Non-commutative Algebra 3, SS 2020

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Solutions 7

Throughout K will be an algebraically closed field. Recall the degeneration order $M \leq N$ provided $\mathcal{O}_N \subset \overline{\mathcal{O}}_M$.

1. Let $A = K[x,y]/(x^2, y^2)$, and let M_t be the two-dimensional module where the x and y actions are given by

$$M_t(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_t(y) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

(a) Let $V \subset \mathrm{Mod}(A,4)$ be the subset consisting of the modules $M_s \oplus M_t$, so pairs of matrices

$$\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & & \\ s & 0 & & \\ & & 0 & 0 \\ & & t & 0 \end{pmatrix}, \qquad s,t \in K.$$

Consider the morphism

$$\theta \colon \mathrm{GL}_4(K) \times V \to \mathrm{Mod}(A,4),$$

given by the restriction of the usual $\mathrm{GL}_4(K)$ action on $\mathrm{Mod}(A,4)$.

Show that $\mathrm{GL}_4(K) \times V$ is irreducible and has dimension 18.

Show further the generic fibre of θ over the image has dimension 6 (it is the Zariski dimension of $\mathrm{Aut}(M_s \oplus M_t)$, equivalently the K-dimension of $\mathrm{End}(M_s \oplus M_t)$, for $s \neq t$).

Deduce that $\overline{\mathrm{Im}(\theta)}$ is irreducible of dimension 12.

(b) Show that the orbit closure $\overline{\mathcal{O}}_A$ is irreducible of dimension 12.

(c) Deduce that there is some s,t such that A does not degenerate to $M_s \oplus M_t$.

Proof. (a) We have the isomorphism $\mathbb{A}^2 \xrightarrow{\sim} V$, $(s,t) \mapsto M_s \oplus M_t$. The inverse is given by the projection onto the relevant entries of the second matrix. Thus V is irreducible of dimension 2, so $\mathrm{GL}_4(K) \times V$ is irreducible of dimension 18.

For $s \neq t$ the fibre of θ over $M_s \oplus M_t$ is $\mathrm{Aut}(M_s \oplus M_t) \sqcup \mathrm{Aut}(M_s \oplus M_t)\sigma$, where σ is the permutation matrix corresponding to $(13)(24)$. This has dimension $\dim \mathrm{End}(M_s \oplus M_t)$. (For $s = t$ we just have $\mathrm{Aut}(M_s^2)$, which has dimension $\dim \mathrm{End}(M_s^2) = 4 \dim \mathrm{End}(M_s)$.)

Now $\mathrm{End}(M_s)$ consists of matrices of the form $\left(\begin{smallmatrix} p & 0 \\ q & p \end{smallmatrix}\right)$, and if $s \neq t$, then $\mathrm{Hom}(M_s, M_t)$ consists of matrices of the form $\left(\begin{smallmatrix} 0 & 0 \\ q & 0 \end{smallmatrix}\right)$. Thus away from the diagonal, so for $s \neq t$, the endomorphism algebra has dimension 6 (and on the diagonal it has dimension 8).

By the Main Lemma on fibre dimension, we deduce that $\overline{\mathrm{Im}(\theta)}$ has dimension $18 - 6 = 12$.

(b) We have $\mathrm{End}(A) \cong A$, so has dimension 4. Using Theorem 5.1 we see that the orbit closure $\overline{\mathcal{O}}_A$ is irreducible of dimension $16 - 4 = 12$.

(c) We know that A is not isomorphic to any $M_s \oplus M_t$ (for example the endomorphism algebras have different dimensions). Thus $\mathrm{Im}(\theta) \cap \overline{\mathcal{O}}_A$ is disjoint from the open dense $\mathcal{O}_A$. Since the orbit closure is irreducible, the proper closed subset $\overline{\mathcal{O}}_A - \mathcal{O}_A$ has dimension strictly smaller than 12.

Thus if $\mathrm{Im}(\theta) \subset \overline{\mathcal{O}}_A$, then $\overline{\mathrm{Im}(\theta)}$ is contained in $\overline{\mathcal{O}}_A - \mathcal{O}_A$, a contradiction by dimensions. Thus there must exist some s, t such that A does not degenerate to $M_s \oplus M_t$. □

2. Let A be a finite dimensional algebra, and $\binom{a}{b}\colon X \to M \oplus X$ a monomorphism.

(a) Show that $f_t := b + t \cdot \mathrm{id}_X$ is an automorphism of X for all but finitely many $t \in K$.

(b) Set $M_t := \mathrm{Coker}\binom{a}{f_t}$. Deduce that $M_t \cong M$ for all but finitely many $t \in K$.

(c) Deduce that $M \leq M_0$.

Proof. (a) We know that $b + t \cdot \mathrm{id}_X$ is not invertible if and only if $-t$ is an eigenvalue of b.

(b) If f_t is invertible, then $\binom{a}{f_t}$ is a split mono, with retract $(0, f_t^{-1})\colon M \oplus X \to X$. In this case $M_t \cong M$.

(c) Let $T \subset K$ consist of those t for which $\binom{a}{f_t}$ is injective. In particular, T is open and contains 0.

As for Grassmannians (for example Exercise 6.3), we have the injective maps $\binom{a}{f_t}$ for $t \in T$. Choose a set I of columns such that the I-minor $\Delta_I\binom{a}{b}$ is non-zero. We then replace T by the non-empty open subset of K containing those t for which $\Delta_I\binom{a}{f_t}$ is non-zero. We can then construct a linear map $\theta\colon K^m \to M \oplus X$, where $m = \dim M$, and a morphism $\Phi\colon T \to \mathrm{Hom}_K(M \oplus X, K^m)$ ($m = \dim M$) such that $\Phi(t)$ is a cokernel of $\binom{a}{f_t}$, and $\Phi(t)\theta = \mathrm{id}$, for all $t \in T$.

We now obtain a morphism $\lambda\colon T \to \mathrm{Mod}(A, m)$. For, given $\alpha \in A$, we have the endomorphism $\bar{\alpha}$ of $M \oplus X$. Then $\lambda(t)$ sends $\alpha \in A$ to $\alpha_t := \Phi(t)\bar{\alpha}\theta$. Note that α_t is the unique endomorphism α_t such that $\alpha_t\Phi(t) = \Phi(t)\bar{\alpha}$, so the corresponding module is isomorphic to the cokernel of $\binom{a}{f_t}$.

Thus $\lambda(t)$ is isomorphic to M_t for all $t \in T$, so is isomorphic to M for almost all $t \in T$, and hence M degenerates to M_0. □

3. Let $k = \mathbb{F}_q$ be a finite field with q elements. Set $\mathcal{N}_d$ to be the set of all nilpotent matrices in $\mathbb{M}_d(k)$. In this question we want to prove $|\mathcal{N}_d| = q^{d(d-1)}$.

Recall that every matrix $M \in \mathbb{M}_d(K)$ determines a $K[T]$-module of dimension d, by letting T act as M. Then, given $M, M' \in \mathbb{M}_d(K)$, we have

$$\mathrm{Hom}_{K[T]}(M, M') = \{\theta \in \mathbb{M}_d(K) : \theta M = M'\theta\}.$$

Set $N = J_d(0)$ to be the Jordan block

$$N := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

and consider the set $\mathcal{S}_d$ of pairs (A, θ) such that $A \in \mathcal{N}_d$ and $\theta \in \mathrm{Hom}_{K[T]}(N, A)$.

(a) Show that for every nilpotent matrix A we have $\dim \mathrm{Hom}_{K[T]}(N, A) = d$. Thus the projection $\mathcal{S}_d \to \mathcal{N}_d$ on to the first co-ordinate is surjective and every fibre has size q^d. In other words, $|\mathcal{S}_d| = q^d|\mathcal{N}_d|$.

(b) Now consider the projection $\mathcal{S}_d \to \mathbb{M}_d(K)$ on to the second co-ordinate. Show that the fibre over θ has the same size as the fibre over $g\theta$ for every $g \in \mathrm{GL}_d(K)$. Thus we may assume that θ is in row-reduced form.

(c) Take $(A, \theta) \in \mathcal{S}_d$. Since $\theta N = A\theta$, we know that $N(\mathrm{Ker}(\theta)) \subset \mathrm{Ker}(\theta)$. Assuming θ is in row reduced form, show that we must have $\theta = E_r := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $I_r \in \mathbb{M}_r(K)$ is the identity matrix and $r = \mathrm{rank}\,\theta$.

(d) Show that for $\theta = E_r$, the number of nilpotent matrices A for which $\theta N = A\theta$ is $q^{r(d-r)}|\mathcal{N}_{d-r}|$. By induction this equals $q^{(d-1)(d-r)}$ for $r > 0$.

(e) Show that the number of θ which have row reduced form E_r is $(q^d - 1)(q^d - q)\cdots(q^d - q^{r-1})$.

(f) It follows that

$$|\mathcal{S}_d| - |\mathcal{N}_d| = \sum_{r>0} (q^d - 1)(q^d - q)\cdots(q^d - q^{r-1})q^{(d-1)(d-r)}.$$

Prove that this equals $(q^d - 1)q^{d(d-1)}$, and hence that $|\mathcal{N}_d| = q^{d(d-1)}$.

Proof. (a) As a $K[T]$-module we have $N \cong K[T]/(T^d)$. So $\mathrm{Hom}_{K[T]}(N, A)$ is in bijection with those $x \in K^d$ such that $A^d(x) = 0$. Since A is nilpotent, we have $A^d = 0$, and hence we can take any $x \in K^d$. Thus $\mathrm{Hom}_{K[T]}(N, A)$ has dimension d.

(b) The fibre over θ consists of those nilpotent matrices A for which $A\theta = \theta N$. Thus conjugation by g yields a bijection between the fibre over θ and the fibre over $g\theta$.

(c) Let θ is a morphism from $K[T]/(T^d)$ to A. Then 1 is sent to the first column v of θ, and T^{i-1} is sent to the i-th column, which must therefore equal $A^{i-1}v$. Thus $\theta = (v, Av, A^2v, \ldots, A^{d-1}v)$. In particular, if the i-th column is zero, so too is every column to the right.

Now assume that $\theta \neq 0$ is in row-reduced form. Then $v \neq 0$, so $v = e_1$. Take r minimal such that $A^r v = 0$. Suppose $A^i v$ is contained in the span of $v, Av, \ldots, A^{i-1}v$. Applying A^{r-1} we see that the coefficient of v is zero. Applying A^{r-2} we get that the coefficient of Av is zero. Continuing we see that all coefficients are zero, and hence $A^i v = 0$.

Thus if $A^i v \neq 0$, then we must have that $A^i v = e_i$. It follows that $\theta = E_r$.

(d) Suppose $AE_r = E_r N$. Now $E_r N = \left(\begin{smallmatrix} J_r(0) & 0 \\ 0 & 0 \end{smallmatrix}\right)$, whereas AE_r is just the first r columns of A. Thus $A = \left(\begin{smallmatrix} J_r(0) & B \\ 0 & A' \end{smallmatrix}\right)$. Then A is nilpotent if and only if A' is nilpotent, so we have a bijection between the fibre over E_r and $\mathbb{M}_{r\times(d-r)}(K) \times \mathcal{N}^{d-r}$.

If $r > 0$, then $d-r < d$, so by induction this fibre has size $q^{r(d-r)}q^{(d-r)(d-r-1)} = q^{(d-r)(d-1)}$.

(e) A matrix θ has row reduced form E_r if and only if the first r columns are linearly independent, and the remaining columns are zero. Thus the number of such θ is $(q^d-1)(q^d-q)\cdots(q^d-q^{r-1})$.

(f) The fibre over the zero matrix is of course $\mathcal{N}_d$. This gives the formula for $|\mathcal{S}_d| - |\mathcal{N}_d|$. We rewrite this as

$$(q^d-1)\sum_{r>0}(q^d-q)\cdots(q^d-q^{r-1})q^{(d-1)(d-r)}.$$

Consider the set $\mathbb{M}_{d\times(d-1)}(K)$. We partition these matrices as follows. Given $r > 0$, take those matrices whose i-th column is not in the span of $e_1, \ldots, e_i$ for $i < r$, the r-th column is in the span of $e_1, \ldots, e_r$, and the remaining columns are arbitrary. There are $(q^d-q)\cdots(q^d-q^{r-1})q^{(d-1)(d-r)}$ such matrices. Also, every matrix lies in precisely one of these subsets: we take r minimal such that the r-th column lies in the span of $e_1, \ldots, e_r$. This shows that

$$\sum_{r>0}(q^d-q)\cdots(q^d-q^{r-1})q^{(d-1)(d-r)} = q^{d(d-1)}.$$

Now combine with the formula $|\mathcal{S}_d| = q^d|\mathcal{N}_d|$ to deduce $|\mathcal{N}_d| = q^{d(d-1)}$. $\square$

4. Fix a finite dimensional A-module X. We want to show that for each i, the map $Y \mapsto \dim \operatorname{Ext}^i(X,Y)$ is upper semi-continuous on $\operatorname{Mod}(A,d)$. We fix a free resolution of X,

$$\cdots \to A^{r_2} \xrightarrow{f_2} A^{r_1} \xrightarrow{f_1} A^{r_0} \to X \to 0.$$

We also set

$$C(U,V,W) := \{(\theta,\phi) : \phi\theta = 0\} \subset \operatorname{Hom}(U,V) \times \operatorname{Hom}(V,W).$$

In the lectures we saw that the function

$$C(U,V,W) \to \mathbb{Z}, \quad (\theta,\phi) \mapsto \dim(\operatorname{Ker}(\phi)/\operatorname{Im}(\theta))$$

is upper semi-continuous.

(a) We fix a surjective algebra homomorphism $K\langle p_1,\dots,p_k\rangle \twoheadrightarrow A$. Then $\operatorname{Mod}(A,d) \subset \mathbb{M}_d(K)^k$. Given $a \in A$, lift it to a non-commutative polynomial $\alpha \in K\langle p_1,\dots,p_k\rangle$. Right multiplication by a gives a map $\rho_a \colon A \to A$.

Now let $(y_1,\dots,y_k) \in \operatorname{Mod}(A,d)$, corresponding to a d-dimensional A-module Y. Show that, under the standard identification $\operatorname{Hom}(A,Y) \cong Y$, the induced homomorphism $\rho_a^* \colon \operatorname{Hom}(A,Y) \to \operatorname{Hom}(A,Y)$ corresponds to the linear map $\alpha(y_1,\dots,y_k) \colon K^d \to K^d$.

(b) A homomorphism $f \colon A^s \to A^r$ corresponds to a matrix $(f_{ij}) \in \mathbb{M}_{r\times s}(A)$. We lift each f_{ij} to a non-commutative polynomial $f_{ij} \in K\langle p_1,\dots,p_k\rangle$. Show that the induced homomorphism $f_Y^* \colon \operatorname{Hom}(A^r,Y) \to \operatorname{Hom}(A^s,Y)$ corresponds to the block matrix $(f_{ij}(y_1,\dots,y_k)) \colon (K^d)^r \to (K^d)^s$.

(c) Deduce that for each i there is a morphism of varieties

$$\operatorname{Mod}(A,d) \to C(K^{dr_{i-1}}, K^{dr_i}, K^{dr_{i+1}}), \quad Y \mapsto ((f_i)_Y^*, (f_{i+1})_Y^*).$$

Conclude that the function $Y \mapsto \dim \operatorname{Ext}^i(X,Y)$ is upper semi-continuous on $\operatorname{Mod}(A,d)$.

Proof. (a) Under the usual identification $\operatorname{Hom}(A,Y) \cong Y$, the map ρ_a^* corresponds to left multiplication by a on Y. Next we have algebra generators $a_1,\dots,a_k$, given by the images of $p_1,\dots,p_k$. Thus $a = \alpha(a_1,\dots,a_k)$. Finally, a_i acts on Y as y_i, so a acts on Y as $\alpha(y_1,\dots,y_k)$.

(b) As in (a), the map f_Y^* corresponds to left multiplication on Y^r by the matrix f, and so by the block matrix $(f_{ij}(y_1,\dots,y_k))$.

(c) Choosing representatives for the matrices f_i, f_{i+1}, we obtain the morphism

$$\operatorname{Mod}(A,d) \to C(\operatorname{Hom}(A^{r_i-1},Y), \operatorname{Hom}(A^{r_i},Y), \operatorname{Hom}(A^{r_i+1},Y)).$$

The homology of this is $\operatorname{Ext}^i(X,Y)$, and so the function $Y \to \dim \operatorname{Ext}^i(X,Y)$ is upper semi-continuous. □