On areas of attraction and repulsion in finite time dynamical systems and their numerical approximation

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ABSTRACT
Stable and unstable fiber bundles with respect to a fixed point or a bounded trajectory are of great dynamical relevance in (non)autonomous dynamical systems. These sets are defined via an infinite limit process. However, the dynamics of several real world models are of interest on a short time interval only. This task requires finite time concepts of attraction and repulsion that have been recently developed in the literature. The main idea consists in replacing the infinite limit process by a monotonicity criterion and in demanding the end points to lie in a small neighborhood of the reference trajectory. Finite time areas of attraction and repulsion defined in this way are fat sets and their dimension equals the dimension of the state space. We propose an algorithm for the numerical approximation of these sets and illustrate its application to several two- and three-dimensional dynamical systems in discrete and continuous time. Intersections of areas of attraction and repulsion are also calculated, resulting in finite time homoclinic orbits.

KEYWORDS
Invariant fiber bundles, finite time dynamical systems, finite time hyperbolicity, areas of attraction and repulsion, contour algorithm, homoclinic orbits

AMS CLASSIFICATION
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1. Introduction
Stable and unstable fiber bundles in autonomous and nonautonomous dynamical systems on a bi-infinite time interval, consist of all points that converge in forward time respectively in backward time towards a reference solution. Due to the dynamical importance of invariant fiber bundles, various concepts have been developed for their approximation, cf. [8, 10, 14, 17, 19, 21, 23, 29, 34, 35, 40, 47, 48]. However, the infinite limit process becomes meaningless, when one is interested in short time dynamics or if the system is defined on a finite time interval. Models from natural sciences are typically relevant on finite time intervals only. In particular transient dynamics cannot be analyzed via infinite limits. Examples of this type are the formation of a storm, eddies in the sea, cf. [28] and the dynamics of asset prices [9]. In several models, transient behavior exhibits rich and even chaotic dynamics, see [41]. On the one hand,
one may consider these models first in infinite time and apply well developed tools for analyzing underlying dynamics. In a second step, finite time results are obtained using approximation techniques. On the other hand, we get a direct approach to transient dynamics by modeling the system on a finite interval. Then appropriate notions and tools have to be developed.

In this paper, we follow the second strategy and consider a nonautonomous dynamical system \((\mathbb{R}^d, \mathbb{T}, \Psi)\), where \(\mathbb{T}\) denotes a finite interval of \(\mathbb{R}\) or \(\mathbb{Z}\). The operator \(\Psi(t, s; x)\) describes the evolution of the point \(x \in \mathbb{R}^d\) from time \(s \in \mathbb{T}\) to time \(t \in \mathbb{T}\). Note that the evolution operator of a nonautonomous dynamical system is alternatively denoted as a two-parameter semi-group or as a process, see [39, Definition 2.1].

The reference objects for which areas of attraction are computed are hyperbolic trajectories. An appropriate concept of finite time hyperbolicity is introduced in detail in Section 2. It is based on exponential dichotomies and has been developed in [4, 6, 16].

In finite time dynamical systems, the notions area of attraction and area of repulsion allows various definitions that are not equivalent. Several authors define these sets via decay conditions, and we refer to [18] for planar nonautonomous ODEs, to [22] for nonautonomous ODEs in \(\mathbb{R}^n\) and to [38] for nonautonomous processes in \(\mathbb{R}^n\). Areas of attraction and repulsion are called stable and unstable manifolds in autonomous systems. Corresponding sets in the nonautonomous case are denoted as invariant stable and unstable fiber bundles, assuming they exhibit respective topological structures.

In autonomous dynamical systems, generated by two-dimensional velocity fields, Haller computed in [25] finite time invariant manifolds via local maxima of the scalar field \(d_T(x_0) = \max_{t \in [t_0, t_0 + T]} \{ t : \det D_{x_0} f(\Psi(\tau, t_0; x_0), \tau) < 0 \text{ for all } t_0 \leq \tau < t \}\). Contour plots of \(d_t\) provide approximations of finite time manifolds. The authors of [24] critically compare various Lagrangian methods for the detection of coherent structures in fluid flow models. Alternative approaches for computing areas of attraction and repulsion are based on numerical continuation of local approximations. This ansatz is applied in [43] to nonautonomous dynamical systems, generated by aperiodic vector fields.

We propose in this paper a related concept of finite time areas of attraction that is motivated by the definition of stable fibers in infinite time w.r.t. a bounded trajectory \(\xi_T = (\xi(t))_{t \in \mathbb{T}}\). First, we choose an \(\varepsilon > 0\) and replace the condition \(\lim_{t \to \infty} \Psi(t, t_0; x) - \xi(t) = 0\) by \(\| \Psi(t_+, t; x) - \xi(t_+) \| \leq \varepsilon\), where \(\mathbb{T} = [t_-, t_+]\). Note that a trajectory may enter and leave the \(\varepsilon\)-neighborhood of \(\xi_T\) several times. From the last entry point in \(\mathbb{T}\) on, we additionally require the trajectory to decay monotonously towards \(\xi_T\) in an appropriately chosen norm. This condition reflects the fact, that in infinite time, the stable fiber of a hyperbolic trajectory consists locally of those points, whose orbits stay for all positive times in a sufficiently small neighborhood of this trajectory, see [50, Theorem III.7] for autonomous and [46, Corollary 4.6.11] for nonautonomous systems.

Areas of attraction, defined in this way, are numerically accessible and invertibility assumptions on the dynamical system are not needed for their calculation. The approach that we pursue in this paper is motivated by recent contour techniques for dynamical systems in infinite time. Corresponding algorithms allow the computation of stable fibers, cf. [34, 35] as well as the detection of stable hierarchies of fiber bundles, see [36]. In contrast to the infinite time case, finite time systems do not require artificial restrictions to finite intervals in order to apply numerical algorithms. The boundaries of finite time areas of attractions turn out to be explicitly given as zero-contours of specific operators.

When dealing with areas of attraction, we do not assume invertibility of the underlying dynamical system. However, invertibility is assumed for the computation of areas
of repulsion, which are areas of attraction for the inverted system. In the discrete time case, this assumption is critical, while for continuous time ODE models, invertibility is guaranteed under reasonable assumptions.

We introduce the resulting algorithm in Section 2.4. Its application to several three-dimensional discrete time linear models is discussed in Section 3. We particularly observe a substantial reduction of the numerical effort for autonomous systems. Illustrations of stable cones are provided, which are the finite time equivalent of stable subspaces.

Applications to nonlinear systems are presented in Section 4. Moreover, we determine the intersection of areas of attraction and areas of repulsion for finite time ODE models. These sets turn out to be accessible via contour calculations. Points in this intersection belong to finite time homoclinic orbits and we discuss the dynamical relevance of homoclinic orbits in finite time dynamical systems.

2. Hyperbolic trajectories and areas of attraction and repulsion

Finite time dynamical systems

$$(\mathbb{R}^d, \mathbb{T}, \Psi) \quad \text{with} \quad \Psi \in C^1(\mathbb{T} \times \mathbb{T} \times \mathbb{R}^d, \mathbb{R}^d)$$  \hspace{1cm} (1)

are generated, for example, by nonautonomous ODEs on a continuous time interval $\mathbb{T} = [t_-, t_+]$ and by difference equations on a discrete time interval $\mathbb{T} = [t_-, t_+ - 1] \cap \mathbb{Z}$, $\mathbb{T} := [t_-, t_+ - 1] \cap \mathbb{Z}$ with $t_\pm \in \mathbb{Z}$:

$$x'(t) = f(t, x(t)), \quad t \in \mathbb{T} \quad \text{respectively} \quad x(t + 1) = f(t, x(t)), \quad t \in \mathbb{T}. \quad (2)$$

$\Psi$ is the evolution operator of these systems, i.e. if $x(s) = x_0$ then $x(t) = \Psi(t, s; x_0)$ for $t, s \in \mathbb{T}$, $t \geq s$.

For $f \in C^{0,1}(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ a solution of the initial value problem

$$x'(t) = f(t, x(t)), \quad x(s) = x_0$$

locally exists and is unique. Thus, $\Psi$ is invertible. Invertibility is generally not true, if $\Psi$ is generated by a difference equation.

In the following, open and closed intervals in $\mathbb{T}$ are denoted by $(t, s)_\mathbb{T} = (t, s) \cap \mathbb{T}$ and $[t, s]_\mathbb{T} = [t, s] \cap \mathbb{T}$, respectively.

Let $\xi(t) = \Psi(t, t_-; \xi(t_-))$ for $t \in \mathbb{T}$ be a reference solution. We establish a notion of hyperbolicity for the trajectory $\xi_\mathbb{T} = (\xi(t))_{t \in \mathbb{T}}$ and consider for this task the dynamical system

$$(\mathbb{R}^d, \mathbb{T}, \Phi), \quad \text{where} \quad \Phi(t, s) = D_x \Psi(t, s; \xi(s)), \quad t, s \in \mathbb{T}, \ t \geq s. \quad (3)$$

If $\Psi$ is generated by (2), then $\Phi$ is the solution operator of the corresponding variational equation

$$u'(t) = D_x f(t, \xi(t)) u(t), \quad t \in \mathbb{T} \quad \text{respectively}$$

$$u(t + 1) = D_x f(t, \xi(t)) u(t), \quad t \in \mathbb{T}. \quad (4)$$
2.1. Finite time hyperbolicity

The notion of hyperbolicity considered here is based on exponential dichotomies that have been developed in [12, 13, 45]. For noninvertible systems in infinite time, we refer to [1, 30, 37]. The definition of a finite time exponential dichotomy considered here is an extension of [4, 6, 16, 38] to noninvertible systems. In the following, \( \mathcal{R}(P) \) denotes the range of the projector \( P \).

**Definition 2.1.** The dynamical system (3) has an exponential dichotomy on \( \mathbb{T} \) w.r.t. \( \| \cdot \| \), if there exist constants \( \alpha_s, \alpha_u > 0 \) and families of projectors \( P^s_t, P^u_t := I - P^s_t, t \in \mathbb{T} \), such that

(i) \( P^s_t \Phi(t, s) P^s_s = \Phi(t, s) \) for all \( t, s \in \mathbb{T}, t \geq s \),

(ii) \( \Phi(t + 1, t) : \mathcal{R}(P^u_t) \to \mathcal{R}(P^u_{t+1}) \) is invertible for all \( t, t + 1 \in \mathbb{T} \). Denote the inverse of \( \Phi(t, s) \) by \( \Phi(s, t) : \mathcal{R}(P^u_s) \to \mathcal{R}(P^u_t), t, s \in \mathbb{T}, t \geq s \).

(iii) For \( t, s \in \mathbb{T}, t \geq s \), the following estimates hold

\[ \| \Phi(t, s)x \| \leq e^{-\alpha_s(t-s)} \| x \| \text{ for all } x \in \mathcal{R}(P^s_t), \]

\[ \| \Phi(s, t)x \| \leq e^{-\alpha_u(t-s)} \| x \| \text{ for all } x \in \mathcal{R}(P^u_t). \] (5)

For dynamical systems, generated by ODE models, both systems (1) and (3) are invertible under reasonable assumptions. Thus, condition (ii) from Definition 2.1 is satisfied.

Unlike the infinite time case, dichotomy projectors of finite time systems are in general not unique, see [4, Example 4]. The unique objects are stable and unstable cones, cf. [16].

The choice of the norm in (5) is critical since an additional constant \( K \) on the right hand side is not permitted that may compensate for transient dynamics. Typical choices of norms are \( \| x \|_\Gamma := \langle x, \Gamma x \rangle \) with a positive definite and symmetric matrix \( \Gamma \in \mathbb{R}^{d,d} \). The matrix \( \Gamma \) may originate from a similarity transformation, since the Euclidean norm of \( S \Phi(t, s) S^{-1} \) equals the \( \Gamma \)-norm of (3), with \( \Gamma = S^T S \). For autonomous systems, \( \| \cdot \| \) can alternatively be chosen to be Lyapunov adapted, cf. [3, Section 4.1]. Note that a nonautonomous choice of the norm \( \| x \|_{\Gamma_t} := \langle x, \Gamma_t x \rangle \), where \( \Gamma_t = S_t^T S_t \), is also possible. Selecting this norm is particularly motivated by a nonautonomous Lyapunov transformation \( S_t \Phi(t, s) S_s^{-1} \) of (3). In finite time, this concept is problematic, since transformations to arbitrary systems can be achieved in this way, see [6, Remark 7].

Due to its strict monotonicity, the dichotomy concept from Definition 2.1 is referred to as M-hyperbolicity in the literature. We note that further nonequivalent finite time notions of hyperbolicity have been introduced, and connections between these spectral concepts are pointed out in [15]. We particularly mention the stronger concept of D-hyperbolicity, see [5, 26] and the notion of finite time Lyapunov exponents, see [27, 49].

2.2. Finite time areas of attraction

Let \( \xi_T \) be a trajectory of (1) that is hyperbolic, i.e. the corresponding variational equation (3) has an exponential dichotomy in the sense of Definition 2.1. With respect to this trajectory, we define finite time areas of attraction as follows. First, we fix \( \varepsilon > 0 \) and introduce a local characterization of areas of attraction in \( \mathcal{B}_\varepsilon(\xi_T) = \{ x_T : \)

\[ \|x_T - \xi_T\|_\infty \leq \varepsilon \}. \] For \( t \in \mathbb{T} \), \( t < t_+ \), we define

\[
W_{\text{loc}, \varepsilon}^s(\xi_T, t) := \{ x \in B_\varepsilon(\xi(t)) : \forall t_1, t_2 \in [t, t_+]_\mathbb{T}, \ t_1 < t_2 : \\
\|\Psi(t_2, t; x) - \xi(t_2)\| \leq \|\Psi(t_1, t; x) - \xi(t_1)\| \}. \tag{6}
\]

Here, the limit condition from infinite time dynamical systems is replaced by monotonicity within an \( \varepsilon \)-neighborhood.

The definition of (6) is motivated by corresponding notions for infinite time dynamical systems. A point lies in the stable fiber of a hyperbolic reference trajectory \( \xi_T \), if its orbit stays forward in time in a sufficiently small neighborhood of \( \xi_T \). Corresponding results are [50, Theorem III.7] for autonomous and [46, Corollary 4.6.11] for nonautonomous dynamical systems, defined on an infinite time horizon.

**Remark 1.** Orbits of the variational equation (3) converge monotonously for starting points \( x \) in the range of the stable dichotomy projector. Using the dichotomy estimate (5), we get for \( x \in R(P^s_\varepsilon) \cap B_\varepsilon(0) \), \( t \in [t_-, t_+]_\mathbb{T} \) and \( t_1, t_2 \in [t, t_+]_\mathbb{T} \) with \( t_1 < t_2 \)

\[ \|\Phi(t_2, t)x\| = \|\Phi(t_2, t_1)\Phi(t_1, t)x\| \leq e^{-\alpha(t_2-t_1)}\|\Phi(t_1, t)x\| \]

and consequently \( \|\Phi(t_2, t)x\| \leq \|\Phi(t_1, t)x\| \) holds true and \( x \in W_{\text{loc}, \varepsilon}^s(0_T, t) \) follows.

We similarly obtain for \( x \in R(P^s_\varepsilon) \cap B_\varepsilon(0) \) that \( x \notin W_{\text{loc}, \varepsilon}^s(0_T, t) \).

Dependence on the chosen norm that we observed in Definition 2.1 is also critical in (6). For nonautonomous, linear models, strictly monotone convergence towards the fixed point 0 holds true globally, if all stable and unstable directions are invariant and orthogonal to each other. One can achieve this, by applying a nonautonomous Lyapunov transformation, resulting in a nonautonomous norm. However, assuming monotonicity globally for nonlinear systems is too restrictive.

We define the time, when the trajectory with starting point \( x \) at time \( t \in \mathbb{T} \), defined as \( \{\Psi(\ell, t; x) : \ell \in [t, t_+]_\mathbb{T} \} \), enters \( B_\varepsilon(\xi_T) \) and stays from that time on in this neighborhood:

\[
m^s(\xi_T, \varepsilon, x, t) = \inf \{ s \in [t, t_+]_\mathbb{T} : \Psi(\ell, t; x) \in B_\varepsilon(\xi_T) \ \forall \ell \in [s, t_+]_\mathbb{T} \}, \tag{7}
\]

where \( \inf \emptyset := t_+ \).

Our global version of the finite time area of attraction reads for \( t \in \mathbb{T} \), \( t < t_+ \)

\[
W_{\varepsilon}^s(\xi_T, t) := \{ x \in \mathbb{R}^d : s = m^s(\xi_T, \varepsilon, x, t) \in [t, t_+]_\mathbb{T} \text{ and } \Psi(s, t; x) \in W_{\text{loc}, \varepsilon}^s(\xi_T, s) \}. \tag{8}
\]

The definition of (8) mimics the construction of the global stable manifold via a continuation process, starting with a local graph representation. Observe that the evolution operator \( \Psi \) does not have to be invertible in order to define (8). In (8), the length of the finite time area of attraction depends on \( \varepsilon \). This \( \varepsilon \)-dependence characterizes the following lemma.

**Lemma 2.2.** Assume that \( \xi_T \) is a hyperbolic trajectory of (1) and fix \( t \in [t_-, t_+]_\mathbb{T} \).

(i) Let \( \varepsilon_1 > \varepsilon_2 > 0 \), then \( W_{\text{loc}, \varepsilon_1}^s(\xi_T, t) \supset W_{\text{loc}, \varepsilon_2}^s(\xi_T, t) \).

(ii) Let \( \varepsilon_1 > \varepsilon_2 > 0 \), \( x \in W_{\varepsilon_1}^s(\xi_T, t) \) and assume that \( m^s(\xi_T, \varepsilon_2, x, t) \in [t, t_+]_\mathbb{T} \), then \( x \in W_{\varepsilon_2}^s(\xi_T, t) \).
Areas of attraction are positive invariant in the following sense:

**Lemma 2.3.** Assume that $\xi_T$ is a hyperbolic trajectory of (1) and fix $\varepsilon > 0$, $t \in [t_-, t_+)_T$.

(i) Let $x \in W^s_{\text{loc}, \varepsilon}(\xi_T, t)$, then $\Psi(s, t; x) \in W^s_{\text{loc}, \varepsilon}(\xi_T, s)$ for all $s \in [t, t_+)_T$.

(ii) Let $x \in W^s(\xi_T, t)$, then $\Psi(s, t; x) \in W^s(\xi_T, s)$ for all $s \in [t, t_+)_T$.

**Remark 2.** We observe the following relation between finite time and infinite time areas of attraction. Consider a dynamical system, defined on a bi-infinite time interval $\mathbb{T}$. We assume that $T$ is a hyperbolic trajectory of $\xi_T$, then

\[ W^s(\xi_T, \tau) \]

are fat sets for small $t_+ - \tau$. But if $t_+ - \tau$ increases, these sets take more and more the shape of infinite time fibers.

### 2.3. Finite time areas of repulsion

Assuming that the dynamical system (1) is invertible, we observe that areas of repulsion are areas of attraction of the inverse system. Thus, one can define $W^u_{\text{loc}, \varepsilon}$, $W^u$ and $m^u$ in complete analogy to (6), (8) and (7). The version that we introduce here is more general and applies to noninvertible systems, too, since the evolution operator $\Psi$ is evaluated forward in time only. Let

\[ W^u_{\varepsilon}(\xi_T, t) := \left\{ x \in \mathbb{R}^d : \exists y \in \mathbb{R}^d : \Psi(t, t_-; y) = x \text{ and} \right. \]

\[ \|\Psi(t_2, t_-; y) - \xi(t_2)\| \leq \|\Psi(t_1, t_-; y) - \xi(t_1)\|, \]

with

\[ m^u(\xi_T, \varepsilon, y, t) = \sup \left\{ s \in [t_- , t]_T : \Psi(\ell, t_-; y) \in B_{\varepsilon}(\xi(\ell)) \forall \ell \in [t_-, s]_T \right\}, \]

where $\sup \emptyset := t_-$. 

### 2.4. Numerical scheme

We introduce a numerical scheme for the computation of stable and unstable finite time fibers. We assume that $\mathbb{T} = \{ t_{n_-}, \ldots, t_{n_+} \}$ is a discrete set. Note that hyperbolic continuous time dynamical systems turn into hyperbolic discrete time systems by considering the $t$-flow w.r.t. the discrete set $\{t_{n_-}, \ldots, t_{n_+}\}$. But for a given ODE-model, the corresponding $t$-flow is typically not known explicitly. Then, a one-step discretization scheme provides the required discrete time data.

With respect to a bounded trajectory $\xi_T$ and for $t \in [t_{n_-}, t_{n_+ - 1}]_T$, $\varepsilon > 0$ and $z \in \mathbb{R}^d$, we define

\[ G^s_{\varepsilon}(\xi_T, z, t) := \min \{ \varepsilon - \|\Psi(t_{n_- - 1}, t; z) - \xi(t_{n_- - 1})\|, \]

\[ \|\Psi(t_\ell, t; z) - \xi(t_\ell)\| - \|\Psi(t_{\ell + 1}, t; z) - \xi(t_{\ell + 1})\| : \]

\[ m^s(\xi_T, \varepsilon, z, t) \leq t_\ell \leq t_{n_+ - 1} \].

6
and observe that
\[
\partial W^s(\xi_T; t) \subset \{ z \in \mathbb{R}^d : G^s_\varepsilon(\xi_T, z, t) = 0 \} =: C^s_\varepsilon(\xi_T; t). \tag{10}
\]
Equality holds true in (10), if the systems satisfies certain nondegeneracy conditions.

The numerical computation of the zero-contour \(C^s_\varepsilon(\xi_T; t)\) for \(d \in \{2, 3\}\)-dimensional systems requires to define a grid in the area of interest. Then, we calculate for each grid point \(z\) the value of \(G^s_\varepsilon(\xi_T, z, t)\). The resulting data are the input of the MATLAB functions \texttt{contour} respectively \texttt{isosurface} that efficiently detect contours for two- and three-dimensional systems.

For higher dimensional systems, we obtain the intersection with a \(k \in \{2, 3\}\)-dimensional subspace as follows. Denote by \(s : \mathbb{R}^k \to \mathbb{R}^d\) its embedding and apply the techniques from above for computing the zero contour
\[
\{ z \in \mathbb{R}^k : G^s_\varepsilon(\xi_T, s(z), t) = 0 \}.
\]
Similarly, we calculate an approximation of the unstable fiber \(W^u_\varepsilon(\xi_T; t)\) for \(t \in [t_{n+1}, t_n]\), assuming that the system is invertible:
\[
G^u_\varepsilon(\xi_T, z, t) := \min \{ \varepsilon - \| \Psi(t_{n+1}, t; z) - \xi(t_{n+1}) \|, \| \Psi(t_t, t; z) - \xi(t_t) \| - \| \Psi(t_{t-1}, t; z) - \xi(t_{t-1}) \| : \}
\]
\[
t_{n+1} \leq t \leq m^u(\xi_T, \varepsilon, z, t) \}.
\]

The computation of unstable fibers in noninvertible systems requires an additional search for pre-images. We avoid this time consuming search for appropriate pre-images and numerically compute in this paper areas of repulsion for invertible examples only.

Finally, we note that the calculation of \(G^{x,u}_\varepsilon(\xi_T, z, t)\) for all points on a two- or three-dimensional grid is rather expensive due to the nonautonomous nature of the system under consideration. As we will see, the numerical effort reduces substantially for autonomous systems.

3. Linear systems, areas of attraction and their computation

In this section, we consider linear dynamical systems \((\mathbb{R}^d, \mathbb{T}, \Phi)\) that are finite time hyperbolic in the sense of Definition 2.1. An important example within this class of systems is the variational equation (3), (4) w.r.t. a hyperbolic trajectory.

There are structural differences between finite and infinite time dynamical systems. On a bi-infinite time interval \(\mathbb{T}\), one compensates for transient dynamics by allowing a constant \(K \geq 1\) on the right hand sides of the dichotomy estimates (5). Furthermore, dichotomy projectors are uniquely determined in infinite time, see [44]. Stable and unstable fiber bundles turn out to be the ranges of corresponding stable respectively unstable dichotomy projectors. Techniques for the numerical approximation of dichotomy projectors have been proposed in [31, 32]. The corresponding algorithms are based on solving least squares and boundary value problems.

To finite time systems, these techniques are not directly applicable, due to the nonuniqueness of dichotomy projectors. The unique objects, one is interested in, are stable and unstable cones, cf. [16]. We define the closure of the stable cone for \(t \in [t_-, t_+)_\mathbb{T}\)
as
\[ \mathcal{M}^s(t) := \{ x \in \mathbb{R}^d : \forall t_1, t_2 \in [t, t_+], t_1 < t_2 : \| \Phi(t_2, t)x \| \leq \| \Phi(t_1, t)x \| \} \quad (12) \]

and observe the following connection between stable cones and areas of attraction:
\[ \mathcal{M}^s(t) \cap \mathcal{B}_\varepsilon(0) = \mathcal{W}_{\text{loc}, \varepsilon}(0_T, t) \quad \text{for all } \varepsilon > 0. \]

Note that \( \mathcal{M}^s(t) \) does generally not define the stable cone of the finite time dichotomy in the sense of [16, Section 3], which requires monotonicity on \([t_-, t_+]_T\) rather than on \([t, t_+]_T\) as in (12).

### 3.1. Numerical approximation in the linear case

We introduce a simplified version of the algorithm from Section 2.4 that applies to linear systems. As stated above, \( \mathbb{T} \) is a discrete set and ODE-models are discretized using a one-step discretization scheme.

On \( \mathbb{T} = \{ t_{n-}, \ldots, t_n \} \), (12) reads for \( n \in [n-, n+ - 1] \mathbb{Z} \)
\[ \mathcal{M}^s(t_n) = \{ x \in \mathbb{R}^d : \forall \ell \in [n, n+ - 1] \mathbb{Z} : \| \Phi(t_{\ell+1}, t_n)x \| \leq \| \Phi(t_{\ell}, t_n)x \| \}. \quad (13) \]

We observe that \( \partial \mathcal{M}^s(t_n) \subset \mathcal{J}^s(t_n) \) for \( n \in [n-, n+ - 1] \mathbb{Z} \), where
\[ \mathcal{J}^s(t_n) = \{ x \in \mathbb{R}^d : g_n(x) = 0 \}, \quad g_n(x) = \min_{\ell \in [n, n+ - 1] \mathbb{Z}} (\| \Phi(t_{\ell}, t_n)x \| - \| \Phi(t_{\ell+1}, t_n)x \|). \]

Numerically, we calculate \( \mathcal{J}^s(t_n) \) with the contour ansatz from Section 2.4.

### 3.2. Autonomous linear models

We start with the autonomous linear case \( \Phi(t, s) = A^{t-s} \) for \( t, s \in \mathbb{T} = [n-, n+] \mathbb{Z}, t \geq s \) and \( A \in \mathbb{R}^{d,d} \) is a hyperbolic matrix. Assuming additionally that \( A \) is diagonalizable, it turns out that one only has to verify the condition (13) for the last step, i.e.
\[ \| \Phi(t_{n+}, t)x \|_\Gamma \leq \| \Phi(t_{n+ - 1}, t)x \|_\Gamma \]
in order to assure that \( x \in \mathcal{M}^s(t) \) w.r.t. a Lyapunov adapted \( \Gamma \)-norm. Here
\[ \Gamma = S^T S, \quad \text{where} \quad \text{SAS}^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_d) \quad (14) \]
with eigenvalues \( \lambda_1, \ldots, \lambda_d \) of \( A \). As described at the end of Section 2.1, the corresponding norm is given as \( \| x \|_{\Gamma}^2 := \langle x, \Gamma x \rangle \). The following lemma justifies this simplification that reduces computational costs for autonomous linear systems substantially.

**Lemma 3.1.** Let \( A \in \mathbb{R}^{d,d} \) be diagonalizable and hyperbolic. Let \( N \in \mathbb{N}, x \in \mathbb{R}^d \) and assume that \( \| A^{N+1}x \|_\Gamma \leq \| A^N x \|_\Gamma \). Then \( \| A^{\ell+1}x \|_\Gamma \leq \| A^\ell x \|_\Gamma \), for all \( \ell = 0, \ldots, N \).

**Proof.** Fix \( x \in \mathbb{R}^d \) and note that \( \| Ax \|_{\Gamma}^2 = \| SAS^{-1}Sx \|_2^2 = \sum_{i=1}^d \lambda_i^2(Sx)_i^2 \), where \( (Sx)_i \) denotes the \( i \)-th component of \( Sx \).
We prove the converse statement $\|x\|_\Gamma < \|Ax\|_\Gamma \Rightarrow \|Ax\|_\Gamma < \|A^2x\|_\Gamma$ which yields the claim inductively.

Assuming $\|x\|_\Gamma < \|Ax\|_\Gamma$ it follows that

$$0 > \|x\|_\Gamma^2 - \|Ax\|_\Gamma^2 = \sum_{i=1}^d (1 - \lambda_i^2)(Sx)_i^2$$

$$= \sum_{i=1}^d \lambda_i^2(1 - \lambda_i^2)(Sx)_i^2 + \sum_{i=1}^d (1 - \lambda_i^2)(Sx)_i^2.$$

As a consequence

$$0 > \sum_{i=1}^d \lambda_i^2(1 - \lambda_i^2)(Sx)_i^2 = \|Ax\|_\Gamma^2 - \|A^2x\|_\Gamma^2.$$

For an illustration, we consider the three examples

$$A_1 = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}. \quad (15)$$

For these examples, we compute the stable cones $M^*(t)$ for $t \in \{n_+ - 3, n_+ - 2, n_+ - 1\}$ in Figure 1. Note that these models are autonomous and $M^*(t)$ only depends on the distance from $t$ to the right boundary $n_+$ of the finite interval. The actual choices of $n_+$ and $t$ are therefore irrelevant.

Figure 1. Upper row: Boundaries of stable cones $M^*(n_+ - 1)$ (brown), $M^*(n_+ - 2)$ (red), $M^*(n_+ - 3)$ (blue) for the three examples (15). Lower row: Intersection of the unit sphere with $M^*(n_+ - 1)$ for $A_{1,2}$ and with $M^*(n_+ - 2)$ for $A_3$.

Of particular interest is the third case. The noninvertibility causes for $t \in \mathbb{T}$, $t \leq
\(n_+ - 2\) degenerate stable cones \(\mathcal{M}^s(t)\). Components in \(x_2\)-direction are zero after one iteration step. If, in addition \(x_1 = 0\), then the monotonicity condition in (12) is violated in the next iteration steps if \(x_3 \neq 0\).

The stable cone \(\mathcal{M}^s(t)\) converges for increasing \(n_+ - t\) towards the stable subspace of the respective matrix \(A_i, i \in \{1, 2, 3\}\).

### 3.3. Nonautonomous linear models

We observed for autonomous models in Lemma 3.1 that it suffices to analyze monotonicity in the last step of an iteration to ensure monotonicity in each step. However, Lemma 3.1 does not apply to nonautonomous models. The nonautonomous case requires to verify monotonicity in each step. For an illustration, we consider the following system in discrete time

\[
u(n + 1) = A(n)\nu(n), \quad \text{where } A(n) = D(n + 1) \cdot B \cdot D(-n), \quad \text{(16)}
\]

\[
B = \begin{pmatrix}
0.9 & 0 & 0 \\
0 & 0.95 & 0 \\
0 & 0 & 1.1
\end{pmatrix}, \quad D(n) = \begin{pmatrix}
\cos n\varphi & 0 & -\sin n\varphi \\
\sin n\varphi & 0 & \cos n\varphi \\
0 & 1 & 0
\end{pmatrix}, \quad \varphi = \frac{\pi}{3}.
\]

**Figure 2.** Intersection of the Euclidean unit-ball with stable cones \(\mathcal{M}^s(t)\) of (16).

We note that system (16) is not a nonautonomous similarity transformation of the constant system \(v(n + 1) = Bv(n)\), since \(D(-n)D(n) \neq I\). Thus, the eigenvalues of \(A(n), n \in \mathbb{T}\) are meaningless for an analysis of the stability behavior of the fixed point 0. We refer to [20, Example 4.17] and [11, Section 2.6] for counterexamples in discrete
and continuous time. Indeed, the stable subspaces of the exponential dichotomy on \( \mathbb{Z} \) of (16) turns out be one-dimensional. Furthermore, \( M^s(t) \) converges for increasing \( n_+ - t \) towards the corresponding one-dimensional stable subspace. With respect to the finite interval \( T = [1, 11] \mathbb{Z} \) we compute the stable cones \( M^s(t) \) for \( t \in \{3, 4, 5, 7, 9, 10\} \) in Figure 2. This figure particularly illustrates that each half of the stable (and unstable) cone is generally not convex.

4. Nonlinear systems, areas of attraction and repulsion and their computation

In this section, we apply the contour algorithm from Section 2.4 to nonlinear systems. We prove that the observation from Lemma 3.1 carries over to autonomous nonlinear models, resulting in reduced computing times. Finally, intersections between stable and unstable fibers are detected. Orbits within these intersections are called finite time homoclinic orbits.

4.1. Autonomous nonlinear models

We first consider the autonomous nonlinear case \( \Psi(t, s; \cdot) = f^{t-s}(\cdot) \) for \( t, s \in T = \{t_n, \ldots, t_n\}, t \geq s \) and define \( t_n = n \), if the underlying system is discrete and generated by the map \( f \). If the underlying system is an ODE, we introduce some grid \( T \subset \mathbb{R} \) and define \( f \) as a corresponding one-step map that originates from a numerical discretization scheme.

We assume that
\[
(A0) \quad f \in C^2(\mathbb{R}^d, \mathbb{R}^d), f(0) = 0 \text{ and } A := Df(0) \text{ is a hyperbolic matrix.}
\]

As in Section 3.2, we additionally assume that \( A \) is diagonalizable in order to define \( \Gamma \) as in (14). Similar to the linear setup in Lemma 3.1, we obtain for sufficiently small \( \varepsilon \) that \( x \in W^{\text{loc}, \varepsilon}_s(0_T, t) \) for \( x \in B_\varepsilon(0) \), provided
\[
\|\Psi(t_{n+}, t; x)\|_{\Gamma} \leq \|\Psi(t_{n+}, t; x)\|_{\Gamma}
\]
holds true. This results in an efficient algorithm, since we do not have to verify monotonicity in each step of the iteration within (6).

Lemma 4.1. Assume \( (A0) \) and let \( A \) be diagonalizable and fix \( N \in \mathbb{N} \). Then there exists an \( \varepsilon > 0 \) such that for each \( x \in B_\varepsilon(0) \), satisfying \( \|f^{N+1}(x)\|_{\Gamma} \leq \|f^N(x)\|_{\Gamma} \) it follows for any \( \ell \in \{0, \ldots, N\} \) with \( f^i(x) \in B_\varepsilon(0) \), \( i = \ell, \ldots, N \) that
\[
\|f^{\ell+1}(x)\|_{\Gamma} \leq \|f^\ell(x)\|_{\Gamma}.
\]

Proof. First, observe that \( \|f^i(x)\|_{\Gamma}^2 = \|Ax\|_{\Gamma}^2 + g_1(x) \) and \( \|f^2(x)\|_{\Gamma}^2 = \|A^2x\|_{\Gamma}^2 + g_2(x) \) with \( g_{1,2}(x) = O(\|x\|_{\Gamma}^3) \).

Choose \( \varepsilon > 0 \) such that
\[
\sum_{i=1}^{d} (1 - \lambda_i^2)^2(Sx)^2_i \geq 2g_1(x) - g_2(x) \quad \text{for all } x \in B_\varepsilon(0).
\]

(17)
Similar to the proof of Lemma 3.1, we show the converse statement.

Let \( x \in B_\epsilon(0) \) such that \( \|x\|_\Gamma - \|f(x)\|_\Gamma < 0 \) then \( \|f(x)\|_\Gamma - \|f^2(x)\|_\Gamma < 0 \),
which yields the claim inductively.

Assuming \( \|x\|_\Gamma < \|f(x)\|_\Gamma \) it follows that
\[
0 > \|x\|_\Gamma^2 - \|f(x)\|_\Gamma^2 = \|x\|_\Gamma^2 - \|Ax\|_\Gamma^2 - g_1(x) = \sum_{i=1}^{d} \lambda_i^2 (1 - \lambda_i^2)(Sx)_i^2 + \sum_{i=1}^{d} (1 - \lambda_i^2)^2(Sx)_i^2 - g_1(x).
\]

Using (17), we conclude for \( x \in B_\epsilon(0) \) that
\[
0 > \sum_{i=1}^{d} \lambda_i^2 (1 - \lambda_i^2)(Sx)_i^2 + g_1(x) - g_2(x) = \|f(x)\|_\Gamma^2 - \|f^2(x)\|_\Gamma^2.
\]

In the autonomous case, Lemma 4.1 allows in an appropriate norm \( \|\cdot\| \) and for sufficiently small \( \epsilon > 0 \) the efficient approximation of \( W^s_\epsilon(0_{\Gamma}, t) \). Instead of computing the zero-contour of \( G^s_\epsilon(0_{\Gamma}, \cdot, t) \), it suffices to consider
\[
\tilde{G}^s_\epsilon(z, t) := \min\{\|\Psi(t_{n+1}, t; z)\| - \|\Psi(t_{n+1}, t; z)\|, \epsilon - \|\Psi(t_{n+1}, t; z)\|\}.
\]

If the system is invertible, we similarly obtain an approximation of the area of repulsion for \( t \in [t_{n+1}, t_{n+1}]_\Gamma \) by computing the zero-contour of
\[
\tilde{G}^u_\epsilon(z, t) := \min\{\|\Psi(t_{n+1}, t; z)\| - \|\Psi(t_{n+1}, t; z)\|, \epsilon - \|\Psi(t_{n+1}, t; z)\|\}.
\]

Finally, we consider homoclinic areas, which are the intersections of areas of attraction and repulsion, see Section 4.2 for a formal introduction. This set turns out to be numerically accessible at time \( t \in [t_{n+1}, t_{n+1}]_\Gamma \) by computing the zero-contour of
\[
\tilde{G}(z, t) := \min\{\tilde{G}^s_\epsilon(z, t), \tilde{G}^u_\epsilon(z, t)\}
\]
in the autonomous case.

### 4.1.1. A three-dimensional map with one stable eigenvalue

For an illustration, we consider the system \( x(n+1) = f(x(n)) \), where
\[
f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1 \\ 3x_2 - \frac{7}{3}x_1^2 \\ 3x_3 - \frac{11}{3}x_1^2 \end{pmatrix}.
\]

On an infinite time interval, the one-dimensional stable manifold of the hyperbolic fixed point 0 has the global graph representation
\[
W^s(0_{\Gamma}) = \{(y, y^2, y^3)^T : y \in \mathbb{R}\}.
\]
We compute the boundary of \( W^s(0_\Gamma, t) \) for \( \varepsilon = 0.2 \) and \( t \in \{ n_+ - 3, n_+ - 2, n_+ - 1 \} \) in Figure 3. For this task, we evaluate for all starting points on a 400 \( \times \) 400 \( \times \) 400 grid around the fixed point, the function \( G^s_\varepsilon(\cdot, t) \), defined in (18). From the resulting data, the MATLAB routine \text{isosurface} \ computes the zero-contour, shown in Figure 3. We observe that for decreasing \( t \), the approximation of \( W^s(0_\Gamma, t) \) becomes longer and narrower. These three-dimensional sets converge towards the one-dimensional manifold (22) as \( t \to -\infty \).

\[
4.1.2. \text{A three-dimensional map with two stable eigenvalues}
\]

Next, we analyze the dynamical system \( x(n + 1) = f(x(n)) \) that is generated by

\[
f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1 \\ \frac{1}{3}x_2 \\ 2x_3 - \frac{7}{4}x_1^2 - \frac{17}{16}x_2^2 \end{pmatrix}.
\]

The hyperbolic fixed point 0 possesses two stable and one unstable eigenvalue. When considering this system on an infinite time frame, the stable manifold has the global graph representation

\[
W^s(0_\Gamma) = \{ (y_1, y_2, y_1^2 + y_2^2)^T, y \in \mathbb{R}^2 \}.
\]

We detect the boundary of \( W^s(0_\Gamma, t) \) for \( \varepsilon = 0.2 \) and \( t \in \{ n_+ - 3, n_+ - 2, n_+ - 1 \} \). For all points \( z \) from a 400 \( \times \) 400 \( \times \) 400 grid we evaluate \( G^s_\varepsilon(z, t) \), see (18). Zero-contours of \( G^s_\varepsilon(\cdot, t) \), computed with the MATLAB routine \text{isosurface}, are shown in Figure 4. We observe that these three-dimensional finite time areas of attraction converge towards the two-dimensional manifold (24) as \( t \to -\infty \).
4.1.3. Areas of attraction and repulsion for a two-dimensional ODE-model

Our algorithm also applies to continuous time dynamical systems. For an illustration we consider the finite time ODE-model

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1.6 & x_1 + x_2^2 \\ -1 & -x_1^2 + x_2 \end{pmatrix}, \quad \nu = -1.6016, \quad \text{for } t \in T = [-2.5, 2.5].
\]

\[ (25) \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Stable manifold $W^s(0)$ (green) and unstable manifold $W^u(0)$ (red) of (25), w.r.t. a bi-infinite time interval with $\nu = -1.6016$. The manifolds are computed up to arc length 5.}
\end{figure}

On an infinite time axis, this model does not possess homoclinic orbits, see Figure 5.
Only for the slightly modified parameter $\nu = -1.6$, stable and unstable manifolds of the hyperbolic fixed point 0 have a whole orbit in common.

When we study this model on a finite interval, intersections of areas of attraction and repulsion of the hyperbolic fixed point 0 exist for $\nu = -1.6016$, provided that $\varepsilon > 0$ is sufficiently large. As a consequence, we obtain finite time homoclinic dynamics in this case.

We first discretize (25) with the classical Runge-Kutta scheme – which is a one-step method of order 4 – with step size $h = 0.01$. Note that continuous time dynamical systems that are generated by ODEs are invertible under reasonable assumptions. In practice, we apply the classical Runge-Kutta scheme with the negative step size $h = -0.01$ in order to compute areas of repulsion.

We proceed by calculating the zero-contour of $\tilde{G}_\varepsilon^{s,u}(\cdot,t)$ with $\varepsilon = 0.1$, cf. (18), (19), on a $1000 \times 1000$ grid, using the MATLAB routine contour. Finite time homoclinic points lie in the intersection of areas of attraction and repulsion that are shown in yellow in Figure 6. The boundary of this set is given as the zero-contour of the operator $\tilde{G}_\varepsilon(\cdot,t)$, defined in (20). Additionally, we apply in a neighborhood of the fixed point the contour-algorithm from [34, 35] for computing nearly one-dimensional parts of areas of attraction and repulsion. We refer to the end of Section 4.3 for more details on numerical difficulties that we observe in this case.

Figure 6. Approximation of $\mathcal{W}_s^\varepsilon(0,T,t)$ (green) and of $\mathcal{W}_u^\varepsilon(0,T,t)$ (red) with $t = -2.5 + nh$ for (25). Homoclinic points lie in the intersection of these sets (yellow).
4.1.4. A three-dimensional ODE-model with finite time homoclinic orbits

We illustrate that the techniques from Section 4.1.3 are not restricted to two-dimensional models. Consider the three-dimensional version of (25)

\[
\begin{pmatrix}
    x_1' \\
    x_2' \\
    x_3'
\end{pmatrix} = \begin{pmatrix}
    1.6(x_1 + x_2^2) \\
    \nu(-x_1^2 + x_2) \\
    0.01x_3
\end{pmatrix}, \quad \nu = -1.6016, \quad \text{for } t \in T = [-2.5, 2.5].
\] (26)

Using the same approach as in Section 4.1.3 we compute in Figure 7 the three-dimensional analog of the lower right diagram from Figure 6. For this task, we choose \(\varepsilon = 0.1, h = 0.01, t = 2.5 - h\) and apply the isosurface algorithm, w.r.t. a \(400 \times 400 \times 400\) grid, to \(\tilde{G}_s^s(\cdot, t), \tilde{G}_u^u(\cdot, t)\) respectively \(\tilde{G}_e^e(\cdot, t)\).

![Figure 7](image)

**Figure 7.** The left panel shows an approximation of \(W^s_s(0, t)\) (green) and of \(W^u_u(0, t)\) (red) for (26) for \(t = 2.5 - h\). The right panel illustrates the intersection of these sets (blue).

4.2. A note on homoclinic dynamics

Let \(\xi_T\) be a bounded trajectory of the dynamical system \((\mathbb{R}^d, \mathbb{T}, \Psi)\) on a bi-infinite time interval \(T\). A second trajectory \(\tilde{x}_T \neq \xi_T\) is homoclinic w.r.t. \(\xi_T\) if \(\lim_{t \to \pm \infty, t \in T} \|\tilde{x}(t) - \xi(t)\| = 0\). Under reasonable hyperbolicity assumptions, numerical approximations of \(\xi_T\) and \(\tilde{x}_T\) can be computed by solving boundary value problems. Precise error estimates justify the approximation process, see [7, Section 4] for autonomous and [33] for nonautonomous systems. Persistence of homoclinic orbits under time discretization has been analyzed in [54, Theorem 4.3] and [23, Theorem 4.2].

Homoclinic orbits are of great dynamical relevance. For autonomous discrete time systems, it is well known that in the maximal invariant neighborhood of a homoclinic orbit, the dynamics are conjugate to a subshift on bi-infinite sequences. This symbolic dynamical system is a prototype for chaotic dynamics. We refer to the famous theorem of Smale [52] and Shilnikov [51] and to the articles [53] and [7] for recent discussions of the symbolic coding in infinite time. The latter reference also considers the finite time case, including numerical techniques that provide an approximation of the maximal invariant set.
Let $\tilde{T}$ be a bounded interval in discrete or continuous time. An orbit $\tilde{x}_T$ of the finite time dynamical system $(\mathbb{R}^d, \tilde{T}, \Psi)$ is homoclinic w.r.t. the reference trajectory $\xi_T$ if

$$\tilde{x}(t) \in \mathcal{W}^s_\varepsilon(\xi_T, t) \cap \mathcal{W}^u_\varepsilon(\xi_T, t) \quad \text{for all } t \in (t_-, t_+)_{\tilde{T}}.$$

Homoclinic orbits of infinite time dynamical systems result in finite time homoclinic orbits for a finite time restriction of the system, under two additional assumptions. First, the homoclinic orbit must eventually converge monotonously towards the reference trajectory and secondly, these orbits must reach the $\varepsilon$-neighborhood within the finite time horizon.

Formally, we assume on the infinite time interval $\mathbb{T}$ without loss of generality that $\xi(t) = 0$ for all $t \in \mathbb{T}$, which can be achieved by a nonautonomous Lyapunov transformation. In an $\varepsilon$-neighborhood of 0, points on the stable fiber bundle at time $t \in \mathbb{T}$ have the graph representation $x = x_s + h^s_\varepsilon(x_s)$, where $x_s$ lies in the stable subspace at time $t$ of the exponential dichotomy and $h^s_\varepsilon(0) = 0$, $Dh^s_\varepsilon(0) = 0$, see [2, Theorem 4.1]. A corresponding graph representation also exists for points on the unstable fiber bundle: $x = x_u + h^u_\varepsilon(x_u)$ for $t \in \mathbb{T}$.

We state the following assumptions on the infinite time dynamical systems (1) and (3).

(A1) The fixed point 0 is M-hyperbolic in the sense of Definition 2.1.

(A2) $\tilde{x}_T$ is a homoclinic orbit w.r.t. the fixed point 0 of $\Psi$.

(A3) There exists a $C > 0$ such that $\Psi(t, s; x) = \Phi(t, s)x + g(t, s, x)$, where $\|g(t, s, x)\| \leq C\|x\|^2$ for all $x \in B_2(0)$ and all $s, t \in \mathbb{T}$ such that $|t - s| \leq 1$.

(A4) $\|\Phi(t, s)\| \leq C$ for all $s, t \in \mathbb{T}$ such that $|t - s| \leq 1$.

(A5) $\|h^u_\varepsilon(x)\| \leq C\|x\|^2$ for all $t \in \mathbb{T}$ and $x \in B_2(0)$.

(A6) The (possibly nonautonomous) norm satisfies $\|x_{s,u}(t)\| \leq \|x(t)\|$ for all $x(t) = x_s(t) + x_u(t) \in B_2(0)$, where $x_{s,u}(t)$ lies in the stable respectively unstable subspace of the exponential dichotomy at time $t \in \mathbb{T}$.

**Theorem 4.2.** Let the assumptions (A1)-(A6) be satisfied. Then there exists an $\tilde{\varepsilon} > 0$ such that for any $0 < \varepsilon < \tilde{\varepsilon}$, there is an $N$ such that for all $N \geq \tilde{N}$, $T_N := [-N, N]_{\tilde{T}}$ it follows that $\tilde{x}_{T_N}$ is a finite time homoclinic orbit of the system $(\mathbb{R}^d, T_N, \Psi)$ w.r.t. the fixed point 0.

**Proof.** First, we verify monotonicity of the homoclinic orbit $\tilde{x}_T$ in a sufficiently small neighborhood of 0. With $\tilde{\varepsilon}$ from (A3), one finds an $\tilde{s} \in \mathbb{T}$ such that $\tilde{x}(t) \in B_2(0)$ for all $t \in \mathbb{T}$, $t \geq \tilde{s}$. We conclude for all $t, s \in \mathbb{T}$, $t = s + 1 > s > \tilde{s}$ that

$$\|\Phi(t, s; \tilde{x}(s))\| \leq \|\Phi(t, s)\tilde{x}(s)\| + \|g(t, s, \tilde{x}(s))\|$$

$$\leq \|\Phi(t, s)\tilde{x}_s(s)\| + \|\Phi(t, s)h^s_\varepsilon(\tilde{x}_s(s))\| + \|g(t, s, \tilde{x}(s))\|$$

$$\leq e^{-\alpha_s(t-s)}\|\tilde{x}_s(s)\| + C\|\tilde{x}_s(s)\|^2 + C\|\tilde{x}(s)\|^2$$

$$\leq (e^{-\alpha_s} + 2C\|\tilde{x}(s)\|)\|\tilde{x}(s)\|$$

for $0 < \tilde{\varepsilon} \leq \tilde{\varepsilon}$ sufficiently small and $\tilde{x}(s) \in B_2(0)$. A similar computation proves monotone convergence for $t \to -\infty$, provided $\tilde{\varepsilon}$ is decreased even further.

Thus, we find for each $0 < \varepsilon < \tilde{\varepsilon}$ an $\tilde{N}$ such that the endpoints of the homoclinic orbit $\tilde{x}_{T_N}$ lie in $B_2(0)$ for all $N \geq \tilde{N}$. As a consequence, $\tilde{x}_{T_N}$ lies in the intersection of finite time areas of attraction and repulsion of the fixed point 0. $\square$
Homoclinic points of (26) at time $t = 2.5 - h$ are located in the blue area, shown in Figure 7. The assumptions of Theorem 4.2 are satisfied for this example. However, the variational equation (4) along this homoclinic orbit does not possess an exponential dichotomy in the sense of Definition 2.1. A dichotomy estimate of the form (5) only holds, if a reasonably small constant $K > 1$ is accepted on the right hand side of (5) or if a nonautonomous $\Gamma$-norm is permitted.

In finite time models, homoclinic orbits are not isolated. It follows from [7, Theorem 8] that the distance between two homoclinic orbits shrinks exponentially fast towards the midpoint of the finite time interval. The homoclinic areas that are shown in Figure 6 clearly illustrate this characteristic.

Finally, we note that the occurrence of homoclinic chaos in nonautonomous discrete time systems – on infinite time intervals – depends on the way in which stable and unstable sets intersect each other, cf. [42]. The main reason for non-chaotic dynamics near homoclinic orbits constitutes the observation that stable and unstable fibers may intersect only once in an isolated point, see [23] for more details. We introduce a finite time example of this type in the next section.

4.3. A nonautonomous nonlinear model

We consider the following model for homoclinic dynamics from [23, Section 5.1]:

$$
\begin{pmatrix}
  x_1' \\
  x_2
\end{pmatrix} = \begin{pmatrix}
  x_2 + x_1^2 + 6x_1\text{sech}^2(t) \\
  x_1^2 + 4x_1 + x_2^2 -12x_2\text{sech}^2(t) \tanh(t)
\end{pmatrix}.
$$

(27)

With respect to the finite time interval $T = [-1.5, 1.5]$, we detect areas of attraction and repulsion of the hyperbolic fixed point 0 at $t = -1$ and $t = -0.5$. On a bi-infinite time interval, this model exhibits a homoclinic orbit, see [23, Fig. 3]. Restricted to the finite interval $T$, the end points of this orbit segment are at a distance of 2.24 from the fixed point 0. Therefore, we choose $\varepsilon \in \{1.8, 2, 2.2\}$ rather large. This setting guarantees an intersection of stable and unstable sets on the short time interval $T$. Furthermore, our numerical calculations for various values of $\varepsilon$ illustrate the influence of $\varepsilon$ on areas of attraction, areas of repulsion and on homoclinic sets, cf. Lemmas 2.2 and 2.3. Figure 8 displays these sets and their intersections. We apply the classical Runge-Kutta scheme for discretizing the ODE (27) with step size $h = 0.01$.

During this computation, we have to verify monotonicity in the $\varepsilon$-neighborhood of 0 in each step, i.e. we have to compute the zero-contour of

$$
G_\varepsilon(0_T, z, t) := \min\{G^s_\varepsilon(0_T, z, t), G^u_\varepsilon(0_T, z, t)\}
$$

with $G^s,u_\varepsilon(0_T, z, t)$ from (9) and (11), respectively. Unlike the autonomous case, it does not suffice to verify monotonicity in the last step only, since Lemma 4.1 does not apply to nonautonomous models.

The widths of the green sets shrink in a neighborhood of the fixed point and we cannot expect to detect these narrow areas by computing the zero contour of $G^s_\varepsilon(0_T, \cdot, t)$. For this task, points from the chosen $2000 \times 2000$ grid have to lie inside and outside of the contour, which is not satisfied in a neighborhood of 0. Alternatively, one may apply techniques for infinite time dynamical systems to compute this nearly one-dimensional part of the stable set. The contour-algorithm from [34, 35] provides an approximation of invariant fiber bundles for infinite time systems. We apply this algorithm for com-
puting parts of Figure 6.

\begin{align*}
t &= -1, \epsilon = 1.8 \quad t = -1, \epsilon = 2 \quad t = -1, \epsilon = 2.2 \\
t &= -0.5, \epsilon = 1.8 \quad t = -0.5, \epsilon = 2 \quad t = -0.5, \epsilon = 2.2
\end{align*}

**Figure 8.** Approximation of $W^s(0_T, t)$ (green) and of $W^u(0_T, t)$ (red) for (27). Homoclinic points lie in the intersection of these sets (yellow).

### 5. Conclusion

In finite time dynamical systems, the absence of an infinite limit process leads to several meaningful, but nonequivalent definitions of areas of attraction and repulsion. The definition proposed here is based on two characteristics: $\epsilon$-closeness of the end points to the reference trajectory and eventual monotonicity. This ansatz allows numerical computations, since the boundaries of these areas are essentially the zero-set of specific operators. By evaluating these operators numerically and applying the MATLAB-routines `contour` and `isosurface` we obtain a graphical representation of areas of attraction and repulsion. These techniques apply to autonomous and nonautonomous dynamical systems. Calculations in the nonautonomous case are more expensive, since the test for monotonicity is more involved. Intersections of areas of attraction and repulsion reveal homoclinic points that are of great dynamical relevance. For an illustration, we choose two- and three-dimensional models in discrete and continuous time. We note that our techniques also apply to higher dimensional models and enable the computation of the intersection of areas of attraction (areas of repulsion) with a two- respectively three-dimensional subspace, see [36, Section 3.6] for corresponding calculations in infinite time dynamical systems.
References


[18] L.H. Duc and S. Siegmund, Hyperbolicity and invariant manifolds for planar nonau-


