Non-commutative Algebra 3, SS 2020

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Solutions 6

Throughout K will be an algebraically closed field.

1. Let an algebraic group G act on a variety X, and assume that a geometric quotient $\pi\colon X \to X/G$ exists.

(a) Given a point $x \in X$, we have the action $\rho_x\colon G \to X$, $g \mapsto gx$. We therefore get maps on tangent spaces

$$\mathfrak{g} \xrightarrow{d(\rho_x)} T_x X \xrightarrow{(d\pi)_x} T_{Gx} X/G.$$

Show that the composition is zero.

(b) Show that

$$T_{(u,v)}(U \times V) \cong T_u U \times T_v V$$

for any varieties U, V.

(c) Now suppose that π is Zariski locally trivial. Deduce that $(d\pi)_x$ is a cokernel for $d(\rho_x)$, so we have a natural isomorphism

$$T_x X / \operatorname{Im}(d(\rho_x)) \xrightarrow{\sim} T_{Gx} X/G.$$

Proof. (a) By the chain rule we have $d(\pi\rho_x)_1 = d\pi_x d(\rho_x)$. On the other hand, $\pi\rho_x$ is the constant map $g \mapsto \pi(x)$, so its derivative is zero.

(b) Let A, B be two K-algebras, with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively. Then $\mathfrak{m}\otimes B + A\otimes\mathfrak{n}$ is a maximal ideal of $A \otimes_K B$. We have the algebra homomorphism $A \to A/\mathfrak{m} \cong K$, $a \mapsto a_{\mathfrak{m}}$, and similarly for B, and hence also $A \otimes_K B \to K$.

We claim that $\operatorname{Der}(A \otimes B, K) \cong \operatorname{Der}(A, K) \times \operatorname{Der}(B, K)$.

Given a derivation ξ of $A \otimes B$, we get a derivation of A by composing with the algebra homomorphism $A \to A \otimes B$, and similarly for B. Conversely, given derivations α, β of A and B, we get a derivation of $A \otimes B$ by taking $a \otimes b \mapsto a_{\mathfrak{m}}\beta(b) + \alpha(a)b_{\mathfrak{n}}$. These maps are mutually inverse, proving the result.

An easier way of seeing this is to introduce the algebra $K[\varepsilon] = K[t]/(t^2)$. Then we have a bijection $\operatorname{Der}(A, K) \cong \operatorname{Hom}_{K-\mathrm{alg}}(A, K[\varepsilon])$ sending a deriviation α to the algebra homomorphism $a \mapsto a_{\mathfrak{m}} + \alpha(a)$. The claim then follows from the bijection

$$\operatorname{Hom}_{K-\mathrm{alg}}(A \otimes B, K[\varepsilon]) \cong \operatorname{Hom}_{K-\mathrm{alg}}(A, K[\varepsilon]) \times \operatorname{Hom}_{K-\mathrm{alg}}(B, K[\varepsilon]).$$

Using this we see that $T_{(u,v)}(U \times V) \cong T_u U \times T_v V$. For, we may first replace U and V by open affine neighbourhoods of u and v (by 3.8 Lemma C), and then apply the above isomorphism.

(c) Assume that π is Zariski locally trivial. Given $x \in X$ we can find an open (affine) neighbourhood V of Gx such that $\pi^{-1}(V) \cong G \times V$. Since all points of G look the same, we may further assume for simplicity that x corresponds to the point $(1, v) \in G \times V$. Now $T_x X \cong \mathfrak{g} \times T_{Gx}(X/G)$.

The action map ρ_x is then $G \to G \times V$, $g \mapsto (g, v)$. Its derivative $d(\rho_x)$ corresponds to $(\mathrm{id}, 0)$ as a map $\mathfrak{g} \to \mathfrak{g} \times T_{Gx}(X/G)$. The morphism π corresponds to the second projection, so its derivative is $(0, \mathrm{id})$. Thus $d\pi_x$ induces an isomorphism $T_x X / \operatorname{Im}(d(\rho_x)) \xrightarrow{\sim} T_{Gx}(X/G)$. □

2. Consider the Grassmannian $\operatorname{Gr}(M, d) = \operatorname{Inj}(K^d, M)/\operatorname{GL}_d(K)$. Let $\theta \in \operatorname{Inj}(K^d, M)$ have image U.

(a) Show that $T_\theta \operatorname{Inj}(K^d, M) \cong \operatorname{Hom}(U, M)$.

(b) Choose a cokernel $\phi \colon M \to M/U$ for θ. Show that the map

$$T_\theta \operatorname{Inj}(K^d, M) \to \operatorname{Hom}(U, M/U), \quad \theta' \mapsto \phi\theta',$$

is a cokernel for the map $d(\rho_\theta) \colon \operatorname{End}(K^d) \to T_\theta \operatorname{Inj}(K^d, M)$.

(c) Deduce that $T_U \operatorname{Gr}(M, d) \cong \operatorname{Hom}(U, M/U)$.

Proof. (a) $\operatorname{Inj}(K^d, M)$ is open in $\operatorname{Hom}(K^d, M)$, which is just affine space. Thus $T_\theta \operatorname{Inj}(K^d, M) = \operatorname{Hom}(K^d, M)$, which is isomorphic to $\operatorname{Hom}(U, M)$ via $\theta \colon K^d \xrightarrow{\sim} U$.

(b) We use Lemma 3.8 E. We have the morphism $\rho_\theta \colon \operatorname{GL}_d(K) \to \operatorname{Hom}(K^d, M)$, $g \mapsto \theta g^{-1}$. Given $\gamma \in \mathfrak{g} = \operatorname{End}(K^d)$, we have $(1 + \gamma\varepsilon)(1 - \gamma\varepsilon) = 1 - \gamma^2\varepsilon^2$, so $(1 + \gamma\varepsilon)^{-1} = 1 - \gamma\varepsilon + O(\varepsilon^2)$, and hence

$$\rho_\theta(1 + \gamma\varepsilon) = \theta(1 - \gamma\varepsilon) + O(\varepsilon^2) = \theta - \theta\gamma\varepsilon + O(\varepsilon^2).$$

Hence $d(\rho_\theta)(\gamma) = -\theta\gamma$.

We thus have the (injective) map $d(\rho_\theta) \colon \operatorname{End}(K^d) \to \operatorname{Hom}(K^d, M)$, $\gamma \mapsto -\theta\gamma$. On the other hand, we have the short exact sequence

$$0 \to K^d \xrightarrow{\theta} M \xrightarrow{\phi} M/U \to 0$$

and applying $\operatorname{Hom}(K^d, -)$ gives the short exact sequence

$$0 \to \operatorname{End}(K^d) \to \operatorname{Hom}(K^d, M) \to \operatorname{Hom}(K^d, M/U) \to 0.$$

Thus $\operatorname{Hom}(K^d, M) \to \operatorname{Hom}(K^d, M/U)$, $h \mapsto \phi h$ is a cokernel for $d(\rho_\theta)$.

Again, identifying $\theta \colon K^d \xrightarrow{\sim} U$, we can write this as $\operatorname{Hom}(U, M) \to \operatorname{Hom}(U, M/U)$.

(c) Using the previous exercise (which is allowed since $\operatorname{Inj}(K^d, M) \to \operatorname{Gr}(M, d)$ is Zariski locally trivial), we obtain

$$T_U \operatorname{Gr}(M, d) \cong \operatorname{Hom}(U, M/U).$$ □

Such computations become easier in the language of functors. Each commutative K-algebra A yields a (covariant) functor $R \mapsto \mathrm{Hom}_{K-\mathrm{alg}}(A,R)$ from the category of commutative K-algebras to the category of sets. If $R = K$, then we just get back the points of $\mathrm{Spec}\,A$, but now we have more flexibility. For $\mathrm{GL}_d(K)$, we have the functor $R \mapsto \mathrm{GL}_d(R)$, which is those matrices $g \in \mathbb{M}_d(R)$ such that $\det(g)$ is a unit in R. Similarly $\mathrm{Inj}(K^d,K^m)(R)$ consists of those matrices $\theta \in \mathbb{M}_{m\times d}(R)$ such that all d-minors generate the unit ideal.

The natural map $K[\varepsilon] \to K$ induces a map $\mathrm{Hom}_{K-\mathrm{alg}}(A,K[\varepsilon]) \to \mathrm{Hom}_{K-\mathrm{alg}}(A,K)$, so a map $\mathrm{Spec}\,A(K[\varepsilon]) \to \mathrm{Spec}\,A(K) = \mathrm{Spec}\,A$. Then $T_p\,\mathrm{Spec}\,A$ is identified with the fibre over p of this map.

If $f\colon A \to B$, corresponding to $\phi\colon \mathrm{Spec}\,B \to \mathrm{Spec}\,A$, then we get the commutative square

$$\begin{array}{ccc} \mathrm{Spec}\,B(K[\varepsilon]) & \longrightarrow & \mathrm{Spec}\,B(K) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,A(K[\varepsilon]) & \longrightarrow & \mathrm{Spec}\,A(K) \end{array}$$

and the differential $d\phi_q$ for $q \in \mathrm{Spec}\,B$ is the induced map on fibres.

In our example, we have $\theta \in \mathrm{Inj}(K^d,M)$ and the map $\rho_\theta\colon \mathrm{GL}_d(K) \to \mathrm{Hom}(K^d,K^m)$. Thus the map

$$\mathrm{GL}_d(K[\varepsilon]) \to \mathbb{M}_{m\times d}(K[\varepsilon]), \quad 1+\gamma\varepsilon \mapsto \theta(1-\gamma\varepsilon),$$

using that $1-\gamma\varepsilon$ is really the inverse of $1+\gamma\varepsilon$ in $\mathrm{GL}_d(K[\varepsilon])$.

3. More generally, consider a quiver Grassmannian $\mathrm{Gr}_A(M,d)$, where A is a K-algebra and M is an m-dimensional A-module.
   Set $\mathrm{Inj}_A(K^d,M)$ to be those $\theta \in \mathrm{Inj}(K^d,M)$ such that $\phi a_M \theta = 0$ for all $a \in A$, where ϕ is a cokernel for θ, and $a_M \in \mathrm{End}_K(M)$ is the map $m \mapsto a\cdot m$.

(a) Show that if $\theta \in \mathrm{Inj}_A(K^d,M)$ has image U, then a_M restricts to an endomorphism of U, so that $U \leq M$ is an A-submodule.

(b) For $I \subset \{1,\dots,m\}$ of size d, let Δ_I be the corresponding d-minor on $\mathrm{Hom}(K^d,M)$ (so the determinant of the $d\times d$-matrix having rows from I), and $U_I = D(\Delta_I)$ the corresponding distinguished open affine. Show that $\mathrm{Inj}_A(K^d,M)\cap U_I$ is closed in U_I. Deduce that $\mathrm{Inj}_A(K^d,M)$ is a closed subset of $\mathrm{Inj}(K^d,M)$.

(c) Show that $\mathrm{Gr}_A(M,d) = \mathrm{Inj}_A(M,d)/\mathrm{GL}_d(K)$ is a geometric quotient, and that $\mathrm{Inj}_A(M,d) \to \mathrm{Gr}_A(M,d)$ is Zariski locally trivial.

(d) Deduce that $T_U\,\mathrm{Gr}_A(M,d) \cong \mathrm{Hom}_A(U,M/U)$.

Proof. (a) Given $u \in U$, write it as $\theta(v)$ for some $v \in K^d$. Then $\phi(a_M(u)) =$

$\phi a_M \theta(v) = 0$, so that $a_M(u) \in \mathrm{Ker}(\phi) = U$.

Alternatively, since $\phi a_M \theta = 0$ we know that there is a unique $a'_M \in \mathrm{End}(K^d)$ such that $a_M \theta = \theta a'_M$. These a'_M then determine an algebra homomorphism $A \to \mathrm{End}(K^d)$.

(b) Observe that $\mathrm{Inj}(K^d, M) = \bigcup_I U_I$, since a matrix is injective if and only if it has rank d, if and only if some d-minor is invertible.

Now $U_I \cong \mathrm{GL}_d(K) \times \mathbb{M}_{(m-d)\times d}(K)$, by rearrranging rows. Given

$$\begin{pmatrix} g \\ h \end{pmatrix} \in \mathrm{GL}_d(K) \times \mathbb{M}_{(m-d)\times d}(K) \subset \mathbb{M}_{m\times d}(K),$$

this has cokernel

$$(-hg^{-1}, 1) \in \mathbb{M}_{(m-d)\times d}(K) \times \mathrm{GL}_{(m-d)}(K) \subset \mathbb{M}_{(m-d)\times m}(K),$$

and rearranging columns we obtain a map $\Phi_I \colon U_I \to \mathrm{Hom}(M, K^{(m-d)})$ such that $\Phi_I(\theta)$ is a cokernel for $\theta \in U_I$.

Observe that each $\mathrm{Inj}_A(K^d, M) \cap U_I = \{\theta : \Phi_I(\theta) a_M \theta = 0 \text{ for all } a \in A\}$ is closed in U_I. Thus $\mathrm{Inj}_A(K^d, M)$ is closed in $\bigcup_I U_I = \mathrm{Inj}(K^d, M)$.

(c) We know that $\pi_I \colon U_I \to V_I := \pi(U_I)$ is a trivial $\mathrm{GL}_d(K)$-bundle. Set $U_I^A := \mathrm{Inj}_A(K^d, M) \cap U_I$. This is closed and $\mathrm{GL}_d(K)$-stable in U_I, so its image V_I^A is closed in V_I. The isomorphism $U_I \cong \mathrm{GL}_d(K) \times V_I$ thus restricts to an isomorphism $U_I^A \cong \mathrm{GL}_d(K) \times V_I^A$. This shows that the restriction $\mathrm{Inj}_A(K^d, M) \to \mathrm{Gr}_A(M, d)$ is a Zariski locally trivial $\mathrm{GL}_d(K)$-bundle, and in particular it is a geometric quotient.

(d) Recall the map $\Phi \colon \mathrm{GL}_d(K) \times \mathrm{Hom}(K^d, K^{m-d}) \to \mathrm{Hom}(M, K^{m-d})$, $(g, h) \mapsto (-hg^{-1}, 1)$. Identifying U_I with $\mathrm{GL}_d(K) \times \mathrm{Hom}(K^d, K^{m-d})$, the matrix a_M corresponds to a block matrix $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, and the closed subset U_I^A corresponds to those (g, h) such that $(-hg^{-1}, 1)\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right) = 0$.

For simplicity, we may assume that $\theta = (1, 0)$, so that $\phi = (0, 1)$, in which case $r = 0$. Now (γ, ν) lies in the tangent space provided

$$(-\nu\varepsilon(1 - \gamma\varepsilon), 1) \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 + \gamma\varepsilon \\ \nu\varepsilon \end{pmatrix} = O(\varepsilon^2),$$

equivalently $s\nu = \nu p$. Since the block p gives the action of $a \in A$ on U, and the block s gives the action of a on M/U, we see that $T_\theta \,\mathrm{Inj}_A(K^d, M) \cong \mathfrak{g} \times \mathrm{Hom}_A(U, M/U)$, and hence that $T_U \,\mathrm{Gr}_A(M, d) \cong \mathrm{Hom}_A(U, M/U)$. □

More precisely, we have computed the tangent space $T_\theta^S \,\mathrm{Inj}_A(K^d, M)$ where S is the collection of polynomials coming from $\Phi_I(\theta) a_M \theta = 0$ for a a (finite) set of generators for A. We haven't shown that this yields a space with functions, since there may be some further functions which vanish on $\mathrm{Inj}_A(K^d, M)$, so in the radical of the ideal (S). This will in general be the case.

4. Let $A = KQ$ be the path algebra of a quiver, and M a finite dimensional A-module.

(a) Using the long exact sequences for hom, show that if $\operatorname{Ext}^1(M, M) = 0$ and $U \leq M$ is a submodule, then $\operatorname{Ext}^1(U, M)$, $\operatorname{Ext}^1(M, M/U)$ and $\operatorname{Ext}^1(U, M/U)$ all vanish.

(b) Use the Ringel form to deduce that $\dim \operatorname{Hom}_A(U, M)$ depends only on the dimension vectors of U and M.

(c) Deduce that $\operatorname{Gr}_A(M, \underline{d})$ is smooth.

Proof. (a) We have $0 \to U \to M \to M/U \to 0$. Since $A = KQ$ is hereditary, applying $\operatorname{Hom}(M, -)$ gives an epimorphism $\operatorname{Ext}^1(M, M) \twoheadrightarrow \operatorname{Ext}^1(M, M/U)$, and applying $\operatorname{Hom}(-, M)$ gives an epimorphism $\operatorname{Ext}^1(M, M) \twoheadrightarrow \operatorname{Ext}^1(U, M)$. Applying $\operatorname{Hom}(U, -)$ gives an epimorphism $\operatorname{Ext}^1(U, M) \twoheadrightarrow \operatorname{Ext}^1(U, M/U)$.

Thus, if $\operatorname{Ext}^1(M, M) = 0$, then $\operatorname{Ext}^1(U, M)$, $\operatorname{Ext}^1(M, M/U)$ and $\operatorname{Ext}^1(U, M/U)$ all vanish.

(b) We have

$$\dim \operatorname{Hom}(U, M) = \langle \underline{\dim}\, U, \underline{\dim}\, M \rangle + \dim \operatorname{Ext}^1(U, M) = \langle \underline{\dim}\, U, \underline{\dim}\, M \rangle.$$

(c) For a fixed dimension vector $\underline{d}$, the tangent spaces of $\operatorname{Gr}_A(M, \underline{d})$ all have the same dimension. This is not enough to deduce smoothness, however, since we have not actually computed the tangent space of the space with functions; just as for $\operatorname{Inj}_A(K^d, M)$ we computed the possibly larger $T_\theta^S \operatorname{Inj}_A(K^d, M)$, what we have computed here is the equivalent for the geometric quotient. $\square$

We can recover the smoothness result as follows. Let A be a finitely generated K-algebra, $\mathfrak{m}$ a maximal ideal, and $A \to A/\mathfrak{m} \cong K$ the quotient map. Recall that the tangent space $T_p \operatorname{Spec} A$ at the point p corresponding to $\mathfrak{m}$ is the fibre over $\operatorname{Hom}_{K-\mathrm{alg}}(A, K[t]/(t^2)) \to \operatorname{Hom}_{K-\mathrm{alg}}(A, K)$. We say that p (or the local ring $A_\mathfrak{m}$) is (formally) smooth provided for each $n \geq 2$, the image of the map $\operatorname{Hom}_{K-\mathrm{alg}}(A, K[t]/(t^n)) \to \operatorname{Hom}_{K-\mathrm{alg}}(A, K[t]/(t^2))$ contains all points in the tangent space $T_p \operatorname{Spec} A$.

N.B. For a local ring, formally smooth implies regular, but the converse is false in general. For example, if L/K is a finite purely inseparable field extension, then L is regular, but L/K is not formally smooth, and after base change $L \otimes_K L$ contains nilpotents so will no longer be regular.

If we fix $\theta \in \mathrm{Inj}_A(K^d, M)$, then as in the previous question we may assume that $\theta = (1, 0)$, its cokernel is $\phi = (0, 1)$, and $a \in A$ acts as $\left(\begin{smallmatrix} a' & \bar{a} \\ 0 & a'' \end{smallmatrix}\right)$. To compute the tangent space we took $(1+\gamma\varepsilon, \nu\varepsilon)$ with cokernel $(-\nu\varepsilon, 1)$, and obtained the equations $a''\nu = \nu a'$ for all $a \in A$. We are now considering elements in $K[t]/(t^{n+1})$. Thus we take $(1, \sum_i \nu_i t^i)$, with cokernel $(-\sum_i \nu_i t^i, 1)$. The equations then become

$$a''\nu_i - \nu_i a' = \sum_{i=j+k} \nu_j \bar{a} \nu_k \quad \text{for all } i \text{ and all } a \in A.$$

We consider the map

$$\phi_0\colon \mathrm{Hom}(K^d, K^{m-d}) \to \mathrm{Hom}(A, \mathrm{Hom}(K^d, K^{m-d})),$$
$$\nu \mapsto \big(a \mapsto a''\nu - \nu a'\big).$$

If the elements $\sum_{i=j+k} \nu_j \bar{a} \nu_k$ always lie in the image of ϕ_0, then we can inductively define ν_i, and hence the image of the map $\mathrm{Hom}_{K-\mathrm{alg}}(A, K[t]/(t^{n+1}) \to \mathrm{Hom}_{K-\mathrm{alg}}(A, K[t]/(t^n))$ contains all points in the fibre over p, for all n. Thus p will be a smooth point.
As in 4.1 Lemma D we have the A-bimodule $\mathrm{Hom}_K(U, M/U)$, and the map ϕ_0 we just constructed is the zeroth map in the Hochschild complex. Now $H^1(A, \mathrm{Hom}(U, M/U)) \cong \mathrm{Ext}^1_A(U, M/U)$, which vanishes when $\mathrm{Ext}^1(M, M) = 0$. Also, consider some $\xi := \sum_{i=j+k} \nu_j \bar{a} \nu_k$. Then $\phi_1(\xi)(a \otimes b) = a''\xi(b) - \xi(ab) + \xi(a)b'$. Since the A-action comes from an algebra homomorphism we have $\overline{ab} = a'\bar{b} + \bar{a}b''$. Thus

$$\phi_1(\xi)(a \otimes b) = \sum_{i=j+k} (a''\nu_j - \nu_j a')\bar{b}\nu_k - \sum_{i=j+k} \nu_j \bar{a}(b''\nu_k - \nu_k b').$$

By induction on i we have

$$a''\nu_j - \nu_j a' = \sum_{j=p+q} \nu_p \bar{a} \nu_q \quad \text{and} \quad b''\nu_k - \nu_k b' = \sum_{k=p+q} \nu_p \bar{b} \nu_q.$$

Thus

$$\phi_1(\xi)(a \otimes b) = \sum_{i=j+k+l} \nu_j \bar{a} \nu_k \bar{p} \nu_l - \sum_{i=j+k+l} \nu_j \bar{a} \nu_k \bar{b} \nu_l = 0.$$

So each element ξ lies in $\mathrm{Ker}(\phi_1) = \mathrm{Im}(\phi_0)$, and we deduce that each point p is smooth. Thus $\mathrm{Inj}_A(K^d, M)$ is smooth, and hence so too is the quiver Grassmannian $\mathrm{Gr}_A(K^d, M)$.

5.

(a) Consider the surjective morphism $\phi\colon V(t^2 - xt + y) \to \mathbb{A}^2$, $(x, y, t) \mapsto (x, y)$. For which $q \in V(t^2 - xt + y)$ is the differential $(d\phi)_q$ surjective?

(b) Suppose $\operatorname{char} K = p > 0$ and consider the surjective morphism $\theta\colon \mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^p$. Compute the differential $(d\theta)_q$.

Proof. (a) We first compute T_qV. Consider $q + r\varepsilon$. Writing $q = (q_1, q_2, q_3)$, and similarly for r, we compute

$$\begin{aligned}(q_3 + r_3\varepsilon)^2 &- (q_1 + r_1\varepsilon)(q_3 + r_3\varepsilon) + (q_2 + r_2\varepsilon)\\ &= (q_3^2 - q_1q_3 + q_2) + (2q_3r_3 - q_1r_3 - q_3r_1 + r_2)\varepsilon + O(\varepsilon^2).\end{aligned}$$

We know that $q \in V$, so the first summand vanishes, and so T_qV consists of those r such that $(2q_3 - q_1)r_3 = q_3r_1 - r_2$.

We now compute the differential $d\phi_q$. Since ϕ is just the restriction of the projection map, $d\phi_q$ is the projection $(r_1, r_2, r_3) \mapsto (r_1, r_2)$.

So, the differential is onto if and only if, given (r_1, r_2) we can solve for r_3, which is if and only if $(2q_3 - q_1) \neq 0$. So the differential is onto at all points q with $2q_3 \neq q_1$, equivalently those q not of the form $(2a, a^2, a)$.

This shows that, even in very nice situations, the differential will in general not be surjective at all points, but only on a dense open subset.

(b) Here we have both tangent spaces being K. To compute the differential, take $q + r\varepsilon$. This is sent to $(q + r\varepsilon)^p = q^p + O(\varepsilon^2)$, so the differential is identically zero at all points.

The corresponding map of function fields is the inseparable field extension $K(t^p) \subset K(t)$. This shows that we need some separability hypothesis in order to conclude that the differential is surjective on a dense open set.

□