ADDENTUM TO "A NOTE ON FOURIER-MUKAI PARTNERS OF ABELIAN VARIETIES OVER POSITIVE CHARACTERISTIC FIELDS"

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There are two remarks should be made for this paper.

(1) Throughout this paper, we only consider abelian varieties of dimension ≥ 2. For example, in the statement Theorem 1.2 and Section 4.2, this condition is required.

(2) In Proposition 4.5, we also need to assume that $\dim A_i \geq 2$ as before. As pointed by Riku Kurama and Saket Shah, a similar statement doesn't hold for elliptic curves.

Moreover, few statements in Section 4 need to be modified. In the proof Proposition 4.5 (p.10), the claim that "...we can find an isomoprhism $f \in \mathrm{Hom}(A_1 \times \widehat{A}_1, A_2 \times \widehat{A}_2)$ such that $\varphi = \mathrm{H}^1_{\mathrm{crys}}(f)$, ..." should be replaced by "...we can find an isogeny $f \in \mathrm{Hom}(A_1 \times \widehat{A}_1, A_2 \times \widehat{A}_2)$ such that $\mathrm{H}^1_{\mathrm{crys}}(f)$ is an isomorphism of abelian crystals, ...". Thus we cannot conclude from the rest part of the original proof. We now complete the proof for Proposition 4.5 with a strong approximation argument as follows.

Proof. As there is an isometry between abelian crystals $\mathrm{H}^1_{\mathrm{crys}}(A_1 \times \widehat{A}_1) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{crys}}(A_2 \times \widehat{A}_2)$, we have an isomorphism $A_1 \times \widehat{A}_1 \cong A_2 \times \widehat{A}_2$, by Ogus crystalline Torelli theorem for supersingular abelian varieties. This implies that A_1 and A_2 are isogenous. Choose an isogeny $\alpha\colon A_1 \to A_2$. Suppose $\deg \alpha = d$ and an isogeny $\beta\colon A_2 \to A_1$ such that $\alpha \circ \beta = [d]_{A_2}$. Let $\alpha^{-1} = \frac{\beta}{d} \in \mathrm{Hom}(A_2, A_1) \otimes \mathbb{Q}$, that is an invertible element. Consider the quasi-isogeny

$$f = \begin{pmatrix} \alpha & 0 \\ 0 & \widehat{\alpha^{-1}} \end{pmatrix} \in \mathrm{Hom}(A_1 \times \widehat{A}_1, A_2 \times \widehat{A}_2) \otimes \mathbb{Q}.$$

We can see $\deg(f) = 1$ and $\widetilde{f} = f^{-1}$. Now it is sufficient to find $g \in U(A_2)(\mathbb{Q})$ such that

$$\prod_{\ell \neq p} \mathrm{H}^1_{\text{ét}}(g \circ f, \mathbb{Z}_\ell) \times \mathrm{H}^1_{\mathrm{crys}}(g \circ f) \in U(A_1, A_2)(\widehat{\mathbb{Z}}).$$

This is because it this will force $g \circ f$ to be an isomorphism of abelian varieties and satisfies the symplecity condition.

For simplicity of notation, let $A = A_2$. Since A is isogenous to a product of supersingular elliptic curves, we have

$$\mathrm{End}(A) \otimes \mathbb{Q} \cong \mathrm{End}(\widehat{A}) \otimes \mathbb{Q} \cong M_n(D)$$

the set of $n \times n$ matrices in a quaternion algebra D. The quaternion algebra D is split at every place except p and ∞. Thus, we can see the group of self isogenis f of $A \times \widehat{A}$ such that $\widetilde{f} = f^{-1}$ is given as

$$U(A)(\mathbb{Q}) \cong \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_{2n}(D) \;\middle|\; \begin{pmatrix} {}^tA^\dagger & {}^tC^\dagger \\ {}^tB^\dagger & {}^tD^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \right\}$$

Since $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$, we can see $U(A)(\mathbb{R}) \cong \mathrm{SO}^*_{4n}$ as $n \geq 2$ (see [3, (3.3.2.3)] for example), a non-compact $\mathbb{R}$-form of $\mathrm{SO}(4n)$ of type $D^{\mathbb{H}}_{2n}$. This implies that $U(A)(\mathbb{C}) \cong \mathrm{SO}(4n)_{\mathbb{C}}$. Actually, $U(A)(\mathbb{Q})$ can be viewed as the $\mathbb{Q}$-rational points of an algebraic group over $\mathbb{Q}$ of type ${}^2D_{2n}$, denoted by G. The general theory of classical groups of type D implies that

$$G \cong \mathrm{SO}(V, h)$$

for some skew-hermitian form h on a free D-module V of dimension $2n$.

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Let $\pi\colon \mathrm{Spin} \to G$ be the universal covering of G, i.e., a simply-connected central isogeny of algebraic groups. It induces an exact sequence of finite adelic points:

$$\mathrm{Spin}(\mathbb{A}_f) \to G(\mathbb{A}_f) \xrightarrow{\mathrm{sn}} \mathbb{A}_f^\times/\mathbb{A}_f^{\times 2} \to 1$$

since the vanishing of Galois cohomology $H^1(\mathbb{Q}_\ell, \mathrm{Spin}) = 0$ for any prime ℓ. With the similar argument as Lemma 7.7 in [4], we can see that the spinor norm image

$$\mathrm{sn}(G(\mathbb{Q}) \cdot G(\widehat{\mathbb{Z}})) = \mathbb{A}_f^\times/\mathbb{A}_f^{\times 2}, \tag{0.0.1}$$

where $G(\widehat{\mathbb{Z}}) \subset G(\mathbb{A}_f)$ is the open compact group of $\widehat{\mathbb{Z}}$-points. Then π induces a surjection

$$\mathrm{Spin}(\mathbb{Q})\backslash \mathrm{Spin}(\mathbb{A}_f)/\pi^{-1}(G(\widehat{\mathbb{Z}})) \twoheadrightarrow G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}}),$$

and $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}})$ is a singleton by the srong approximation property of Spin (who is simply connected semisimple with non-compact real part). Actually, (0.0.1) can be deduced from the following facts.

(i) For any field F in characteristic zero, $G(F)$ is generated by reflections:

$$(s, \sigma)\colon x \mapsto x - h(s,s)\sigma^{-1}x,$$

with $\sigma \in (D \otimes_{\mathbb{Q}} F)^\times$, $s \in V_F$ and $h(s,s) = \sigma - \sigma^\dagger$. The spinor norm $\mathrm{sn}(s,\sigma) = \mathrm{Nrd}(\sigma) \cdot F^\times$, with the reduced norm $\mathrm{Nrd}\colon D \otimes_{\mathbb{Q}} F \to F$. (see [1, (12),(13)]). The reduced norm $D \to \mathbb{Q}$ maps $D^\times$ onto $\mathbb{Q}^+$.

(ii) As $\dim_D V = 2n \geq 3$ (under the assumption that $n \geq 2$), if $h_{\mathbb{R}}$ represents zero, then the skew-Hermitian form h represents all non-zero skew-quaternions in D, i.e., for all $a \in D^\times$ such that $a = -a^\dagger$ there is $s \in V$ such that $h(s,s) = a$ (see §4.3 Lemma 4 and §5.10 in [2]).

Fix integral isometries $\theta_\ell\colon H^1_{\text{ét}}(A_2 \times \widehat{A}_2, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(A_1 \times \widehat{A_1}, \mathbb{Z}_\ell)$ for all ℓ (for $\ell = p$, taking the crsytalline realization). This can be done for any $\ell \neq p$ since $\mathrm{H}^1_{\text{ét}}(A_i \times \widehat{A}_i, \mathbb{Z}_\ell)$ are both isometric to $(U \otimes \mathbb{Z}_\ell)^{\oplus n}$ for hyperbolic plane $U \otimes \mathbb{Z}_\ell = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and for $\ell = p$, this is given by our assumption. Let

$$\varphi \coloneqq \left(\prod_{\ell \neq p} \mathrm{H}^1_{\text{ét}}(f, \mathbb{Z}_\ell) \times \mathrm{H}^1_{\text{crys}}(f) \right) \circ (\theta_\ell) \in G(\mathbb{A}_f)$$

As we have seen the double quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}}) = \{1\}$, there is $g \in G(\mathbb{Q})$ such that $g \circ \varphi \in G(\widehat{\mathbb{Z}})$, and in particular

$$\prod_{\ell \neq p} \mathrm{H}^1_{\text{ét}}(g \circ f, \mathbb{Z}_\ell) \times \mathrm{H}^1_{\text{crys}}(g \circ f) \in U(A_1, A_2)(\widehat{\mathbb{Z}}). \qquad \square$$

Remark 0.1. If $n = 2$, then we have $D_2 = A_1 \times A_1$. In this case, the strong approximation argument fails.

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