Non-commutative Algebra 3, SS 2020

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Solutions 9

Throughout K will be an algebraically closed field.

1.

(a) Consider the family of algebras $A_t := K[x]/(x^2 - tx)$ for $t \in K$. Compute $\operatorname{gl.dim} A_t$.

(b) Consider the family of algebras $B_t := K[x,y]/(x^2, y^2 - ty)$ for $t \in K$. For which t is B_t representation finite?

Proof. (a) For $t \neq 0$ we have $A_t \cong K[x]/(x) \times K[x]/(x-t) \cong K \times K$ by the Chinese Remainder Theorem. This is semisimple, so has global dimension 0.

For $t = 0$ we have $A_0 \cong K[x]/(x^2)$, which is self-injective, so has infinite global dimension. In fact, there are two indecomposables: A_0 itself, which is projective-injective, and the simple K. Using the short exact sequence $0 \to K \to A_0 \to K \to 0$, we see that $\operatorname{p.dim} K = \infty$.

(b) For $t \neq 0$ we have $B_t \cong \big(K[y]/(y(y-t))\big)[x]/(x^2) \cong (K \times K)[x]/(x^2) \cong K[x]/(x^2) \times K[x]/(x^2)$. This has four indecomposables.

For $t = 0$ we have $B_0 \cong K[x,y]/(x^2, y^2)$. We saw in Exercise 7.1 that this has indecomposables M_s for all $s \in K$, so is representation infinite. $\square$

2. Let G be a connected algebraic group acting on a variety X. For a G-stable constructible subset $Y \subset X$, set

$$Y_{(d)} := \{y \in Y : \dim Gy = d\} \quad \text{and} \quad Y_{(\leq d)} := \{y \in Y : \dim Gy \leq d\}.$$

We define

$$\dim_G Y := \max\{\dim Y_{(d)} - d : d \geq 0\} \text{ and } \operatorname{top}_G Y := \sum_{\dim Y_{(d)} = \dim_G Y + d} \operatorname{top} Y_{(d)}.$$

Let $Y, Y_i \subset X$ be G-stable constructible subsets.

(a) Show that $\dim_G(Y_1 \cup Y_2) = \max\{\dim_G Y_1, \dim_G Y_2\}$.

(b) Show that $\dim_G Y = \max\{\dim Y_{(\leq d)} - d : d \geq 0\}$.

(c) Suppose $Z \subset Y$ is constructible and meets every orbit in Y. Show that $\dim_G Y \leq \dim Z$.

(d) Show that $\dim_G Y = 0$ if and only if Y is a finite union of G-orbits, in which case $\operatorname{top}_G Y$ equals the number of orbits.

(e) Define $Z := \{(g, x) : gx = x\} \subset G \times X\}$, and let $\pi\colon Z \to X$ be the projection. Show that $\dim_G Y = \dim \pi^{-1}(Y) - \dim G$. Show further that if $\operatorname{Stab}_G(y)$ is connected for all $y \in Y$, then $\operatorname{top}_G Y = \operatorname{top} Z$.

Proof. (a) We have $(Y \cup Z)_{(d)} = Y_{(d)} \cup Z_{(d)}$ and $\dim(Y \cup Z) = \max\{\dim Y, \dim Z\}$. The result follows.

(b) We have $\dim Y_{(d)} \leq \dim Y_{(\leq d)}$, so that $\dim_G Y \leq \max\{\dim Y_{(\leq d)} - d\}$.

On the other hand, $\dim Y_{(\leq d)} - d = \max\{\dim Y_{(s)} - d : s \leq d\} \leq \max\{\dim Y_{(s)} - s : s \leq d\} \leq \dim_G Y$.

(c) The map $G \times Z_{(d)} \to Y_{(d)}$ is onto. The fibre over $y = gz$ is $\{(h, z') : hz' = gz\} \cong \{h \in G : hz \in Z\}$. The isomorphism sends h with $hz \in Z$ to the pair (gh^{-1}, hz), and sends the pair (h, z') to $h^{-1}g$. Since $z \in Z_{(d)}$ we know that $\{h : hz = z\}$ has dimension $\dim G - d$. Thus each fibre has dimension at least $\dim G - d$, so by the Main Lemma on fibre dimension, $\dim G + \dim Z_{(d)} - \dim Y_{(d)} \geq \dim G - d$, and hence $\dim Y_{(d)} - d \leq \dim Z_{(d)}$. This holds for all d, so $\dim_G Y \leq \dim Z$.

(d) Suppose $Y = Gy_1 \cup \cdots \cup Gy_n$ is a finite union of orbits. We have $\dim_G Gy = 0$ for each orbit, so by (a) also $\dim_G Y = 0$. Moreover, $Y_{(d)}$ is the union of those orbits of dimension d. Since each orbit is irreducible, these must be the irreducible components of $Y_{(d)}$, so $\operatorname{top} Y_{(d)}$ equals the number of orbits of dimension d, and hence $\operatorname{top}_G Y = n$ is the total number of orbits.

Conversely, suppose $\dim_G Y = 0$. Then $\dim Y_{(d)} = d$ for all d such that $Y_{(d)} \neq \emptyset$. Now each orbit in $Y_{(d)}$ is irreducible of dimension d, so its closure must be an irreducible component of $\bar{Y}_{(d)}$, of which there are only finitely many. Thus $Y_{(d)}$ contains only finitely many orbits.

(e) The fibre over each point $y \in Y_{(d)}$ is isomorphic to $\operatorname{Stab}_G(y)$, so has dimension $\dim G - d$. By the Main Lemma on fibre dimension, $\dim \pi^{-1}(Y_{(d)}) =$

$\dim Y_{(d)} + \dim G - d$, so $\dim_G Y_{(d)} = \dim \pi^{-1}(Y_{(d)}) - \dim G$. Taking the union over all d we get $\dim_G Y = \dim \pi^{-1}(Y) - \dim G$.

We now recall from Exercise 5.2 that if G acts on a variety, then every constructible subset can be written as a finite disjoint union of G-stable irreducible and locally closed subsets. This obviously applies to each $Y_{(d)}$, but also to $\pi^{-1}(Y)$, using the G-action $h \cdot (g, x) = (hgh^{-1}, hx)$.

Assume further that each stabiliser is connected, so each fibre of π is irreducible. Then, since the map π admits a section $X \to Z$, $x \mapsto (1, x)$, we know from Exercise 5.3 that the preimage of a (G-stable) irreducible subset of $Y_{(d)}$ is again (G-stable) irreducible.

Write $Y_{(d)} = Y^1 \cup \cdots \cup Y^n$ as a finite disjoint union of G-stable irreducible locally closeds. Then $\pi^{-1}(Y_{(d)}) = Z^1 \cup \cdots \cup Z^n$ is a disjoint union, where $Z^i = \pi^{-1}(Y^i)$ is G-stable irreducible locally closed. Moreover, $\dim Z^i = \dim_G Y^i + \dim G$. Now $\operatorname{top} Y_{(d)}$ is the number of Y^i of maximal dimension, and similarly $\top \pi^{-1}(Y_{(d)})$ is the number of $\dim Z^i$ of maximal dimension. We thus see that $\operatorname{top}_G Y_{(d)} = \operatorname{top} \pi^{-1}(Y_{(d)})$.

Since

$$\operatorname{top}_G Y = \sum_{\dim_G Y_{(d)} = \dim_G Y + d} \operatorname{top} Y_{(d)}$$

and

$$\operatorname{top} \pi^{-1}(Y) = \sum_{\dim \pi^{-1}(Y_{(d)}) = \dim \pi^{-1}(Y)} \operatorname{top} \pi^{-1}(Y_{(d)}),$$

the result follows. $\square$

3. Consider the path algebra KQ of the Kronecker quiver. The indecomposable modules of dimension vector smaller than $\alpha = (1, 2)$ are, up to isomorphism, given by the following list

$$S_1 \colon K \rightrightarrows 0 \qquad S_2 \colon 0 \rightrightarrows K \qquad P_1 \colon K \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} K^2$$

together with

$$R_a \colon K \underset{a}{\overset{1}{\rightrightarrows}} K \quad \text{for } a \in K, \qquad R_\infty \colon K \underset{1}{\overset{0}{\rightrightarrows}} K$$

Set $X := \operatorname{Mod}(KQ, \alpha)$. Describe $X_{(d)}$ for all d, giving representatives for the orbits it contains and computing its dimension. Hence compute the number of parameters $\dim_{\mathrm{GL}(\alpha)} X$.

Proof. The orbits in X are represented by the modules P_1, $S_2 \oplus R_\lambda$ for $\lambda \in K \cup \{\infty\}$, and $S_1 \oplus S_2^2$.

The dimensions of the endomorphism rings are, respectively, 1, 3, 5. Since the dimension of the group $\mathrm{GL}_1(K) \times \mathrm{GL}_2(K)$ is 5, the dimensions of the orbits are, respectively, 4, 2, 0.

Thus $X_{(4)} = \mathcal{O}_{P_1}$, $X_{(2)} = \bigcup_\lambda \mathcal{O}_{S_2\oplus R_\lambda}$, $X_{(0)} = \mathcal{O}_{S_1\oplus S_2^2}$.

The first and last are just single orbits, so have respective dimensions 4, 0.

Now the representation given by the pair $\left(\binom{a}{c}, \binom{b}{d}\right)$ is isomorphic to some $S_2 \oplus R_\lambda$ if and only if the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has rank one. Thus $\bar{X}_{(2)} = V(ad - bc)$, so is irreducible of dimension 3 (as in Section 3.3 (F) of the lecture notes).

The numbers $\dim X_{(d)} - d$ are thus, respectively, 0, 1, 0. Hence $\dim_G X = 1$ and $\mathrm{top}_G X = 1$. □

4. Set $A := K[x, y]/(x, y)^2$ and KQ the path algebra of the Kronecker quiver. Consider the functor $F\colon A\text{-mod} \to KQ\text{-mod}$ defined as follows. Given an A-module M, we can regard this as a K-vector space M equipped with two endomorphisms x_M and y_M. The Jacobson radical of A is $J = A(x, y) = Kx + Ky$, so $x_M M, y_M M \subset JM$ and $x_M JM = 0 = y_M JM$. We thus have induced maps $\bar{x}_M, \bar{y}_M\colon M/JM \to JM$. The functor F then sends M to the Kronecker representation

$$F(M)\colon M/JM \underset{\bar{y}_M}{\overset{\bar{x}_M}{\rightrightarrows}} JM$$

(a) How does F act on morphisms?

(b) Deduce that F sends indecomposable A-modules to indecomposable KQ-modules. Moreover, $M \cong N$ if and only if $FM \cong FN$.

(c) Show further that if X is an indecomposable KQ-module other than the simple S_2, then $X \cong FM$ for some indecomposable A-module.

(d) Using that KQ is tame, deduce that A is tame.

(e) Is the functor F a representation embedding?

Proof. (a) Let $f\colon M \to N$ be A-linear. Then $f(JM) \subset JN$, and so we have the induced maps $M/JM \to N/JN$ and $JM \to JN$. These commute with $\bar{x}$ and $\bar{y}$, and so we obtain a morphism of Kronecker representations.

Clearly $F(\mathrm{id}) = \mathrm{id}$ and $F(gf) = F(g)F(f)$. Also, $F(f + g) = F(f) + F(g)$, so we have an additive functor.

(b) There is an additive functor G from Kronecker modules to A-modules, sending the Kronecker module $U \underset{B}{\overset{A}{\rightrightarrows}} V$ to the A-module $U \oplus V$, where x acts as $\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right)$ and y acts as $\left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right)$. The action on morphisms is clear.

We claim that $GFM \cong M$. For, take a vector space retract r for $JM \rightarrowtail M$, and consider the linear isomorphism $M \to (M/JM) \oplus (JM)$, $m \mapsto (\bar{m}, rm)$. Note that $x \cdot (\bar{m}, m') = (0, \bar{x}_M \bar{m})$. On the other hand, given $m \in M$ we know that $xm \in JM$, and equals $\bar{x}_M \bar{m}$, so is sent to $(0, \bar{x}_M \bar{m}) = x \cdot (\bar{m}, rm)$. Similarly for y, so this is an isomorphism of A-modules.

(It is not canonical, however, so we do not have $GF \cong \mathrm{id}$, and F and G are not adjoint functors.)

Thus if $FM \cong X \oplus Y$, then $M \cong GFM \cong GX \oplus GY$. This shows that M indecomposable implies FM indecomposable.

Also, if $FM \cong FN$, then $GFM \cong GFN$, so $M \cong N$.

(c) Take an indecomposable Kronecker module $X\colon U \underset{B}{\overset{A}{\rightrightarrows}} V$. If $\mathrm{Im}(A) + \mathrm{Im}(B) \neq V$, then there is a non-zero map $X \to S_2$. This is necessarily a split epimorphism, since S_2 is simple projective, and hence $X \cong S_2$. Thus if $X \not\cong S_2$, then $\mathrm{Im}(A) + \mathrm{Im}(B)$ is surjective. Now $GX = U \oplus V$, and $xU + yU = V$, so $V = JGX$. Hence $FGX \cong X$.

(d) We have a bijection between isomorphism classes of indecomposable A-modules of dimension d and isomorphism classes of indecomposable Kronecker modules of dimension d, for all $d > 1$. (For $d = 1$ there is a unique simple A-module.)

The Kronecker quiver is tame, as can be seen by taking the bimodules

$$X_d\colon K[T]^d \underset{J_n(T)}{\overset{1}{\rightrightarrows}} K[T]^d \ .$$

Thus A is tame, using the bimodules GX_d.

(e) The functor F is not left exact, so cannot be a representation embedding.

To see this, consider the inclusion $JA \to A$. Then $F(JA) \cong S_1^2$ and $F(A) \cong P_1$, but there is no non-zero map $S_1^2 \to P_1$. $\square$