb{M}_d(K) : MN = NM\}$ is a subspace, so in particular an irreducible cone. Deduce that the map $M \mapsto \dim Z_M$ is upper semicontinuous. Show further that its minimal value is d, and $\dim Z_M = d$ if and only if $M \in U$, the set of regular matrices (that is, those matrices whose Jordan Normal Form has Jordan blocks with pairwise distinct eigenvalues). In particular, U is open dense in $\mathbb{M}_d(K)$.

(b) Set $C'_d := \{(M, N) \in C_d : N \in U\}$ and let $p\colon C'_d \to \mathbb{M}_d(K)$ be the projection onto the second co-ordinate. Use the previous exercise to show that C'_d is irreducible of dimension $d^2 + d$.

(c) As in the lectures, given $(M, N) \in C_d$ there exists $R \in U$ commuting with M. Consider the morphism $f\colon \mathbb{A}^1 \to C_d$, $\lambda \mapsto (M, N + \lambda R)$. Then $f^{-1}(C'_d)$ is non-empty open, so $\mathrm{Im}(f) \subset \bar{C}'_d$. Hence C'_d is dense in C_d, so C_d is irreducible of dimension $d^2 + d$.

Proof. (a) Clearly $N, N' \in Z_M$ and $\lambda \in K$ implies $N + \lambda N' \in Z_M$. Thus Z_M is a subspace. Moreover, the set C_d is closed in $\mathbb{M}_d(K)^2$. Thus the Cones Theorem implies $M \mapsto \dim Z_M$ is upper semi-continuous.

As in the lectures, if $M = J_r(\lambda)$ is a Jordan block, then Z_M consists of those upper-triangular matrices N which are constant on each diagonal. For example

$$M = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad \text{commutes with} \quad N = \begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

A direct proof: clearly $Z_M = Z_J$, where $J = J_r(0) = E_{12} + E_{23} + \cdots + E_{r-1r}$. Thus $(NJ)_{ij} = N_{ij-1}$ and $(JN)_{ij} = N_{i+1j}$. Thus N is constant on all diagonals, and zero if $j < i$.

Thus, $\dim Z_M = r$. In general, writing $M \in \mathbb{M}_d(K)$ in Jordan Normal Form, we see that $\dim Z_M \geq d$. On the other hand, if M is diagonal with distinct eigenvalues, then it is easy to see that Z_M is the set of all diagonal matrices, so $\dim Z_M = d$. Thus d is the generic value, and $\{M : \dim Z_M = d\}$ is open.

We still need to check that this is precisely the set U of regular matrices. Suppose $M = J_r(\lambda) \oplus J_s(\mu)$. If $\lambda = \mu$, then M commutes with the elementary matrix $E_{1,r+1}$. Hence if M does not have distinct eigenvalues, then $\dim Z_M > d$.

If $\lambda \neq \mu$, then only the zero matrix satisfies $J_r(\lambda)N = NJ_s(\mu)$. For, take (i,j) such that $n_{ij} \neq 0$ and all entries below and to the left are zero. Then the (i,j) entry in $J_r(\lambda)N - NJ_s(\mu)$ is $(\lambda-\mu)n_{ij}$. Thus if M in Jordan Normal Form has distinct eigenvalues, then every $N \in Z_M$ has the same block-diagonal form, and hence $\dim Z_M = d$.

This shows that $\{M : \dim Z_M = d\} = U$ is the set of regular matrices.

> A much better way of thinking about this is that each matrix M determines a module for $K[t]$, and the space Z_M equals $\mathrm{End}_{K[t]}(M)$. Now a Jordan block $M = J_r(\lambda)$ gives the module $K[t]/(t-\lambda)^r$, which is uniserial with all composition factors isomorphic to the simple module $K[t]/(t-\lambda)$. Thus $\mathrm{Hom}_{K[t]}(K[t]/(t-\lambda)^r, K[t]/(t-\mu)^s)$ is zero unless $\lambda = \mu$, in which case it has dimension $\min\{r,s\}$.

(b) Clearly want the projection onto the second co-ordinate. (I changed the order to align with what was done in the lectures, but obviously forgot to change 'first' to 'second'.

We have the morphism $p\colon C'_d \to U$. This has a section $s(N) = (0,N)$, and the fibres are all vector spaces of dimension d, so are irreducible. Now U is open dense in $\mathbb{M}_d(K)$, so is irreducible of dimension d. Thus by Q3 we know that C'_d is irreducible, and of dimension d^2+d by the Main Lemma.

(c) Consider $g\colon K \to \mathbb{M}_d(K)$, $t \mapsto R + tN$. Then $0 \in g^{-1}(U)$, so $g^{-1}(U)$ is non-empty open, hence dense. Also, if $t \in K^\times$, then $R + tN \in U$ if and only if $N + t^{-1}R \in U$. Now consider $h\colon K \to \mathbb{M}_d(K)$, $t \mapsto N + tR$. It follows that $h^{-1}(U)$ is again non-empty open, so dense.

Using this we see that $f^{-1}(C'_d)$ is non-empty open, and is disjoint from the open $f^{-1}(C_d - \overline{C}'_d)$. By irreducibility we deduce that the latter is empty, so in particular $(M,N) = f(0)$ lies in $\overline{C}'_d$. Thus $C_d = \overline{C}'_d$ is irreducible of dimension d^2+d. □