Non-commutative Algebra 3, SS 2020

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Solutions 8

Throughout K will be an algebraically closed field. Recall that we have three partial orders on $\mathrm{Mod}(A,d)$.

Ext $M \leq_{\mathrm{ext}} N$ if there exists a sequence $M = M_0, M_1, \ldots, M_n = N$ and short exact sequences

$$0 \to L_i' \to M_{i-1} \to L_i'' \to 0, \quad M_i \cong L_i' \oplus L_i''.$$

Deg $M \leq_{\mathrm{deg}} N$ if $N \in \overline{\mathcal{O}}_M$.

Hom $M \leq_{\mathrm{hom}} N$ if $\dim \mathrm{Hom}(X,M) \leq \dim \mathrm{Hom}(X,N)$ for all finite dimensional A-modules X.

We saw in the lectures that $M \leq_{\mathrm{ext}} N$ implies $M \leq_{\mathrm{deg}} N$ implies $M \leq_{\mathrm{hom}} N$.

1. Let A be a (finite dimensional) K-algebra, and $M, N \in \mathrm{Mod}(A,d)$.

(a) Show that for finite dimensional A-modules M, N we have $M \cong N$ if and only if $\dim \mathrm{Hom}(X,M) = \dim \mathrm{Hom}(X,N)$ for all finite dimensional A-modules X.

Hint. We may proceed as follows. Take a basis $f_1, \ldots, f_d$ for $\mathrm{Hom}(M,N)$. Use these to construct a morphism $f\colon M \to N^d$, and show that the induced map $\mathrm{Hom}(N^d,N) \to \mathrm{Hom}(M,N)$ is onto. Deduce that the map $\mathrm{Hom}(N^d,M) \to \mathrm{End}(M)$ is onto, and hence that f is a split monomorphism. By Krull-Remak-Schmidt, M and N have a common direct summand. Finish by induction on $\dim M$.

(b) Given a minimal projective presentation $Q \to P \to X \to 0$, show that

$$\dim \mathrm{Hom}(X,M) - \dim \mathrm{Hom}(M,\tau X) = \dim \mathrm{Hom}(P,M) - \dim \mathrm{Hom}(Q,M).$$

Deduce that $M \leq_{\mathrm{hom}} N$ if and only if $\dim \mathrm{Hom}(M,X) \leq \dim \mathrm{Hom}(N,X)$ for all finite dimensional A-modules X.

Hint. We have the (minimal) projective presentation $P^\vee \to Q^\vee \to \mathrm{Tr}X \to 0$ as right A-modules, where $P^\vee := \mathrm{Hom}(P,A)$. Now tensor with M to obtain an exact sequence

$$0 \to \mathrm{Hom}(X,M) \to \mathrm{Hom}(P,M) \to \mathrm{Hom}(Q,M) \to D\,\mathrm{Hom}(M,\tau X) \to 0.$$

Proof. (a) Given $h\colon M \to N$ we can write $h = \sum_i \lambda_i f_i$ for some $\lambda_i \in K$. Then the map $\mathrm{Hom}(N^d,N) \to \mathrm{Hom}(M,N)$ sends $(\lambda_i \cdot \mathrm{id}_N)$ to h, proving that f^* is onto.

We have an exact sequence $M \xrightarrow{f} N^d \to C \to 0$. Applying $\mathrm{Hom}(-, N)$ yields the short exact sequence

$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(N^d, N) \to \mathrm{Hom}(M, N) \to 0.$$

Applying $\mathrm{Hom}(-, M)$ yields the exact sequence

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(N^d, M) \to \mathrm{End}(M).$$

Comparing dimensions we see that $\mathrm{Hom}(N^d, M) \to \mathrm{End}(M)$ is onto, and hence that f is a split mono. By KRS M and N have a common direct summand T. Writing $M = M' \oplus T$ and $N = N' \oplus T$, we see that $\dim \mathrm{Hom}(X, M') = \dim \mathrm{Hom}(X, N')$ for all X. By induction on $\dim M$ we deduce that $M' \cong N'$, and hence that $M \cong N$.

(b) Since P is a finitely generated projective we know that

$$P^\vee \otimes M = \mathrm{Hom}(P, A) \otimes_A M \cong \mathrm{Hom}(P, M).$$

Also,

$$\mathrm{Hom}(M, DY) \cong \mathrm{Hom}(Y \otimes M, K) = D(Y \otimes M),$$

so taking duals again we get $\mathrm{Tr}X \otimes M \cong D\,\mathrm{Hom}(M, \tau X)$. Since the kernel of $\mathrm{Hom}(P, M) \to \mathrm{Hom}(Q, M)$ is $\mathrm{Hom}(X, M)$, we get the required four term exact sequence. Rearranging thus gives

$$\dim \mathrm{Hom}(X, M) - \dim \mathrm{Hom}(M, \tau X) = \dim \mathrm{Hom}(P, M) - \dim \mathrm{Hom}(Q, M).$$

If $M \leq_{\mathrm{hom}} N$, then $\dim \mathrm{Hom}(Ae, M) \leq \dim \mathrm{Hom}(Ae, N)$, so $\dim eM \leq \dim eN$, for all idempotents $e \in A$. Since $\dim M = \dim N$, we deduce that $\dim eM = \dim eN$ for all idempotents e. Hence $\dim \mathrm{Hom}(P, M) = \dim \mathrm{Hom}(P, N)$ for all finitely generated projectives P. Dually, $\dim \mathrm{Hom}(M, I) \leq \dim \mathrm{Hom}(N, I)$ for all finitely generated injectives I.

From the first part, $\dim \mathrm{Hom}(M, \tau X) \leq \dim \mathrm{Hom}(N, \tau X)$ for all X. Since every finite dimensional module is of the form $\tau X \oplus I$ for some X and some injective I, it follows that $\dim \mathrm{Hom}(M, Y) \leq \dim \mathrm{Hom}(N, Y)$ for all Y.

The other implication is analogous. $\square$

2. Consider the path algebra KQ, where Q is the quiver $1 \longrightarrow 3 \longleftarrow 2$

(a) Show that this has Auslander-Reiten quiver

$$\begin{tikzcd}
 & P_1 \arrow[dr] & & S_2 \\
S_3 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & \\
 & P_2 \arrow[ur] & & S_1
\end{tikzcd}$$

(b) Consider a finite dimensional module M. By the Krull-Remak-Schmidt Theorem, we can write

$$M \cong S_3^a \oplus P_1^b \oplus P_2^c \oplus I_3^d \oplus S_2^e \oplus S_1^f,$$

and so we can use the shorthand $M \leftrightarrow (a, b, c, d, e, f)$.

Compute the dimension vector $\underline{\dim}\, M$.

Compute $\dim \operatorname{Hom}(X, M)$ as X runs through all six indecomposable modules.

(c) Suppose $\underline{\dim}\, M = \underline{\dim}\, N$ with $N \leftrightarrow (a', b', c', d', e', f')$. Write out the conditions that $M \leq_{\text{hom}} N$, that is, $\dim \operatorname{Hom}(X, M) \leq \dim \operatorname{Hom}(X, N)$ for all six indecomposable modules X.

(d) Using the three Auslander-Reiten sequences

$$0 \to S_3 \to P_1 \oplus P_2 \to I_3 \to 0$$

and

$$0 \to P_1 \to I_3 \to S_2 \to 0 \quad 0 \to P_2 \to I_3 \to S_1 \to 0$$

show that $M \leq_{\text{hom}} N$ implies $M \leq_{\text{ext}} N$.

Proof. (a) Simple knitting.

(b) We have $\dim M = (b + d + f, a + b + c + d, c + d + e)$.

$\dim \operatorname{Hom}(S_3, M) = a+b+c+d$, $\dim \operatorname{Hom}(P_1, M) = b+d+f$, $\dim \operatorname{Hom}(P_2, M) = c + d + e$.

$\dim \operatorname{Hom}(I_3, M) = d + e + f$, $\dim \operatorname{Hom}(S_2, M) = e$, $\dim \operatorname{Hom}(S_1, M) = f$.

(c) Given that they have the same dimension vector, the conditions become $e \leq e'$, $f \leq f'$ and $d + e + f \leq d' + e' + f'$.

(d) Set $\delta := (e' - e) + (f' - f) + ((d' + e' + f') - (d + e + f)) = (d' - d) + 2(e' - e) + 2(f' - f) \geq 0$.

Suppose $d > d'$. Then $(e' - e) + (f' - f) > 0$. If $f' > f$, then we can use $0 \to P_2 \to I_3 \to S_1 \to 0$ to get $M \leq_{\text{ext}} M'$ with $M' \leftrightarrow (a, b, c+1, d-1, e, f+1)$.

Now $M' \leq_{\mathrm{hom}} N$, and by induction on δ we know that $M' \leq_{\mathrm{ext}} N$. Similarly if $e' > e$.

Suppose instead that $d < d'$. Then $b+d+f = b'+d'+f'$, so $b-b' = (d'-d)+(f'-f) > 0$, and similarly $c-c' > 0$. We can then use $0 \to S_3 \to P_1 \oplus P_2 \to I_3 \to 0$ to get $M \leq_{\mathrm{ext}} M'$ with $M' \leftrightarrow (a+1, b-1, c-1, d+1, e, f)$. Now $M' \leq_{\mathrm{hom}} N$, and again by induction on δ we get $M' \leq_{\mathrm{ext}} N$.

Finally, suppose $d = d'$. Then $b-b' = f'-f$. If this is positive, then we can use $0 \to S_3 \to P_1 \to S_1 \to 0$ to get $M \leq_{\mathrm{ext}} M'$ with $M' \leftrightarrow (a+1, b-1, c, d, e, f+1)$. Again, $M' \leq_{\mathrm{hom}} N$, so by induction $M' \leq_{\mathrm{hom}} N$. Similarly for $c - c' = e' - e$.

This proves that $M \leq_{\mathrm{hom}} N$ implies $M \leq_{\mathrm{ext}} N$.

Note also that replacing $P_1 \rightsquigarrow S_3 \oplus S_1$ can be regarded as the composition $P_1 \oplus P_3 \rightsquigarrow S_3 \oplus I_3$ with $I_3 \rightsquigarrow P_2 \oplus S_1$. In this sense, the Auslander-Reiten sequences determine the partial order $\leq_{\mathrm{hom}}$. $\square$

3. Consider the algebra $A = KQ/I$ given by the quiver

$$\begin{tikzcd} 2 \arrow[r, "b"] & 1 \arrow[loop right, "a"] \end{tikzcd}$$

and I is the ideal generated by a^2 (c.f. Exercise 6.3 last semester). We know that this algebra is representation finite, having precisely seven indecomposables up to isomorphism. Moreover, there are two non-isomorphic indecomposables of dimension vector $2e_1 + e_2$, namely

$$X\colon \begin{tikzcd} K \arrow[r, "\binom{1}{0}"] & K^2 \arrow[loop right, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] \end{tikzcd} \qquad \text{and} \qquad Y\colon \begin{tikzcd} K \arrow[r, "\binom{0}{1}"] & K^2 \arrow[loop right, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] \end{tikzcd}$$

Compute $\mathrm{End}(X)$ and $\mathrm{End}(Y)$. This again shows that $X \ncong Y$.
Set

$$X_t\colon \begin{tikzcd} K \arrow[r, "\binom{t}{1}"] & K^2 \arrow[loop right, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] \end{tikzcd}$$

Show that $X_t \cong X$ for all $t \neq 0$.
This proves that $X \leq_{\mathrm{deg}} Y$. Explain why $X \not\leq_{\mathrm{ext}} Y$.

Proof. $\mathrm{End}(X) \cong K$.

$\mathrm{End}(Y) \cong K[t]/(t^2)$, consisting of pairs of matrices $\left(p, \left(\begin{smallmatrix} p & 0 \\ q & p \end{smallmatrix}\right)\right)$.

The pair of matrices $\left(1, \left(\begin{smallmatrix} t & 0 \\ 1 & t \end{smallmatrix}\right)\right)$ determines a morphism $X \to X_t$. This is an isomorphism for all $t \neq 0$.

Since $X_0 \cong Y$, it follows that $X \leq_{\mathrm{deg}} Y$.

Since Y is indecomposable and not isomorphic to X, we cannot have $X \leq_{\mathrm{ext}} Y$. $\square$

4. We introduce a new partial order by saying $M \leq_{\text{v.ext}} N$ (virtual extension) provided $M \oplus X \leq_{\text{ext}} N \oplus X$ for some finite dimensional module X.

(a) Show that $M \leq_{\text{v.ext}} N$ implies $M \leq_{\text{hom}} N$.

As in Exercise 7.1, let $A = K[x,y]/(x^2,y^2)$, and let M_t be the two-dimensional module where the x and y actions are given by

$$M_t(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_t(y) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

We have shown that $A \not\leq_{\text{deg}} M_s \oplus M_t$ for some s, t.

(b) Show that we have a short exact sequence (in fact an Auslander-Reiten sequence)

$$0 \to \operatorname{rad} A \to A \oplus \operatorname{rad} A/\operatorname{soc} A \to A/\operatorname{soc} A \to 0.$$

(c) Show that for all s, t we have short exact sequences

$$0 \to M_s \to \operatorname{rad} A \to K \to 0 \quad \text{and} \quad 0 \to K \to A/\operatorname{soc} A \to M_t \to 0.$$

(d) Using that $\operatorname{rad} A/\operatorname{soc} A \cong K^2$ deduce that

$$A \oplus K^2 \leq_{\text{ext}} M_s \oplus M_t \oplus K^2 \quad \text{for all } s, t \in K,$$

and hence that $A \leq_{\text{v.ext}} M_s \oplus M_t$.

Proof. (a) We know that $\leq_{\text{ext}}$ implies $\leq_{\text{hom}}$ and that $\leq_{\text{hom}}$ has cancellation, so $M \oplus X \leq_{\text{hom}} N \oplus X$ implies $M \leq_{\text{hom}} N$.

(b) A is four dimensional and local, with basis $1, x, y, xy$. It has radical $Ax + Ay$, and socle Axy.

We have the natural map $A \oplus \operatorname{rad} A/\operatorname{soc} A \to A/\operatorname{soc} A$, $(a, \bar{b}) \mapsto \bar{a} - \bar{b}$. This is clearly surjective. Its kernel is $(a, \bar{a})$ for $a \in \operatorname{rad} A$, so is isomorphic to $\operatorname{rad} A$.

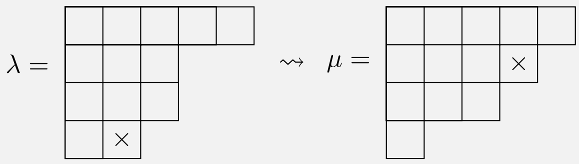
(c) The elements $sx + y, xy$ span a submodule of $\operatorname{rad} A$ isomorphic to M_s. Similarly, the submodule of $A/\operatorname{soc} A$ spanned by $t\bar{x} - \bar{y}$ has cokernel isomorphic to M_t.

(d) We have $A \oplus K^2 \leq_{\text{ext}} \operatorname{rad} A \oplus \operatorname{soc} A \leq_{\text{ext}} (M_s \oplus K) \oplus (M_t \oplus K)$. Thus $A \leq_{\text{v.ext}} M_s \oplus M_t$ for all $s, t \in K$. $\square$

5. The dominance order on partitions of n is given by $\lambda \lhd \mu$ provided $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i.

(a) Show that $\lambda \lhd \mu$ if and only if $\mu' \lhd \lambda'$.

Hint. Suppose that μ is obtained from λ by moving a 'bottom right corner' block to the next available space to the upper left. For example



Show that, taking dual partitions, the inverse move is of the same type, but now from μ' to λ'. As in the lectures, the covers in the dominance order are all of this type.

Let λ and μ be two partitions. Their sum $\lambda + \mu$ is the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$; their cup product $\lambda \cup \mu$ is the partition given by rearranging the parts of λ and μ into decreasing order.

(b) Show that $\lambda' + \mu' = (\lambda \cup \mu)'$.

Recall that for each partition λ we have a nilpotent $K[t]$-module $M(\lambda)$, having Jordan blocks of sizes λ_i'. In particular, $M(d)$ is semisimple, and $M(1^d)$ is indecomposable.

(c) Show that $M(\lambda) \oplus M(\mu) \cong M(\lambda + \mu)$.

(d) Let $U \leq M(\lambda)$ be a submodule such that $\dim \operatorname{soc} U \leq r$. Show that $\dim U \leq \lambda_1' + \cdots + \lambda_r'$.

Hint. Show that $\dim \operatorname{soc}(U/\operatorname{soc} U) \leq r$. Show that $M(\lambda)/\operatorname{soc} M(\lambda) \cong M(\lambda_{\geq 2})$, where $\lambda_{\geq 2} = (\lambda_2, \lambda_3, \ldots)$. Now use $U/\operatorname{soc} U \leq M(\lambda_{\geq 2})$ and induction.

(e) Deduce that $\lambda_1' + \cdots + \lambda_r'$ is the maximum dimension of a submodule $U \leq M(\lambda)$ having $\dim \operatorname{soc} U \leq r$.

(f) Suppose now that we have a short exact sequence

$$0 \to M(\lambda) \to M(\xi) \to M(\mu) \to 0.$$

Use the previous part to deduce that $\xi' \lhd \lambda' + \mu'$, and hence that $\lambda \cup \mu \lhd \xi$.

Proof. (a) This is clear. (The hard part is showing that the dominance order is generated by such moves.)

(b) The number λ_i' is the length of the i-th column of λ. The cup product $\lambda \cup \mu$ is formed by placing λ on top of μ, and then rearranging rows so that we get a partition. It is thus clear that $(\lambda \cup \mu)_i' = \lambda_i' + \mu_i'$.

(c) $M(\lambda)\oplus M(\mu)$ has Jordan blocks given by the entries of $\lambda'\cup\mu'=(\lambda+\mu)'$, and hence is isomorphic to $M(\lambda+\mu)$.

(d) We know that $K[T]$-modules are uniserial, so $\dim\operatorname{soc}U$ is precisely the number of indecomposable summands, and hence $\dim\operatorname{soc}(U/\operatorname{soc}U)\leq\dim\operatorname{soc}U$.

$\dim\operatorname{soc}M(\lambda)$ is the number of Jordan blocks, which is λ_1. In the quotient $M(\lambda/\operatorname{soc}M(\lambda)$, the size of each Jordan block decreases by one. Thus this quotient is isomorphic to $M(\lambda_{\geq 2})$.

$\operatorname{soc}U=U\cap\operatorname{soc}M(\lambda)$, so $U/\operatorname{soc}U\leq M(\lambda_{\geq 2})$. By induction on λ we have $\dim U/\operatorname{soc}U\leq(\lambda_1'-1)+\cdots+(\lambda_r'-1)$, so $\dim U\leq r+\dim U/\operatorname{soc}U\leq\lambda_1'+\cdots+\lambda_r'$.

(e) There is a submodule $U\leq M(\lambda)$ given by the largest r Jordan blocks. This has socle of dimension r, and total dimension $\lambda_1'+\cdots+\lambda_r'$.

(f) Take $U\leq M(\xi)$ given by the largest r Jordan blocks. Then $U':=U\cap M(\lambda)$ has socle of dimension at most r, and the image U'' of U in $M(\mu)$ also has at most r summands, so its socle is again of dimension at most r since it is uniserial. Now $\dim U=\dim U'+\dim U''$, so by the previous part we have

$$\xi_1'+\cdots+\xi_r'\leq(\lambda_1'+\cdots+\lambda_r')+(\mu_1'+\cdots+\mu_r').$$

This holds for all r, so $\xi'\lhd\lambda'+\mu'$. Taking duals gives $\lambda\cup\mu\lhd\xi$. $\square$