MONODROMY RANK AND THE SEMISIMPLE MUMFORD–TATE CONJECTURE FOR HYPER-KÄHLER VARIETIES

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ABSTRACT. In this paper, we establish two main results concerning the Mumford–Tate conjecture for hyper-Kähler varieties. First, we prove the conjecture for the semisimplified ℓ-adic Galois representations attached to hyper-Kähler varieties with second Betti number $b_2 \geq 4$. As a direct consequence, we deduce that the Hodge conjecture implies the Tate conjecture for powers of hyper-Kähler varieties. Second, we show that the Mumford–Tate conjecture for hyper-Kähler varieties is invariant under deformation. The proofs rely on comparing the ranks of ℓ-adic algebraic monodromy groups in higher degrees to those in degree 2 via the theory of Frobenius tori and the Looijenga–Lunts–Verbitsky Lie algebra.

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1. INTRODUCTION

In the study of Galois representations of abelian varieties, Mumford and Tate [Mum66] formulated the following conjecture (see also [Ser64]). Let X be a smooth projective variety over a finitely generated field K over $\mathbb{Q}$.

Conjecture (MTC$_i$)**.** Fix a field embedding $K \subseteq \mathbb{C}$. Let $\overline{K}$ be the algebraic closure of K inside $\mathbb{C}$. For any prime ℓ, the isomorphism $\mathrm{GL}(\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)) \cong \mathrm{GL}(\mathrm{H}^i_B(X))_{\mathbb{Q}_\ell}$ via the Artin comparison isomorphism gives an identification

$$\mathbf{G}_{\ell,i}(X)^\circ = \mathbf{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_\ell}.$$

Here $\mathbf{G}_{\ell,i}(X)$ is the *ℓ-adic algebraic monodromy group* of $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$; and $\mathbf{MT}_i(X_{\mathbb{C}})$ is the *Mumford–Tate group* of the Hodge structure on the Betti cohomology $\mathrm{H}^i_B(X, \mathbb{Q})$.

In this paper, we propose a general approach to tackle the Mumford–Tate conjecture for higher-degree cohomologies of *hyper-Kähler varieties*, which are known to have a close relationship with abelian varieties. As a consequence, we show that the conjecture holds

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after taking semisimplification of the Galois representation. Moreover, we can show the validity of (MTC_i) is deformation invariant for hyper-Kähler varieties.

1.1. **Abelian varieties and their relatives.** To begin, we recall the case of abelian varieties. Let K be a field in characteristic zero. For simplicity, we assume that $K \subseteq \mathbb{C}$, a subfield of complex numbers.

1.1.1. For an abelian variety A over K, a celebrated theorem due to Deligne states that every Hodge cycle on A is an absolute Hodge cycle (see [DMOS82, I 2.11]). Consequently, there exists a natural injective homomorphism

$$\mathbf{G}_{\ell,i}(A)^\circ \hookrightarrow \mathbf{MT}_i(A_{\mathbb{C}})_{\mathbb{Q}_\ell} \tag{1.1.1}$$

for an abelian variety over $K \subseteq \mathbb{C}$ and any integer $0 \leq i \leq 2\dim(A)$. Therefore, to verify the Mumford–Tate conjecture for an abelian variety, it suffices to show that $\mathbf{G}_{\ell,i}(A)$ and $\mathbf{MT}_i(A_{\mathbb{C}})_{\mathbb{Q}_\ell}$ are isomorphic (or simply of equal dimension) as abstract $\mathbb{Q}_\ell$-algebraic groups. However, this problem remains open, even for abelian fourfolds (see [MZ95, Pin98]).

Deligne observed that (1.1.1) can be extended to a more general setting by considering the *abelian motives*. For example, the motives of curves, Fermat hypersurfaces, or K3 surfaces (see [DMOS82, II Proposition 6.26.]). Here, "motive" may refer to a Chow motive, or a motive in a weak sense of Deligne [DMOS82, II. §6] or André [And96b].

From a motivic perspective, one can also formulate the Mumford–Tate conjecture for Shimura varieties, viewing them as moduli spaces of Hodge structures of abelian motives with additional structure. Indeed, Deligne's theory of canonical models provides natural Galois representations for special subvarieties of Shimura varieties, allowing the Mumford–Tate conjecture to be stated for their generic points (see e.g., [UY13, Vas08]).

1.1.2. Beyond the case of abelian varieties, the Mumford–Tate conjecture faces more essential challenges. One of the primary difficulties for general smooth projective varieties is the lack of a proof for the absolute Hodge conjecture. Consequently, the existence of the injective homomorphism (1.1.1) remains unestablished in general. Without this inclusion, it is not clear how to identify these two algebraic groups via the comparison isomorphism

$$\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \cong \mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell,$$

even if one could show that $\mathbf{G}_{\ell,i}(X)^\circ \cong \mathbf{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ holds abstractly as an isomorphism of $\mathbb{Q}_\ell$-algebraic groups.

Another challenge for the conjecture is about the semisimplicity of Galois representations. In fact, the Mumford–Tate conjecture implies that $\mathbb{Q}_\ell$-algebraic groups $\mathbf{G}_{\ell,i}(X)^\circ$ are all reductive groups over $\mathbb{Q}_\ell$, since the Mumford–Tate groups are intrinsically reductive. Since the reductivity of $\mathbf{G}_{\ell,i}(X)^\circ$ is equivalent to the semisimplicity of the ℓ-adic étale cohomology $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ as a $\mathrm{Gal}(\overline{K}/K)$-representation, it is also predicted by

- the Tate conjecture regarding the algebraicity of Tate classes (see [Moo19]), which remains open except in a few cases; or
- Tate's conjecture on the semisimplicity of Frobenius over finite fields ([Tat65, §3 Conjecture (d)]) together with the Chebotarev density theorem.

We also refer to the reductivity of $\mathbf{G}_\ell(X)^\circ$ as the *semisimplicity conjecture* for X. Like the algebraicity of Tate classes, this has been verified in only a limited number of cases. For example, considering abelian varieties over a finitely generated field over $\mathbb{Q}$, it can be deduced from the Tate conjecture on the endomorphisms, proved by Faltings (see [FWG+92, IV, §1, 1.1, 1.4]).

1.2. **Hyper-Kähler varieties and the semisimple Mumford–Tate conjecture.** In higher dimensions, a natural direction is to study so-called *hyper-Kähler varieties*, which are K-trivial varieties that bear a close relationship with abelian varieties.

Definition 1.2.1. A hyper-Kähler variety over K is a smooth projective variety X over K such that

(1) $\mathrm{H}^1(X, \mathcal{O}_X) = 0$; and

(2) $\mathrm{H}^0(X, \Omega_{X/K})$ is K-linearly spanned by a *symplectic form*[1] $\sigma: \mathcal{O}_X \to \Omega^2_{X/K}$.

By definition, the presence of a symplectic form forces the dimension to be even, i.e., $\dim X = 2n$. The basic example is the surface case ($\dim X = 2$): smooth projective K3 surfaces are 2-dimensional hyper-Kähler varieties, and they are the only such examples among algebraic surfaces.

In higher dimensions ($2n \geq 4$), the picture changes substantially. Beauville [Bea83] constructed two families of deformation types:

- varieties of $\mathrm{K3}^{[n]}$-type, and
- varieties of Kum_n-type.

In dimensions 6 and 10, O'Grady constructed two further exceptional families of deformation types [O'G03, O'G99], commonly called of OG 6-type and OG 10-type. At present, these are all the known deformation types, and producing examples beyond these four remains a very difficult problem.

1.2.2. The Mumford–Tate conjecture has been verified for hyper-Kähler varieties in certain special cases. For degree 2 cohomology, it was established by Tankeev [Tan90, Tan95] (for $\dim X = 2$) and André [And96a].

Theorem 1.2.3 (Tankeev, André)**.** *Let X be a hyper-Kähler variety over a finitely generated field K with $b_2(X) \geq 4$. Then* (MTC_2) *holds for X.*

A natural question is whether one can extend this result to (MTC_i) for $2 < i < 2\dim X - 2$. In this paper, we employ algebraic group–theoretic methods to reduce the problem to degree 2, carefully analyzing the constraints on the ℓ-adic algebraic monodromy group $\mathbf{G}_\ell(X)$ inherent to hyper-Kähler varieties.

Our starting point is a general feature of the Hodge structures of hyper-Kähler varieties. Consider the Mumford–Tate group $\mathbf{MT}(X_\mathbb{C})$ associated with the full cohomology ring $\mathrm{H}^\bullet_B(X_\mathbb{C}, \mathbb{Q})$, viewed as a graded polarizable $\mathbb{Q}$-Hodge structure. For a hyper-Kähler variety, a theorem of Verbitsky (Theorem 3.1.9) establishes that the natural projection to the second cohomology group,

$$\mathbf{MT}(X_\mathbb{C}) \longrightarrow \mathbf{MT}_2(X_\mathbb{C}),$$

is an isogeny of degree ≤ 2 (see Lemma 3.2.5). Inspired by this result, we investigate the second degree projection in the Galois side. Our first main result is:

Theorem A (Theorem 4.3.2). *Let X be a hyper-Kähler variety over a finitely generated field over $\mathbb{Q}$. For any rational prime ℓ, we have*

$$\mathrm{rk}\, \mathbf{G}_\ell(X) = \mathrm{rk}\, \mathbf{G}_{\ell,2}(X)\,.$$

Here, $\mathbf{G}_\ell(X)$ is the ℓ-adic algebraic monodromy group of the full cohomology ring $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, and rk denotes the rank of a connected algebraic group (the dimension of a maximal torus of its geometric form).

[1] a nowhere degenerate closed 2-form.

Notably, Theorem A allows us to avoid appealing to the abelianicity of the motive $\mathfrak{h}(X)$ in order to see the injective homomorphism (1.1.1), at least for the reductive part.

1.2.4. Let $\rho^{ss}\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{\mathbb{Q}_\ell}\left(\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)\right)$ be the semisimplification of the Galois representation on $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, and let $\mathbf{G}^{\mathrm{red}}_{\ell,i}(X)$ denote the identity component of the Zariski closure of the image of ρ^{ss}. The subgroup $\mathbf{G}^{\mathrm{red}}_{\ell,i}(X) \subseteq \mathbf{G}_{\ell,i}(X)^\circ$ can be viewed as a Levi subgroup—a maximal reductive subgroup that, together with the unipotent radical, generates $\mathbf{G}_{\ell,i}(X)^\circ$.

Theorem A, combined with (MTC_2) for X establishes an isogeny $\mathbf{G}^{\mathrm{red}}_{\ell}(X) \sim \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$-algebraic groups. This implies an isomorphism

$$\mathfrak{g}^{\mathrm{red}}_{\ell}(X) \cong \mathfrak{mt}(X_{\mathbb{C}})_{\mathbb{Q}_\ell} \tag{1.2.1}$$

of $\mathbb{Q}_\ell$-Lie algebras, which corresponds to the original formulation of the Mumford–Tate conjecture in [Mum66] (specifically for abelian varieties). We further show that this isomorphism lifts to the level of algebraic groups and is compatible with the comparison isomorphism:

Theorem B (Theorem 5.6.4)**.** *Let X be a hyper-Kähler variety over a finitely generated field K over the prime field $\mathbb{Q}$. Suppose that the second Betti number $b_2(X) \geq 4$. The comparison isomorphism induces an identification of algebraic groups*

$$\mathbf{G}^{\mathrm{red}}_{\ell,i}(X) = \mathbf{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_\ell}\,.$$

for any field embedding $K \hookrightarrow \mathbb{C}$ and integer $0 \leq i \leq 2\dim X$.

We refer to the statement in Theorem B as the *semisimple Mumford–Tate conjecture* for X (in degree i).

1.2.5. We can see that Theorem B has a strong consequence on the relation between Hodge conjecture and Tate conjecture. The semisimple Mumford–Tate conjecture for X particularly implies that

$$\mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell} \subseteq \mathbf{G}_\ell(X)^\circ \tag{1.2.2}$$

for any prime ℓ. Recall that *Hodge tensors* and *Tate tensor* are defined to be the tensors that are invariant under the action of the groups $\mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ and $\mathbf{G}_\ell(X)$ respectively. Thus, the natural inclusion (1.2.2) shows that Tate tensors in any

$$\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\otimes m} \otimes \mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\vee,\otimes n} \otimes \mathbb{Q}_\ell(k)$$

can be written as certain $\mathbb{Q}_\ell$-linear combination of Hodge tensors in the corresponding tensor space of $\mathrm{H}^\bullet_B(X_{\mathbb{C}}, \mathbb{Q})$, under the comparison isomorphism.

With the Künneth formula, we derive the following implication relating the Hodge conjecture and the Tate conjecture for powers of hyper-Kähler varieties.

Corollary 1.2.6. *For any self-product $X^{\times m}$ of a hyper-Kähler variety X, over a finitely generated field K over the prime field $\mathbb{Q}$ with $b_2(X) \geq 4$, we have*

$$\boxed{\textit{Hodge conjecture in degree } 2k} \implies \boxed{\textit{Tate conjecture in degree } 2k}.$$

This result also appears in [Flo22b, Theorem 1.2], where it is obtained under the additional hypothesis that the odd-degree cohomology groups vanish when $m \geq 2$.

1.3. **Mumford–Tate conjecture in a family.** Previous works in [FFZ21] and [Sol22] used the deformation principle for motivated cycles to show that the property of being "abelian" is stable under deformation for André motives arising from hyper-Kähler varieties. This particular result implies that a motivic version of the Mumford–Tate conjecture is stable under deformation.

1.3.1. In this paper, we provide a variant of this deformation principle. Rather than working with motives, we study the behavior of the Mumford–Tate conjecture within a family. This approach offers greater flexibility, as it only requires dealing with the local system of ℓ-adic cohomologies. Our key observation is that *semisimplicity is preserved under geometric deformation* of hyper-Kähler varieties. Consequently, assuming Theorem B, (MTC_i) is also stable under deformation for any degree i.

Recall that two varieties X/K and Y/K' are *deformation equivalent* if there is a smooth projective family $f\colon \mathfrak{X} \to S$ over a geometrically connected smooth variety S (defined over a common subfield $K_0 \subseteq K \cap K'$), such that $X \cong \mathfrak{X}_s$ and $Y \cong \mathfrak{X}_{s'}$ for some $s \in S(K)$ and $s' \in S(K')$.

Theorem C. *Let X be a hyper-Kähler variety over a finitely generated field K over the prime field $\mathbb{Q}$. Suppose that the second Betti number $b_2(X) \geq 4$. For any $0 \leq i \leq 2\dim X$, the following statements are equivalent.*

(a) (MTC_i) *holds for X.*

(b) For all prime ℓ, the $\mathrm{Gal}(\overline{K}/K)$*-representation* $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ *is semisimple.*

(c) For any hyper-Kähler variety Y defined over a finitely generated field K' that is deformation equivalent to X, (MTC_i) *holds for Y/K'.*

By combining Theorem B and Theorem C, we can provide an alternative proof for the Mumford–Tate conjecture for hyper-Kähler varieties belonging to the four deformation types (see Section 5.7 below), a result previously proven in [FFZ21] and [Sol22].

Corollary 1.3.2. *If X is a* $\mathrm{K3}^{[n]}$, Kum_n, $\mathrm{OG}\,6$, *or* $\mathrm{OG}\,10$*-type variety, the* (MTC_i) *holds for any prime ℓ and $0 \leq i \leq 2\dim X$.*

1.4. **Strategy of proof.** Here, we briefly explain the ideas behind the proof of our main results.

1.4.1. *Rank comparison and p-adic methods.* The proof of Theorem A relies on classical arguments in the study of ℓ-adic algebraic monodromy for $\mathbb{Q}$-*compatible system* of global Galois representations, which dates back to the work of Serre [Ser86]. Inspired by the methods used in [Pin98], we reduce the problem to p-adic local Galois representations by considering *Frobenius tori* (see Section 4.1), establishing that

$$\mathrm{rk}(\mathbf{G}_\ell(X)) = \mathrm{rk}(\mathbf{G}_p(X \times_K K_v)),$$

for a "general" p-adic place v of a number field K (see Corollary 4.1.6).

A key hyper-Kähler input in the proof of Theorem A is a p-adic analogue of Verbitsky's theorem for Hodge–Tate representations of hyper-Kähler varieties over p-adic fields (Proposition 4.2.7). Following [IIK+25, Theorem 6.2], it is formulated in terms of the *Sen Lie algebra*. In particular, the *Looijenga–Lunts–Verbitsky (LLV) Lie algebra* [LL97, Ver96] contains the Sen Lie algebra over $\mathbb{Q}_p$.

1.4.2. *Big monodromy and specialization.* The proof of Theorem B is somewhat involved and proceeds in two main steps. For simplicity, we discuss here in the level of Lie algebras.

(1) We consider the decomposition of Mumford–Tate Lie algebra:

$$\mathfrak{mt}(X_{\mathbb{C}}) = \mathfrak{z} \oplus \mathfrak{s}$$

where $\mathfrak{z}$ is the center and $\mathfrak{s}$ is the semisimple part. The (local) Torelli theorem of hyper-Kähler varieties together with Zarhin's classification for Hodge groups of polarized Hodge structures of K3-type implies that $\mathfrak{s}$ is simple (over $\mathbb{Q}$), and is controlled by the geometric monodromy group of a $\mathbb{Q}$VHS with maximal monodromy. In other words, hyper-Kähler varieties have certain "big monodromy" property (see Proposition 5.3.4).

This allows us to identify semisimple parts of $\mathfrak{g}_\ell^{\mathrm{red}}(X)$ and $\mathfrak{mt}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ canonically via geometric monodromy groups. The assumption that the second Betti number $b_2 \geq 4$ is crucial here, ensuring that the geometric monodromy group is sufficiently large.

(2) When $X_{\mathbb{C}}$ is general in the universal family over a moduli space, $\dim \mathfrak{z} = 1$ (given by the weight cocharacter). This implies the compatibility of (1.2.1) for a generic member of the moduli space by Bogomolov's theorem (Lemma 2.2.11).

If $\dim \mathfrak{z} \geq 2$, then we need to apply a specialization argument. The key ingredient is the deformation invariance of the *(twisted) LLV representation* (Lemma 3.2.7). This implies that the action of $\mathfrak{g}_\ell^{\mathrm{red}}(X)$ on the cohomology ring—in particular, the cocharacters generating the center of $\mathfrak{g}_\ell^{\mathrm{red}}(X)$—is also always induced by the LLV representation (Proposition 5.3.8), by reducing to the case $\dim \mathfrak{z} = 1$.

1.4.3. *Semisimplicity at Galois generic fibers.* Theorem C is a direct application of the results by Cadoret in [Cad15], which shows that the reductivity at a single fiber would imply the reductivity at all ℓ-adic Galois generic points. For families of hyper-Kähler varieties, the reductivity at the Zariski generic fiber would also imply that at other fibers (Theorem 5.5.8).

1.5. **Relations with previous works and applications.** André [And96a, Theorem 1.5.1] established that the second-degree motive $\mathfrak{h}^2(X)$ is an abelian motive in the sense of [And96b]. Given the philosophy that hyper-Kähler varieties are largely governed by their second cohomology groups along with associated structures (such as their Hodge structures and Galois representations), it is natural to formulate the following conjecture:

Conjecture (Ab)**.** The motive $\mathfrak{h}(X)$ of a hyper-Kähler variety X is an abelian motive. In particular, there is a canonical injective homomorphism $\mathbf{G}_{\ell,i}(X)^\circ \hookrightarrow \mathbf{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$.

1.5.1. For hyper-Kähler varieties belonging to the four known deformation types ($\mathrm{K3}^{[n]}$, Kum_n, OG 6, and OG 10), Conjecture (Ab) has been verified by Floccari–Fu–Zhang [FFZ21] and Soldatenkov [Sol22]. Using the abelianicity of motives and [FFZ21, Corollary 6.11], the authors deduced the statements in Corollary 1.3.2.

Theorem 1.5.2. *If X is of deformation type* $\mathrm{K3}^{[n]}$, Kum_n*,* OG 6*, or* OG 10*, then* (Ab) *holds for X.*

The proof of Theorem 1.5.2 relies heavily on the specific geometry and topology of these deformation types. In contrast, our statement in Theorem B is independent of the deformation type. While it is strictly weaker than the original Mumford–Tate conjecture and does not directly address the abelianicity of motives, it provides a uniform framework for all hyper-Kähler varieties.

We also note that passing to the semisimplification is a standard and reasonable reduction in arithmetic geometry. The semisimple Mumford–Tate conjecture is sufficient for several applications.

1.5.3. In light of the complex geometry of hyper-Kähler varieties, the following is a natural question: *How much arithmetic information about a hyper-Kähler variety can be extracted from its degree 2 cohomology?*

When formulated in terms of ℓ-adic local monodromy operators, this question is referred to as the *ℓ-adic Nagai conjecture* [IIK+25]. In §6.1, we prove the ℓ-adic Nagai conjecture for hyper-Kähler varieties with Type I reduction over a p-adic local field; previously, this was established in *loc. cit.* only for the four known deformation types.

Theorem D (Theorem 6.1.1)**.** *Let X be a hyper-Kähler variety over a p-adic local field K_v, with $b_2(X) \geq 4$ and Type I reduction. Then* $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ *is potentially unramified for all $0 \leq i \leq 4n$ and all primes $\ell \neq p$.*

1.5.4. A classical question of Serre [Ser94] asks about the maximality of $\mathbb{Q}$-compatible systems $\{\rho_\ell\}$ of Galois representations arising from *maximal motives*. A weaker version of the Galois maximality conjecture for arbitrary smooth projective varieties was formulated by Larsen in [Lar95, §0]. Consider the following diagram of $\mathbb{Q}_\ell$-algebraic groups:

$$\begin{array}{ccc} \mathbf{G}_\ell & & \mathbf{G}_\ell^{\mathrm{sc}} \\ & \searrow^{\mathrm{ad}} \quad \swarrow & \\ & \mathbf{G}_\ell^{\mathrm{ad}} & \end{array}$$

where $\mathbf{G}_\ell^{\mathrm{sc}}$ is the simply connected cover of the adjoint group $\mathbf{G}_\ell^{\mathrm{ad}}$.

Let $\Gamma_\ell = \rho_\ell(G_K) \subseteq \mathbf{G}_\ell(\mathbb{Q}_\ell)$ be the image of the Galois representation, viewed as an ℓ-adic Lie group. Let $\Gamma_\ell^{\mathrm{sc}} \subseteq \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ be the preimage of $\Gamma_\ell^{\mathrm{ad}} = \mathrm{ad}(\Gamma_\ell)$ under the simply connected covering map.

Conjecture 1.5.5. For a $\mathbb{Q}$-compatible system $\{\rho_\ell\colon G_K \to \mathrm{GL}(\mathrm{H}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))\}$ of Galois representations arising from a smooth projective variety X over a finitely generated field K, the ℓ-adic Lie group $\Gamma_\ell^{\mathrm{sc}}$ is a maximal compact subgroup of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ for all sufficiently large primes ℓ. Moreover, $\Gamma_\ell^{\mathrm{sc}}$ is *hyperspecial*; that is, there exists a smooth affine group scheme $\mathscr{G}_\ell^{\mathrm{sc}}$ over $\mathbb{Z}_\ell$ such that $\mathscr{G}_\ell^{\mathrm{sc}}(\mathbb{Z}_\ell) = \Gamma_\ell^{\mathrm{sc}}$.

For hyper-Kähler varieties, the degree 2 case of Conjecture 1.5.5 was confirmed in [HL20, Theorem 1.3]. With Theorem A, we can generalize Hui–Larsen's theorem to the full cohomology of hyper-Kähler varieties with $b_2(X) \geq 4$, which provides new examples for Conjecture 1.5.5.

Theorem E (Theorem 6.2.1)**.** *Let X be a hyper-Kähler variety over a finitely generated field K with $b_2(X) \geq 4$. Then the ℓ-adic Lie group $\Gamma_\ell^{\mathrm{sc}} \subseteq \mathbf{G}_\ell(X)^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is a hyperspecial maximal compact subgroup for all sufficiently large primes ℓ.*

Outline. §2 and §3 are preliminary sections. In §2, we summarize elementary facts concerning André motives and the algebraic groups arising from them. We also recall the $\mathbb{Q}$-compatibility of systems of Galois representations. In §3, we review the definition of the LLV Lie algebra of a smooth projective variety, as well as well-known properties of the LLV Lie algebra of a hyper-Kähler variety.

In §4, we prove Theorem A. To provide the framework for our proof, we recall the construction of Frobenius tori and Pink's comparison theory between the global Frobenius rank and the local crystalline Frobenius rank.

In §5, we prove Theorem B and Theorem C. Furthermore, in §5.7, we apply our main results to reprove the Mumford–Tate conjecture for varieties of $\mathrm{K3}^{[n]}$, Kum_n, OG 6, and OG 10 type.

Finally, §6 is devoted to the proof of the arithmetic Nagai conjecture for Type I reductions (Theorem D) and the maximality of the Galois action (Theorem E).

Notation.

- In this paper, a variety X over K means an integral K-scheme X that is separated and of finite type over K.
- We will use the superscript "cl" to denote the set of closed points of a variety, e.g., X^{cl} for the closed points of X. Note that in the literature, it is also often denoted as $|X|$. We reserve the notation $|-|$ for the cardinality of a set, especially a finite one.
- For an object Y (e.g., an algebraic variety, or a motive) defined over a field E, we denote Y_F for the base change from E to a field extension F/E.

– For any algebraic group $\mathbf{G}$, we denote $\mathbf{G}^\circ$ for its identity connected component.

– For a connected algebraic group $\mathbf{G}$ over a field, we denote $\mathbf{G}^{\mathrm{der}}$ as the derived subgroup of $\mathbf{G}$ (the subgroup generated by the commutator $[\mathbf{G},\mathbf{G}] \subseteq \mathbf{G}$).

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2. Motivic Galois group and compatible system

In this section, we summarize the elementary facts concerning André motives and the algebraic groups arising from them. We also fix the notation that will be frequently used in the subsequent discussion.

2.1. **André motives and realizations.** The concept of motivated motives, now referred to as André motives and developed by Y. André [And96b], offers a practical framework for analyzing algebraic groups arising from various cohomological realizations.

2.1.1. Fix a field embedding $K \subseteq \mathbb{C}$. Let X be a smooth projective variety over K. Recall that a class $\alpha \in \mathrm{H}^{2i}(X_{\mathbb{C}}, \mathbb{Q})$ is *motivated* if there is a smooth projective variety Y over K and algebraic classes β, $\gamma \in \mathrm{H}^\bullet((X \times_K Y)_{\mathbb{C}}, \mathbb{Q})$ such that $\alpha = p_*(\gamma \cup \star\beta)$ (with $p\colon X \times_K Y \to X$ the projection, and $\star$ the Hodge star operator). The category $\mathbf{Mot}_K$ of André motives consists of objects (X, π, m) where

– X is a smooth projective variety over K,

– π is a motivated cycle on $X \times_K X$ such that $\pi \circ \pi = \pi$, and

– m is an integer.

The Hom sets of $\mathbf{Mot}_K$ are given as

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Mot}_K}\big((X, \pi_X, m), (Y, \pi_Y, n)\big)\\ =&\left\{\gamma \in \mathrm{H}^{m-n}\big((X \times_K Y)_{\mathbb{C}}, \mathbb{Q}\big) \;\middle|\; \begin{array}{l}\gamma \text{ is motivated,}\\ \gamma = \pi_Y \circ \gamma \circ \pi_X.\end{array}\right\}.\end{aligned}$$

Denote $\mathfrak{h}(X)$ for the André motive $(X, \Delta_X, 0)$ for a smooth projective variety over K. The category $\mathbf{Mot}_K$ satisfies the following properties (see [And96b, Theorem 0.4]):

– It has a natural tensor product structure and is a graded Tannakian category over $\mathbb{Q}$;

– It is semisimple and polarized.

The gradation of $\mathbf{Mot}_K$ induces the Chow–Künneth decomposition for each smooth projective variety X over K:

$$\mathfrak{h}(X) \cong \bigoplus_{i=0}^{2\dim X} \mathfrak{h}^i(X),$$

where $\mathfrak{h}^i(X) = (X, \Delta_i, 0)$, since the Künneth factor $\Delta_i \in \mathrm{H}^{2\dim X}((X \times_K X)_{\mathbb{C}}, \mathbb{Q})$ is motivated (see [And96b, Proposition 2.2.]).

2.1.2. There is a *Betti realization functor*

$$\mathrm{H}_B\colon \mathbf{Mot}_K \longrightarrow \mathbf{Vect}_{\mathbb{Q}}\,,$$

such that $\mathrm{H}_B(\mathfrak{h}(X)) = \mathrm{H}^\bullet(X_{\mathbb{C}}, \mathbb{Q})$ is the graded $\mathbb{Q}$-algebra of Betti cohomologies for any smooth projective variety X over K.

On the other hand, one may also consider the ℓ-adic realization

$$\mathrm{H}_\ell\colon \mathbf{Mot}_K \longrightarrow \mathbf{Vect}_{\mathbb{Q}_\ell}\,,$$

such that $\mathrm{H}_\ell(\mathfrak{h}(X)) = \mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which are the ℓ-adic étale cohomologies, a graded $\mathbb{Q}_\ell$-algebra endowed with a natural G_K-action, where G_K is the absolute Galois group of K. The Artin comparison provides a natural equivalence from the ℓ-adic realization H_ℓ to the composition of the following functors

$$\mathrm{H}_{B,\mathbb{Q}_\ell}\colon \mathbf{Mot}_K \xrightarrow{\mathrm{H}_B} \mathbf{Vect}_{\mathbb{Q}} \xrightarrow{-\otimes_{\mathbb{Q}}\mathbb{Q}_\ell} \mathbf{Vect}_{\mathbb{Q}_\ell}$$

for any prime ℓ.

2.2. **Motivic Galois groups and Mumford–Tate conjecture.** As before, we fix a subfield $K \subseteq \mathbb{C}$. We denote $\mathbf{G}_{\text{mot}}$ for the motivic Galois group of $\mathbf{Mot}_K$, i.e., the automorphism group of the fiber functor H_B, which is a reductive pro-algebraic group over $\mathbb{Q}$. For a single smooth projective variety X/K, we can similarly define its motivic Galois group as follows.

Definition 2.2.1. Let X be a smooth projective variety over K. The *motivic Galois group* $\mathbf{G}_{\text{mot}}(X)$ of X is the automorphism group of the composition (as a fiber functor) of the Betti realization functor H_B on the Tannakian subcategory generated by the André motives $\mathfrak{h}(X)$ and $\mathfrak{h}(X)^\vee \in \mathbf{Mot}_K$.

2.2.2. Since the sub-Tannakian category $\langle \mathfrak{h}(X)\rangle^{\otimes}$ is semisimple, the identity component of $\mathbf{G}_{\text{mot}}(X)$ is reductive[2], denoted by $\mathbf{G}_{\text{mot}}(X)^\circ$. Similarly, we can replace $\mathfrak{h}(X)$ with $\mathfrak{h}^i(X)(m)$ or an arbitrary object $\mathcal{M} \in \mathbf{Mot}_K$, and the same result holds.

Remark 2.2.3. Since the Betti realization functor H_B is based on the choice of the field embedding $K \subseteq \mathbb{C}$, the formation of the motivic Galois group $\mathbf{G}_{\text{mot}}(\mathcal{M})$ also depends on this choice. However, the group is invariant up to inner twists. See [And96b, Remarks on p.25].

Notation 2.2.4. We use the subscript "$\square$" in $\mathbf{G}_{\text{mot},\square}(X)$ for the motivic Galois group of $\mathfrak{h}^\square(X)$. A typical usage of $\square$ can be any integer $0 \le i \le 2\dim X$, and moreover it can also be $+$ and $-$, which stand for the even degree part

$$\mathfrak{h}^+(X) \coloneqq \bigoplus_{k=0}^{\dim X} \mathfrak{h}^{2k}(X)\,,$$

and the odd degree part

$$\mathfrak{h}^-(X) \coloneqq \bigoplus_{k=0}^{\dim X-1} \mathfrak{h}^{2k+1}(X)\,,$$

respectively. We also follow the same rules of notation for the subsequent Galois groups arising from cohomological realizations.

[2] To the best of our knowledge, it is not clear whether $\mathbf{G}_{\text{mot}}(X)$ is connected or not.

2.2.5. As mentioned above, there is a gradation on $\mathbf{Mot}_K$, which is actually defined by the Betti realization functor, namely, the Betti realization functor H_B factors as

$$\mathbf{Mot}_K \longrightarrow \mathbf{Gr\,Vect}_{\mathbb{Q}} \xrightarrow{\text{for.}} \mathbf{Vect}_{\mathbb{Q}} \,.$$

The Tannakian group of the forgetful functor of the category of graded $\mathbb{Q}$-vector spaces is just $\mathbb{G}_m$, the multiplicative group over $\mathbb{Q}$. The Tannakian formalism induces a homomorphism

$$\mathbb{G}_m \longrightarrow \mathbf{G}_{\mathrm{mot}}\,,$$

denoted by ω. For any smooth projective variety X over K, the action of $\mathbb{G}_m$ on $\mathfrak{h}^i(X) \subseteq \mathfrak{h}(X)$ is given by $z \mapsto z^i$, coinciding with the cohomology degree of X.

2.2.6. Let $\phi\colon \mathbb{S} \to \mathrm{GL}_{\mathbb{R}}(V_{\mathbb{R}})$ be a pure Hodge structure on a $\mathbb{Q}$-vector space V, i.e., an algebraic homomorphism from the Deligne torus $\mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}}$. The *Mumford–Tate group* $\mathbf{MT}(V,\phi)$ associated to (V,ϕ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that $\phi(\mathbb{S}) \subseteq \mathbf{MT}(V,\phi)(\mathbb{R})$. From the definition, it is clear that $\mathbf{MT}(V,\phi)$ is *connected*. When (V,ϕ) is polarizable, the Mumford–Tate group $\mathbf{MT}(V,\phi)$ is, moreover, *reductive*. Denote $\mathbf{HS}_{\mathbb{Q}}$ for the Tannakian category generated by polarizable $\mathbb{Q}$-Hodge structures.

Let $\mathcal{M} \in \mathbf{Mot}_K$. The Betti realization $\mathrm{H}_B(\mathcal{M})$ of $\mathcal{M}$ is naturally a polarizable $\mathbb{Q}$-Hodge structure. For a smooth projective variety X over K. We denote $\mathbf{MT}(X_{\mathbb{C}})$ for the Mumford–Tate group of the $\mathbb{Q}$-Hodge structure on $\mathrm{H}_B(\mathfrak{h}(X)) = \mathrm{H}^{\bullet}(X_{\mathbb{C}},\mathbb{Q})$.

It is also convenient to describe Mumford–Tate groups via the Tannakian formalism. The forgetful functor $\text{for.}\colon \mathbf{HS}_{\mathbb{Q}} \to \mathbf{Vect}_{\mathbb{Q}}$ is a fiber functor from $\mathbf{HS}_{\mathbb{Q}}$. Then the Mumford–Tate group of a pure $\mathbb{Q}$-Hodge structure V is isomorphic to

$$\mathrm{Aut}^{\otimes}(\text{for.}|_{\langle V\rangle})\,,$$

the automorphism group of the restricted fiber functor on the sub-Tannakian category $\langle V\rangle$ generated by V (see [And92, Lemma 2 & Remark]). In general, for any object $V \in \mathbf{HS}_{\mathbb{Q}}$, i.e., a subquotient of a direct sum $\oplus_m V_m$ in which V_m is a polarizable pure $\mathbb{Q}$-Hodge structure of weight m, we can define its Mumford–Tate group in the same way. In summary, via the Tannakian formalism,

(1) We can view the Mumford–Tate group as the *maximal* subgroup of $\mathrm{GL}_{\mathbb{Q}}(V)$, whose induced action on $V^{\otimes}$ fixes all Hodge tensors of type $(0,0)$.

(2) The Betti realization factors as

$$\begin{array}{ccc} \langle \mathfrak{h}(X)\rangle^{\otimes} & \xrightarrow{\quad \mathrm{H}_B \quad} & \mathbf{Vect}_{\mathbb{Q}} \,. \\ & \searrow \qquad \nearrow_{\text{for.}} & \\ & \mathbf{HS}_{\mathbb{Q}} & \end{array}$$

Also note that the Betti realization H_B (with respect to the fixed field embedding $K \subseteq \mathbb{C}$) induces an injective homomorphism

$$\mathbf{MT}(X_{\mathbb{C}}) \hookrightarrow \mathbf{G}_{\mathrm{mot}}(X)^{\circ}\,, \tag{2.2.1}$$

since the motivated cycles are Hodge tensors. Here $\mathbf{MT}(X_{\mathbb{C}})$ is the Mumford–Tate group of the (graded) polarizable $\mathbb{Q}$-Hodge structure on $\mathrm{H}^{\bullet}(X_{\mathbb{C}},\mathbb{Q})$.

- The Tannakian category $\mathbf{HS}_{\mathbb{Q}}$ has a natural $\mathbb{Z}$-gradation, in particular, there is a homomorphism

  $$\omega_{\phi}\colon \mathbb{G}_{m,\mathbb{R}} \longrightarrow \mathbf{MT}(V,\phi)_{\mathbb{R}}$$

  defined over $\mathbb{Q}$ for any Hodge structure (V,ϕ), which is called the *weight cocharacter* of (V,ϕ).

– If $V = \mathrm{H}_B(\mathfrak{h}(X))$, then we can see that the composition of ω_ϕ with the injective homomorphism $\mathbf{MT}(X_\mathbb{C}) \hookrightarrow \mathbf{G}_{\mathrm{mot}}(X)$ is the homomorphism $\omega: \mathbb{G}_m \to \mathbf{G}_{\mathrm{mot}}(X)$ in 2.2.2. Clearly, ω_ϕ is non-trivial if and only if V has non-zero weights; and the image lies in the center, $\omega_\phi(\mathbb{G}_m) \subseteq Z(\mathbf{MT}(V, \phi))$.

2.2.7. Let K be a field in characteristic zero and $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. For simplicity, we may assume that there is a field embedding $K \hookrightarrow \mathbb{C}$, and again we fix it.

Definition 2.2.8. Let $\rho_\ell: G_K \to \mathrm{GL}_{\mathbb{Q}_\ell}(V)$ be a continuous finite dimensional representation of G_K. The *ℓ-adic algebraic monodromy group* of V is the Zariski closure of the image $\mathrm{Im}(\rho_\ell) \subseteq \mathrm{GL}_{\mathbb{Q}_\ell}(V)$, denoted by $\mathbf{G}_\ell(V)$.

With the Tannakian formalism, we can identify the group $\mathbf{G}_\ell(V)$ as the Tannakian group of the restriction of the forgetful functor from the category of G_K-representations to the category of $\mathbb{Q}_\ell$-vector spaces $\mathbf{Vect}_{\mathbb{Q}_\ell}$.

2.2.9. For any $\mathcal{M} \in \mathbf{Mot}_K$, the Artin comparison induces an isomorphism of $\mathbb{Q}_\ell$-algebraic groups

$$\mathrm{Aut}^\otimes\left(\mathrm{H}_\ell|_{\langle\mathcal{M}\rangle^\otimes}\right) \cong \mathrm{Aut}^\otimes\left(\mathrm{H}_{B,\mathbb{Q}_\ell}|_{\langle\mathcal{M}\rangle^\otimes}\right) = \mathbf{G}_{\mathrm{mot}}(\mathcal{M})_{\mathbb{Q}_\ell}$$

for any prime ℓ. Therefore, via the Tannakian duality, there is an injective homomorphism

$$\mathbf{G}_\ell(\mathcal{M})^\circ \hookrightarrow \mathbf{G}_{\mathrm{mot}}(\mathcal{M})_{\mathbb{Q}_\ell}\,. \tag{2.2.2}$$

Therefore, the Mumford–Tate conjecture holds for $\mathcal{M}$ if and only if

$$\mathbf{G}_\ell(\mathcal{M})^\circ = \mathbf{MT}(\mathcal{M}_\mathbb{C})_{\mathbb{Q}_\ell}$$

as $\mathbb{Q}_\ell$-subgroups of $\mathbf{G}_{\mathrm{mot}}(\mathcal{M})_{\mathbb{Q}_\ell}$.

2.2.10. Keep the notation the same as in Notation 2.2.4. Let $V = \mathrm{H}_\ell(\mathfrak{h}(X))$ be a G_K-representation of a smooth projective variety X over K. The image of ρ_ℓ is a compact ℓ-adic Lie group, whose Lie algebra is denoted by $\mathfrak{g}_\ell(X)$ (resp. $\mathfrak{g}_{\ell,i}(X)$). An algebraicity theorem of Bogomolov [Bog80, Theorem 1] (see also [Ser86, §2]) implies that $\mathrm{Im}(\rho_\ell) \subseteq \mathbf{G}_\ell(X)$ is open and the Lie algebra $\mathfrak{g}_\ell(X)$ is algebraic, i.e.,

$$\mathfrak{g}_\ell(X) = \mathrm{Lie}(\mathbf{G}_\ell(X))\,.$$

The following fact is well-known as a consequence of Bogomolov's algebraicity theorem and the Weil conjecture (see [Bog80, Corollarie 1]). We include a proof here for the sake of completeness.

Lemma 2.2.11. *If K is a finitely generated field over $\mathbb{Q}$, then the image of weight cocharacter*

$$\mathbb{G}_{m,\mathbb{Q}_\ell} \subseteq Z(\mathbf{G}_\ell(X)^\circ) \subset \mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell} \tag{2.2.3}$$

coincides with the image $\omega(\mathbb{G}_m)_{\mathbb{Q}_\ell}$ in $\mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell}$.

Proof. It suffices to show that the group of homotheties satisfies $\mathbb{G}_{m,\mathbb{Q}_\ell} \subset \mathbf{G}_\ell(X)^\circ$. We may assume that K is a number field by Serre's spreading out argument. Take v to be a place of K such that X has good reduction at v. The Weil conjecture implies that the eigenvalues of the Frobenius action $\Phi_v \curvearrowright \mathrm{H}^i_{\text{ét}}(X_{\overline{K}_v}, \mathbb{Q}_\ell)$ are Weil q-numbers of weight i. Fix a torus $T \subset \mathbf{G}_\ell(X)^\circ$ that contains the semisimple part Φ_v^{ss}. Then the same argument as in [Chi92, Proposition (3.2)] shows that the group of homotheties $\mathbb{G}_{m,\mathbb{Q}_\ell} \subset T$. □

2.3. **Compatibility of systems of Galois representations.** In this section, we focus on the case where K is a *number field*, and the $\mathbb{Q}$-*compatibility* for a system of G_K-representations.

2.3.1. Consider a profinite group $\mathscr{G}$, and a system of continuous n-dimensional representations

$$\rho_\bullet = \{\rho_\ell : \mathscr{G} \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)\}_{\ell \in \mathbf{L}}\,,$$

indexed by a set $\mathbf{L}$ of rational primes. Suppose that $\mathscr{G}$ is endowed with a *dense* subset of "Frobenius elements",

$$\{F_v\}_{v\in\Sigma} \subset \mathscr{G}\,.$$

For example, if $\mathscr{G} = G_K$ is the absolute Galois group of a number field K and $F_v = \mathrm{Frob}_v$ is the Frobenius representative for the place v of K, then all the Frobenius conjugacy classes ranging over all finite places form a dense subset by the Chebotarev density theorem.

The system $\rho_\bullet$ is called a $\mathbb{Q}$-*compatible system* of ℓ-adic representations if there is a subset $\mathscr{X} \subset \Sigma \times \mathbf{L}$ such that

(1) For every $v \in \Sigma$, $(v, \ell) \in \mathscr{X}$ for all but at most finitely many $\ell \in \mathbf{L}$.

(2) For any primes $\ell_1, \ldots, \ell_m \in \mathbf{L}$, the set $\{F_v \mid (v, \ell_i) \in \mathscr{X}$ for all $i = 1, \ldots, m\}$ is dense in $\mathscr{G}$.

(3) For every $v \in \Sigma$, the characteristic polynomial $ch(\rho_\ell(F_v)) \in \mathbb{Q}[t]$, and is independent of ℓ.

2.3.2. In the geometric context, we consider the following situation.

– Let $\mathcal{M} = \mathfrak{h}^\square(X)$ be a motive associated to a smooth proper variety X over K.

– Σ is the following subset of the places of K:

$$\left\{\begin{matrix}\text{non-archimedean places of } K \text{ such that} \\ K \text{ is unramified at } v\text{; and } X \text{ has good} \\ \text{reduction at } v.\end{matrix}\right\}, \tag{2.3.1}$$

where K is a number field.

– Let $F_v = \mathrm{Frob}_v \in G_K$ be a Frobenius lift, which is unique as K is unramified at v.

Since X has good reduction at v, the system of Galois representations

$$\rho_\bullet = \{\rho_\ell : G_K \longrightarrow \mathrm{GL}_{\mathbb{Q}_\ell}(\mathrm{H}_\ell(\mathcal{M}))\}_{\ell\in\mathbf{L}}$$

is unramified at all $v \in \Sigma$ by smooth proper base changes when $\ell \neq \mathrm{char}(k_v)$.

It is worth noticing that the system $\rho_\bullet$ is $\mathbb{Q}$-compatible, and can be described by p-adic cohomology theory. The p-adic étale realization $\mathrm{H}_p(\mathcal{M}_v)$ is *crystalline* as a G_{K_v}-representation for $\mathcal{M}_v = \mathfrak{h}^\square(X_v)$, i.e.,

$$\dim_{\mathbb{Q}_p} \mathrm{H}_p(\mathcal{M}_v) = \dim_{K_0} \left(\mathrm{H}_p(\mathcal{M}_v) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{crys}}\right)^{G_{K_v}}\,,$$

where $\mathrm{B}_{\mathrm{crys}}$ is the Fontaine's crystalline period ring of K. Fontaine's formalism provides a functor

$$\mathrm{D}_{\mathrm{crys}} : \mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_{K_v}) \longrightarrow \mathbf{MF}_{\varphi, K_v}, \quad V \rightsquigarrow \left(V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{crys}}\right)^{G_{K_v}}$$

from the category of crystalline representations of G_{K_v} to the category of filtered φ-modules over K, which is *fully faithful*. Let K_0 be the maximal unramified subfield of K_v, i.e., the fraction field of the ring of Witt vectors $W(\mathbb{F}_q)$. Let σ be the Frobenius on K_0. For $v \in \Sigma_{\max}$, as v is unramified, we have $K_v = K_0$. Recall that the filtered φ-module

$$\mathrm{D}_{\mathrm{crys}}(\mathcal{M}_v) \coloneqq \mathrm{D}_{\mathrm{crys}}(\mathrm{H}_p(\mathcal{M}_v))\,,$$

is a K_0-vector space, endowed with a σ-semilinear Frobenius φ_v. Recall that being σ-semilinear means that

$$\varphi_v(ax) = \sigma(a)\varphi_v(x), \quad \forall a \in K_0 \text{ and } x \in \mathrm{D}_{\mathrm{crys}}(\mathcal{M}_v)\,.$$

Let $m = [\mathbb{F}_q : \mathbb{F}_p]$. The power $\Phi_v \coloneqq \varphi_v^m$ is a K_0-linear automorphism of $\mathrm{D}_{\mathrm{crys}}(\mathcal{M}_v)$. As X has good reduction at v, then $\mathrm{D}_{\mathrm{crys}}(\mathcal{M}_v)$ (together with Frobenius structure φ_v) is the same as the crystalline cohomology of the special fiber. According to Katz–Messing [KM74, Theorem 1], the eigenvalues of Φ_v are equal to those of $\rho_\bullet(F_v)$.

Remark 2.3.3. In general, it is not clear whether the system of ℓ-adic realizations of an André motive $\mathcal{M}$ is $\mathbb{Q}$-compatible or not, except when $\mathcal{M} \subseteq \mathfrak{h}^\square(X)$ is a submotive cut out by an algebraic cycle.

2.3.4. Let $\rho_\bullet$ be a $\mathbb{Q}$-compatible system of representations of G_K. Denote $\mathbf{G}_\ell$ the ℓ-adic algebraic monodromy group of $\rho_\ell(G_K)$ for each prime $\ell \in \mathbf{L}$ for simplicity. As before, $\mathbf{G}_\ell^\circ$ is the connected component of the identity. The set of connected components $\mathbf{G}_\ell/\mathbf{G}_\ell^\circ$ is finite, so there is a finite field extension K^{conn} over K, corresponding to the finite index subgroup $\mathscr{G} = \rho_\ell^{-1}(\mathbf{G}_\ell^\circ(\mathbb{Q}_\ell)) \subseteq G_K$, such that

$$\rho_\ell(G_{K^{\mathrm{conn}}}) \subset \mathbf{G}_\ell^\circ(\mathbb{Q}_\ell)$$

is Zariski dense. A priori, it is unclear whether there exists a single finite extension that works uniformly for all $\ell \in \mathbf{L}$ when $\mathbf{L}$ is infinite. Nevertheless, we have the following theorem of Serre (see [Ser86], or alternatively, for example, Larsen–Pink [LP92, Proposition 6.14]).

Theorem 2.3.5. *The open subgroup of finite index*

$$\mathscr{G} \subset G_K$$

is independent of ℓ. In particular, the groups $\mathbf{G}_\ell/\mathbf{G}_\ell^\circ$ for different ℓ are canonically isomorphic; if $\mathbf{G}_\ell$ is connected for some ℓ, then it is so for all ℓ.

Remark 2.3.6. Therefore, given a $\mathbb{Q}$-compatible system of representations $\rho_\bullet$ (even when $\mathbf{L}$ is infinite), we can always assume that $K^{\mathrm{conn}} = K$ after a finite extension by Theorem 2.3.5.

Remark 2.3.7. Let $\mathbf{G}_\ell$ be the ℓ-adic algebraic monodromy group of the motive $\mathfrak{h}(X)$. For any submotive $\mathcal{M} \subseteq \mathfrak{h}(X)$, e.g., $\mathfrak{h}^i(X)$, there is a surjection $\mathbf{G}_\ell \twoheadrightarrow \mathbf{G}_\ell(\mathcal{M})$. Therefore, if $K = K^{\mathrm{conn}}$, then $\mathbf{G}_\ell(\mathcal{M})$ is also connected.

2.3.8. Finally, we make some remarks on the case where K is not a number field. If $[K : \mathbb{Q}] = \infty$, then one can also define the (quasi-)compatibility of the representations of G_K after choosing a model of K (see [LP92, §6] or Commelin [Com19, §3]). Let K be a finitely generated field over $\mathbb{Q}$, and B a normal scheme of finite type over $\mathbb{Z}$ with the function field K, called a normal model of K. For a closed point $s \in B$, there is an exact sequence

$$1 \longrightarrow I_s \longrightarrow G_{K_s} \longrightarrow G_{k(s)} \longrightarrow 1\,,$$

where K_s is the function field of the Henselization B_s^h at s. An element F_s in G_K is a *Frobenius element with respect to* s if there is a field embedding $K \hookrightarrow K_s$ such that F_s is the restriction of a Frobenius element in K_s. The Chebotarev density theorem holds: For any finite étale Galois covering $\widetilde{B} \to B$ with $\widetilde{B}$ integral and the Galois group G, the following subset of the closed points of B

$$\left\{x \in B^{\mathrm{cl}} \;\middle|\; \mathrm{Conj}_G(F_s) = C\right\}$$

is of density $\frac{|C|}{|G|}$ for any conjugacy class $C \subset G$ (see [Pin97, Theorem B.9]). Then we can set Σ to be a Zariski dense subset of B^{cl} as discussed in 2.3.1. When K is a number field, the only model of K is $\mathrm{Spec}(\mathcal{O}_K)$ with $\mathcal{O}_K$ the ring of integers in K, and the definition of the Frobenius element here coincides with the one for a number field.

However, it is unclear whether a system of ℓ-adic representations coming from geometry is compatible with those given in 2.3.2. Fortunately, we can still reduce many problems on $\mathbf{G}_\ell$ to the number field case, by the classical spreading-out argument as in [Ser86, §1]. Namely, there are infinitely many closed points $s \in B^{\mathrm{cl}}$ such that $\mathbf{G}_{\ell,s} \cong \mathbf{G}_\ell$ (see [Ser97, §10.6 Theorem]). In particular, we see that Theorem 2.3.5 holds when K is an arbitrary finitely generated field over $\mathbb{Q}$.

3. LLV representations of hyper-Kähler varieties

3.1. Looijenga–Lunts–Verbitsky Lie algebra.

We now briefly review the basic properties of the *Looijenga–Lunts–Verbitsky Lie algebra* (LLV algebra) introduced by Verbitsky [Ver96] and Looijenga–Lunts [LL97].

3.1.1. Let $K \subseteq \mathbb{C}$ be a field embeddable to $\mathbb{C}$, X be a smooth projective variety over K as before. Denote $\mathrm{H}(X) \coloneqq \mathrm{H}^\bullet_\square(X)$ a cohomological realization of $\mathfrak{h}(X)$, with $\square = B, \ell, \mathrm{dR}$ or pst. Let E be the field of coefficients of $\mathrm{H}(-)$.

The LLV algebra $\mathfrak{g}$ for X/K and the associated LLV decomposition of $\mathrm{H}(X)$ generalize the usual Hard Lefschetz $\mathfrak{sl}_2$-decomposition of the cohomology for an ample class ω on X. Recall that the class ω defines two operators on the cohomology, one is the Lefschetz operator $L_\omega = \omega \cup -$ given by the cup product, and the other is the dual Lefschetz operator $\Lambda_\omega = \star^{-1} L_\omega \star$, where $\star$ is the Hodge star operator. Operators L_ω and Λ_ω generate an $\mathfrak{sl}_2 \subset \mathfrak{gl}(\mathrm{H}(X))$ acting on $\mathrm{H}(X)$, and the Hard Lefschetz is the same as the existence of the $\mathfrak{sl}_2$-decomposition of the cohomology. A more formal framework that avoids the use of the Hodge star operator has been formulated in [LL97], where the key observation is the following identity

$$[L_\omega, \Lambda_\omega] = h\,,$$

with h the shifted degree operator

$$h\colon \mathrm{H}(X) \longrightarrow \mathrm{H}(X),\ \ x \longmapsto (k - \dim X)x,\ \ \text{for } x \in \mathrm{H}^k(X)\,. \tag{3.1.1}$$

It is then clear that $\{L_\omega, h, \Lambda_\omega\}$ forms an $\mathfrak{sl}_2$-triple. Moreover, the operator $L_x = x \cup -$ is well-defined for any cohomology class $x \in \mathrm{H}^2(X)$, while h is independent of any choice of x. Due to the Jacobson–Morozov Theorem, the existence (and thus the uniqueness) of an operator Λ_x that completes the pair $\{L_x, h\}$ into an $\mathfrak{sl}_2$-triple is an open algebraic condition on the classes $x \in \mathrm{H}^2(X)$. In other words, the dual Lefschetz operator Λ_x can be defined for almost all classes x, which can go beyond the ample (or even Kähler) classes and is in fact independent of the complex structure of X. Therefore, one can define a Lie algebra $\mathfrak{g}$ (over $\mathbb{Q}$) containing all these operators such that it is a diffeomorphism invariant of $X(\mathbb{C})$. The action of these operators on $\mathrm{H}(X)$ extends to the whole Lie algebra $\mathfrak{g}$, and therefore, any $\mathfrak{sl}_2$-Lefschetz decomposition factors through $\mathfrak{g}$ according to the definition. Note that the projectivity assumption on X is only required to make sure that the set of $x \in \mathrm{H}^2(X)$ for which Λ_x is defined is a *non-empty* (and thus Zariski dense) subset.

More generally, we formulate the definition over an arbitrary field E as follows.

Definition 3.1.2. Let X be a smooth projective variety over K. The *Looijenga–Lunts–Verbitsky (LLV) Lie algebra* $\mathfrak{g}(X)_E$ of X is the smallest E-Lie subalgebra of $\mathfrak{gl}(\mathrm{H}(X))$ generated by all $\mathfrak{sl}_2$-triples

$$\{(L_x, h, \Lambda_x)\}_x\,,$$

where $x \in \mathrm{H}^2(X)$ is any element satisfying the Hard Lefschetz property.

Remark 3.1.3. The definition of LLV Lie algebra is actually valid for any graded Frobenius–Lefschetz algebra as stated in [Ver96, Definition 1.1, Definition 1.2]. Thus one may also consider the LLV Lie algebra for the Betti cohomology ring of a Kähler manifold.

3.1.4. The LLV algebra $\mathfrak{g}(X)_{\mathbb{Q}}$ for the Betti realization $\mathrm{H}^\bullet_B(X)$ is a semisimple Lie algebra defined over $\mathbb{Q}$ (cf. [LL97, (1.9)]). Moreover, we have the comparison isomorphism

$$\mathfrak{g}(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} E \cong \mathfrak{g}(X)_E$$

given in [IIK+25, Example 3.4.]. The case where X is a hyper-Kähler variety is of the most interest to us and will be assumed in the subsequent part of this paper. As $\mathfrak{g}(X)_E$ is invariant for any field embedding $K \hookrightarrow \mathbb{C}$ with $E = \mathbb{Q}_\ell$, and the real form satisfies

$$\mathfrak{g}(X)_{\mathbb{Q}} \otimes \mathbb{R} \cong \mathfrak{so}(4, b_2 - 2)\,,$$

when X is hyper-Kähler (see Theorem 3.1.7 below), we can see that the $\mathbb{Q}$-form $\mathfrak{g}(X)_{\mathbb{Q}}$ is independent of the choice of the field embedding $K \hookrightarrow \mathbb{C}$.

For simplicity, we will abbreviate the notation into $\mathfrak{g} = \mathfrak{g}(X)_E$, if there is no confusion.

3.1.5. The adjoint action of the shifted degree operator $h \in \mathfrak{g}$ induces an eigenspace decomposition of $\mathfrak{g}$. In the case of hyper-Kähler varieties, $\mathrm{H}(X)$ is of Jordan-type as a graded Frobenius–Lefschetz algebra by [LL97, Lemma 4.2., Proposition 4.4.], i.e., the eigenspace decomposition for h is of the form

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2}\,.$$

In particular, the 0-eigenspace $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$ and admits a decomposition

$$\mathfrak{g}_0 = \overline{\mathfrak{g}} \oplus E \cdot h\,,$$

where $\overline{\mathfrak{g}}$ is the semisimple part of $\mathfrak{g}_0$ that satisfies $\overline{\mathfrak{g}} = [\mathfrak{g}_0, \mathfrak{g}_0]$, and the center $\mathfrak{z}(\mathfrak{g}_0)$ is 1-dimensional and is spanned by the shifted degree operator h. The Lie subalgebra $\overline{\mathfrak{g}}$ is called the *reduced LLV algebra* of X.

3.1.6. Note that $\overline{\mathfrak{g}} \subset \mathfrak{g}_0$ consists of degree 0 operators, and thus the induced $\overline{\mathfrak{g}}$-action on $\mathrm{H}(X)$ preserves the cohomology degree. In other words, for any integer $0 \le k \le 2\dim X$, there is an associated representation

$$\rho_k^{\mathrm{LLV}} \colon \overline{\mathfrak{g}} \longrightarrow \mathrm{End}(\mathrm{H}^k(X))\,.$$

In particular, there is an action of $\overline{\mathfrak{g}}$ on $\mathrm{H}^2(X)$ restricted from $\mathfrak{g}$. On the other hand, it acts as a derivation with respect to the cup product:

$$e.(x \cup y) = (e.x) \cup y + x \cup (e.y), \qquad \text{for } e \in \overline{\mathfrak{g}},\ x, y \in \mathrm{H}(X)\,. \tag{3.1.2}$$

One can also see that the action of $\overline{\mathfrak{g}}$ respects the Beauville–Bogomolov–Fujiki form $\bar{q}$ on the second cohomology $\mathrm{H}^2(X)$, and therefore there is an inclusion

$$\overline{\mathfrak{g}} \subset \mathfrak{so}(\mathrm{H}^2(X), \bar{q})\,. \tag{3.1.3}$$

In fact, the above inclusion (3.1.3) is an equality so that $\mathrm{H}^2(X)$ is the standard representation of $\overline{\mathfrak{g}}$. More specifically, one has the following results.

Theorem 3.1.7 ([Ver96, Theorem 2.3], [LL97, Theorem 4.5.])**.** *Let X be a hyper-Kähler variety over K. Consider the quadratic space*

$$\widetilde{\mathrm{H}}(X) = \mathrm{H}^2(X) \oplus E^{\oplus 2}, \qquad q = \bar{q} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

which is called the Mukai extension *of* $\mathrm{H}^2(X)$ *associated to the cohomology of X. Then the (reduced) LLV Lie algebras of X can be determined as follows:*

$$\overline{\mathfrak{g}} \cong \mathfrak{so}(\mathrm{H}^2(X), \bar{q})\,, \quad \mathfrak{g} \cong \mathfrak{so}(\widetilde{\mathrm{H}}(X), q)\,. \tag{3.1.4}$$

Moreover, let $r = \lfloor b_2(X)/2 \rfloor$. Then the LLV algebra $\mathfrak{g}$ is a simple Lie algebra of type B_{r+1} *or* D_{r+1}, *depending on the parity of* $b_2(X)$, *with the reduced form $\overline{\mathfrak{g}}$ a simple Lie algebra of type* B_r *or* D_r. *Moreover,*

$$\overline{\mathfrak{g}}_{\mathbb{R}} \cong \mathfrak{so}(3, b_2(X) - 3)\,, \quad \mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}(4, b_2(X) - 2)\,. \tag{3.1.5}$$

3.1.8. Let $\mathfrak{mt}(X_\mathbb{C})$ be the Mumford–Tate algebra of complex projective variety $X_\mathbb{C}$, i.e., the Lie algebra of the Mumford–Tate group $\mathbf{MT}(X_\mathbb{C})$. Similarly, $\overline{\mathfrak{mt}}(X_\mathbb{C})$ is defined as the special Mumford–Tate Lie algebra of $X_\mathbb{C}$. It is important to note that the Weil operator lies in the LLV Lie algebra $\mathfrak{g}$, as observed by Verbitsky in [Ver96]. Here we recall a refined version given in [GKLR22].

Theorem 3.1.9 (Verbitsky)**.** *Let X be a hyper-Kähler variety over K. Then $\overline{\mathfrak{mt}}(X_\mathbb{C}) \subseteq \overline{\mathfrak{g}}_\mathbb{Q}$ in $\mathrm{End}_\mathbb{Q}(\mathrm{H}_B(X))$.*

Proof. According to [GKLR22, Proposition 2.24], the Weil operator for the Hodge structure on $\mathrm{H}^\bullet(X_\mathbb{C}, \mathbb{Q})$ lies in the real form of the reduced LLV Lie algebra $\overline{\mathfrak{g}}_\mathbb{R}$. As the special Mumford–Tate algebra $\overline{\mathfrak{mt}}(X_\mathbb{C})$ is the smallest $\mathbb{Q}$-Lie algebra whose $\mathbb{C}$-form contains the Weil operator by the definition, $\overline{\mathfrak{mt}}(X_\mathbb{C}) \subseteq \overline{\mathfrak{g}}_\mathbb{Q}$ as $\mathbb{Q}$-Lie algebras. □

Example 3.1.10. If X is a complex $K3$ surface, the full cohomology $\mathrm{H}^\bullet(X, \mathbb{Q})$ is naturally endowed with the Mukai pairing, which is isomorphic to the Mukai extension $\widetilde{\mathrm{H}}(X, \mathbb{Q})$ for $\mathrm{H}^2(X, \mathbb{Q})$. In this particular case, the representation-theoretic explanation is quite clear, $\mathrm{H}^2(X, \mathbb{Q})$ is the standard representation of $\overline{\mathfrak{g}}(X)_\mathbb{Q}$, and $\mathrm{H}^\bullet(X, \mathbb{Q})$ is the standard representation of $\mathfrak{g}(X)_\mathbb{Q}$. Also note that $\mathfrak{g}(X)_\mathbb{Q}$ is realized as the special Mumford–Tate algebra of a generic (non-algebraic) K3 surface.

In the case of Kummer surfaces, the situation is a bit different. Since a Kummer surface is a K3 surface, it still has $\overline{\mathfrak{g}}_\mathbb{R} = \mathfrak{so}(3, 19)$. However, the construction makes sure that there are 16 independent (-2)-curves obtained as the blow-ups, and therefore its generic special Mumford–Tate algebra has the real form $\mathfrak{so}(3, 3)$, which is the LLV Lie algebra of the corresponding 2-dimensional complex torus.

3.2. **(Twisted) LLV representations.** Let X be a hyper-Kähler variety over K of dimension $2n$, and $\mathfrak{g}$ be the LLV Lie algebra of the cohomological realization $\mathrm{H}(X)$.

3.2.1. Here we recall the integrated representations of the LLV representations introduced by Floccari [Flo22a, §2.1]. The connected algebraic groups over E corresponding to the Lie algebras $\overline{\mathfrak{g}} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ are the Spin groups

$$\mathrm{Spin}(\mathrm{H}^2(X), \overline{q}) \subset \mathrm{GSpin}(\mathrm{H}^2(X), \overline{q}) \subset \mathrm{Spin}(\widetilde{\mathrm{H}}(X), q)\,.$$

In the following, we would mainly focus on the action of $\mathfrak{g}_0$ or the reduced part $\overline{\mathfrak{g}}$ which preserves cohomological degrees.

Notation 3.2.2. For simplicity of notation, we set

- $\mathrm{GSpin} = \mathrm{GSpin}(\mathrm{H}^2(X), \overline{q})$;
- $\mathrm{Spin} = \mathrm{Spin}(\mathrm{H}^2(X), \overline{q})$; and
- $\mathrm{SO} = \mathrm{SO}(\mathrm{H}^2(X), \overline{q})$.

In the remaining parts of this paper, we will also use the subscript "ℓ" for any prime to denote the base change of these groups to the corresponding completion field $\mathbb{Q}_\ell$.

Under the identifications in (3.1.4), the LLV representation

$$\rho_i^{\mathrm{LLV}} \colon \mathfrak{g}_0 \longrightarrow \mathrm{End}_E(\mathrm{H}^i(X))$$

integrates to a group homomorphism

$$\rho_i^{\mathrm{LLV}} \colon \mathrm{Spin} \longrightarrow \mathrm{GL}_E\left(\mathrm{H}^i(X)\right).$$

The kernel $\{\delta, \delta^{-1}\} = \mu_2 \subset \mathrm{Spin}$ of the universal covering $\mathrm{Spin} \to \mathrm{SO}$ acts on $\mathrm{H}^i(X)$ via $\rho_i^{\mathrm{LLV}}(\delta) = (-1)^i|_{\mathrm{H}^i(X)}$ (see [Ver99, Corollary 8.2] and [KSV19, Theorem A.10.]). When

$i = 2k$, the LLV representation ρ^{LLV}_{2k} factors through the orthogonal group by a faithful representation

$$\mathrm{SO} \hookrightarrow \mathrm{GL}_E(\mathrm{H}^{2k}(X)), \quad \forall 1 \le k \le 2n,$$

which is the standard representation of SO when $k = 1$. We still denote by ρ^{LLV}_{2k} for representations of SO.

3.2.3. In the usual setting of the LLV representation, the degree operator $h \in \mathfrak{g}_0$ acts on $\mathrm{H}^i(X)$ as the scalar $i-2n$, which a shift of the usual cohomology degree by the dimension of X. To study the interaction between the LLV representation and various cohomological realizations, it is also convenient to consider the *twisted LLV representation*

$$\rho^{\mathrm{tw}}_i \colon \mathfrak{g}^{\mathrm{tw}}_0 \coloneqq \overline{\mathfrak{g}} \oplus E h^{\deg} \longrightarrow \mathfrak{gl}_E(\mathrm{H}^i(X))$$

where $h^{\deg} = h + \dim(X)\,\mathbb{1}$ is the standard degree operator, i.e., $h^{\deg}|_{\mathrm{H}^i(X)} = i$. As E-Lie algebras, $\mathfrak{g}^{\mathrm{tw}}_0 \cong \mathfrak{g}_0$, and the integrated representation is

$$\rho^{\mathrm{tw}}_i \colon \mathrm{GSpin} \longrightarrow \mathrm{GL}_E(\mathrm{H}^i(X)), \tag{3.2.1}$$

such that

- $\rho^{\mathrm{tw}}_i(z)(x) = z^i \cdot x$ for any z in the central torus $\mathbb{G}_m \subset \mathrm{GSpin}$; and
- $\rho^{\mathrm{tw}}_i|_{\mathrm{Spin}} = \rho^{\mathrm{LLV}}_i$.

This implies that $\rho^{\mathrm{tw}}_{2k+1}$ is faithful when $\mathrm{H}^{2k+1}(X) \neq 0$ (see [Flo22a, Lemma 2.6] and its proof for details).

Remark 3.2.4. For a complex hyper-Kähler variety X, Theorem 3.1.9 implies that

$$\mathbf{MT}(X) \subseteq \rho^{\mathrm{tw}}\left(\mathrm{GSpin}(\mathrm{H}^2(X,\mathbb{Q}),\overline{q})\right).$$

The following is a restatement of Verbitsky's theorem (Theorem 3.1.9) in the algebraic group level.

Lemma 3.2.5. *Suppose that X is a complex hyper-Kähler variety.*

(1) The projections in even degrees induce isomorphism $\pi_{2k} \colon \mathbf{MT}_+(X) \xrightarrow{\sim} \mathbf{MT}_{2k}(X)$ for all $0 \le k \le \dim X$. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \overline{\mathbf{MT}}_+(X) & \longrightarrow & \mathrm{SO}(\mathrm{H}^2(X),\overline{q}) \\ \simeq\big\downarrow \pi_2 & & \big\downarrow \rho^{\mathrm{LLV}}_{2k} \\ \overline{\mathbf{MT}}_{2k} & \hookrightarrow & \mathrm{GL}_{\mathbb{Q}}(\mathrm{H}^{2k}(X_{\mathbb{C}},\mathbb{Q})). \end{array}$$

(2) Suppose that the odd part $\mathrm{H}^-(X) \neq 0$. Then the projection

$$\pi_2 \colon \mathbf{MT}(X) \longrightarrow \mathbf{MT}_{2k}(X)$$

is a central isogeny of degree 2. The kernel $\ker(\pi_2) = \rho^{\mathrm{tw}}_2(\mu_2) \subseteq \mathbf{MT}(X)$ for the center $\mu_2 \subseteq \mathrm{Spin}(\mathrm{H}^2(X),\overline{q})$.

Proof. These statements are actually paraphrases of [FFZ21, Proposition 6.4]. The compatibility with (twisted) LLV representation can be seen from its proof. ☐

3.2.6. From the definition of the LLV Lie algebra $\mathfrak{g}$, it follows immediately that $\mathfrak{g}$ is invariant under deformation, since it is completely determined by the algebra structure of $\mathrm{H}(X)$. We also note that the (twisted) LLV representation is also preserved under deformation.

Lemma 3.2.7. *Let $\mathfrak{X} \to S$ be a smooth family of hyper-Kähler varieties over a smooth connected variety S. Let $\eta \rightsquigarrow s$ be an étale path of points in S. We have the following commutative diagram*

$$\begin{array}{ccc} & & \mathrm{GL}_{\mathbb{Q}_\ell}(\mathrm{H}^\bullet_\ell(\mathfrak{X}_{\bar{s}})) \\ & \overset{\rho_s^{\mathrm{tw}}}{\nearrow} & \Big\downarrow \simeq\ \mathrm{sp}^*_{\eta,s} \\ \mathrm{GSpin}_\ell & \underset{\rho_\eta^{\mathrm{tw}}}{\searrow} & \\ & & \mathrm{GL}_{\mathbb{Q}_\ell}(\mathrm{H}^\bullet_\ell(\mathfrak{X}_{\bar{\eta}}))\,. \end{array}$$

Proof. It is sufficient to assume that $\eta \rightsquigarrow s$ is a specialization of points in S. The general theory of étale cohomology ensures that a specialization isomorphism

$$\mathrm{sp}_{\eta,s}\colon \mathrm{H}^\bullet_\ell(\mathfrak{X}_{\bar{s}}) \xrightarrow{\sim} \mathrm{H}^\bullet_\ell(\mathfrak{X}_{\bar{\eta}})$$

is an isomorphism between the Frobenius graded $\mathbb{Q}_\ell$-algebras. Thus it is equivariant under the action of $\mathfrak{g}_0$ or $\mathfrak{g}_0^{\mathrm{tw}}$. □

3.3. **Motivic lifting of LLV representation.** In this subsection, we collect some facts on defect groups of hyper-Kähler varieties, which are ingredients in [Flo22b, Flo22a, FFZ21].

3.3.1. For any integer $0 < i < 2\dim X$, there is a natural surjective homomorphism

$$\pi_i\colon \mathbf{G}_{\mathrm{mot}}(X) \twoheadrightarrow \mathbf{G}_{\mathrm{mot},i}(X)\,.$$

induced by the inclusion $\mathfrak{h}^i(X) \hookrightarrow \mathfrak{h}(X)$ in $\mathbf{Mot}_K$. Under the second degree projection, there are short exact sequences of motivic Galois groups:

$$1 \longrightarrow P \longrightarrow \mathbf{G}_{\mathrm{mot}}(X) \longrightarrow \mathbf{G}_{\mathrm{mot},2}(X) \longrightarrow 1\,, \tag{3.3.1}$$

$$1 \longrightarrow P_+ \longrightarrow \mathbf{G}_{\mathrm{mot},+}(X) \longrightarrow \mathbf{G}_{\mathrm{mot},2}(X) \longrightarrow 1\,. \tag{3.3.2}$$

The kernel P (resp. P_+) is called the *defect group* (resp. *even defect group*) of X.

Lemma 3.3.2. *Let $\mathrm{Aut}_{\mathrm{alg}}(\mathrm{H}(X))$ be the subgroup of $\mathrm{GL}_E(\mathrm{H}(X))$ generated by E-linear automorphisms preserving the graded algebra structure on $\mathrm{H}(X)$. Then $P_E \in \mathrm{Aut}_{\mathrm{alg}}(\mathrm{H}(X))$ and commutes with $\rho^{\mathrm{tw}}(\mathrm{GSpin}_E) \subseteq \mathrm{Aut}_{\mathrm{alg}}(\mathrm{H}(X))$.*

Proof. It is sufficient to consider the case $\mathrm{H}(X)$ is the Betti realization $\mathrm{H}^\bullet(X_{\mathbb{C}}, \mathbb{Q})$ and $E = \mathbb{Q}$. See Lemma 6.8 and the proof of Theorem 6.9 in [FFZ21]. □

3.3.3. According to [Flo22b, Proposition 4.1], there is a decomposition of motivic Galois groups

$$\mathbf{G}_{\mathrm{mot},+}(X) = \mathbf{G}_{\mathrm{mot},2}(X) \times P_+ \tag{3.3.3}$$

given by a splitting $\sigma\colon \mathbf{G}_{\mathrm{mot},2}(X) \to \mathbf{G}_{\mathrm{mot},+}(X)$ such that $\sigma(\mathbf{G}_{\mathrm{mot},2}(X)) = \mathbf{MT}_+(X)$, and $\mathbf{G}_{\mathrm{mot},2}(X)$ commutes with P in $\mathbf{G}_{\mathrm{mot},+}(X)$. This implies that the connected component P_+° is reductive.

The decomposition of motivic Galois groups (3.3.3) implies that, for any $\mathcal{M} \in \langle \mathfrak{h}^+(X)\rangle^\otimes$, the invariant part $\mathcal{M}^{P_+} \in \langle \mathfrak{h}^2(X)\rangle^\otimes$ which exists by the reductivity of P_+°. In other words, there is an injective homomorphism in $\mathbf{Mot}_K$:

$$\mathcal{M}^{P_+} \hookrightarrow \bigoplus_{m,n} \mathfrak{h}^2(X)^{\otimes m} \otimes \mathfrak{h}^2(X)^{\vee,\otimes n}\,. \tag{3.3.4}$$

If P_+ is trivial, then we can take $\mathcal{M} = \mathfrak{h}^{2k}(X)$, and in particular the conjecture (Ab) holds for $\mathfrak{h}^{2k}(X)$ by the André's theorem.

Moreover, the embedding (3.3.4) can be chosen to be compatible with the LLV representation. By Tannakian duality for $\langle \mathfrak{h}^2(X)\rangle^{\otimes}$, (3.3.4) gives an injective homomorphism of finite-dimensional $\mathbf{G}_{\mathrm{mot},2}(X)$-representations on its Betti realization. By definition of the splitting σ, the $\mathbf{G}_{\mathrm{mot},2}(X)$-action on $\mathrm{H}_B(\mathcal{M}^{P_+})$ is that of $\mathbf{MT}_+(X_{\mathbb{C}})$, which factors through the twisted LLV representation. Hence we can choose (3.3.4) as an injective homomorphism of motives, corresponding to an injective homomorphism of GSpin-representations

$$\mathrm{H}_B(\mathcal{M}^{P_+}) \hookrightarrow \bigoplus_{m,n} \mathrm{H}^2_B(X)^{\otimes m} \otimes \mathrm{H}^2_B(X)^{\vee,\otimes n}.$$

3.3.4. According to [FFZ21, Theorem 6.9][3], the motivic Galois group of $\mathfrak{h}(X)$ is an almost-direct product of the Mumford–Tate group $\mathbf{MT}(X)$ and the defect group P:

$$\mathbf{G}_{\mathrm{mot}}(X) = \mathbf{MT}(X_{\mathbb{C}}) \cdot P,$$

where $P \cap \mathbf{MT}(X_{\mathbb{C}}) = \{\delta, \delta^{-1}\} \subset Z(\mathbf{G}_{\mathrm{mot}}(X))$ if X has non-vanishing odd degree cohomologies, and $\delta|_{\mathrm{H}^i} = (-1)^i$; otherwise it is truly a direct product.

Lemma 3.3.5. *The element $\delta \in Z(\mathbf{G}_\ell(X)) \cap P_{\mathbb{Q}_\ell}$ in $\mathbf{G}_{\mathrm{mot}}(X)$. If X has non-vanishing odd degree cohomologies, then $\delta \neq \mathbb{1}$.*

Proof. By Lemma 2.2.11, the group of homotheties $\mathbb{G}_{m,\mathbb{Q}_\ell} \subset Z(\mathbf{G}_\ell(X))$. The image of $-1 \in \mathbb{G}_{m,\mathbb{Q}_\ell}$ in $\mathbf{G}_\ell(X)$ acts on $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ as $(-1)^i$ by the definition, and thus is equal to δ. Clearly, if $b_{2k+1}(X) \neq 0$, then δ acts non-trivially on $\mathrm{H}^{2k+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. □

Remark 3.3.6. If the defect group P is finite, the Mumford–Tate conjecture holds true for $\mathfrak{h}(X)$ (see [FFZ21, Proposition 7.6]).

4. Rank estimation of algebraic monodromy groups

4.1. **Maximal tori of algebraic monodromy groups.** We retain the notation established in Section 2.3. In [Ser86], Serre considered the *Frobenius tori* T_v in $\mathbf{G}_\ell^\circ$ for a $\mathbb{Q}$-compatible system of representations arising from geometry over number fields. The following theorem from Serre constitutes a central ingredient of this paper. Heuristically, the rank of $\mathbf{G}_\ell$ can be determined by considering Frobenius tori at some "general" places.

Theorem 4.1.1 (Serre)**.** *Let K be a number field and $\rho_\bullet$ be a $\mathbb{Q}$-compatible system of representations of G_K.*

(a) For every $v \in \Sigma$, there is a torus T_v over $\mathbb{Q}$ such that the $\mathrm{GL}_{n,\mathbb{Q}_\ell}$-conjugacy classes of F_v lie in $T_v(\mathbb{Q}_\ell)$ for any prime ℓ.

(b) There is a subset

$$\Sigma_{\max} \subseteq \Sigma$$

of Dirichlet-density 1 such that for all $v \in \Sigma_{\max}$, the Frobenius torus $T_{v,\mathbb{Q}_\ell}$ is conjugate to a maximal torus of $\mathbf{G}_\ell^\circ$ under the $\mathrm{GL}_n(\mathbb{Q}_\ell)$-action.

Proof. In [Ser86], the construction of the Frobenius torus T_v is only treated for $\mathbb{Q}$-compatible systems arising from abelian varieties (over number fields). We briefly recall the general construction of T_v in [LP92, §4, §7].

[3] It is wrongly claimed in the published version of [FFZ21] that $\mathbf{MT}(X_{\mathbb{C}})$ intersects trivially with P inside $\mathbf{G}_{\mathrm{mot}}(X)$.

As in Section 2.3, let F_v be a Frobenius lift on K from the finite field $\mathbb{F}_q$ for each place $v \in \Sigma$. By the assumptions on Σ, each place $v \in K$ is unramified, and thus the Frobenius lift is unique.

(1) Firstly, we assume that $\rho_\bullet(F_v)$ are semisimple. Then the ℓ-adic monodromy group $\mathbf{G}_\ell^\circ$ is a connected reductive group over $\mathbb{Q}_\ell$. By taking the characteristic polynomial, there is a morphism

$$ch\colon \mathrm{GL}_{n,\mathbb{Q}_\ell} \longrightarrow (\mathbb{G}_m \times \mathbb{A}^{n-1})_{\mathbb{Q}_\ell}\,,$$

which is actually defined over $\mathbb{Q}$. Under the faithful representation $\mathbf{G}_\ell^\circ \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$, the image $ch(\mathbf{G}_\ell^\circ)$ is defined over $\mathbb{Q}$ as a Zariski closed subset by Proposition (6.12) in *loc. cit.*. Then there is a semisimple element $t_v \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ such that

$$ch(\rho_\ell(F_v)) = ch(t_v) \in ch(\mathbf{G}_\ell^\circ)(\mathbb{Q})\,,$$

which is unique up to GL_n-conjugation. Let H_v be the Zariski closure of the subgroup generated by t_v in $\mathrm{GL}_{n,\mathbb{Q}}$. Since t_v is semisimple, H_v is multiplicative. We denote T_v for the identity component of H_v, which is a torus over $\mathbb{Q}$.

There is a Dirichlet-density 1 subset $\Sigma_{\max} \subseteq \Sigma$ such that all $\rho_\ell(F_v), v \in \Sigma_{\max}$ are Γ-regular by (7.2) in *loc. cit.*. Then the element $\rho_\ell(F_v)$ is Γ-regular and in particular regular in $\mathbf{G}_\ell^\circ$, and thus determines a torus $T_v \subseteq \mathrm{GL}_{n,\mathbb{Q}}$ by (4.7) in *loc. cit.*, such that $t_v \in T_v(\overline{\mathbb{Q}})$, $ch(t_v) = ch(\rho_\ell(F_v))$ for any prime $\ell \in \mathbf{L}$, and

$$T_{v,\mathbb{Q}_\ell} \sim_{\mathrm{GL}_{n,\mathbb{Q}_\ell}} Z_{\mathbf{G}_\ell^\circ}(\rho_\ell(F_v)) \subseteq \mathbf{G}_\ell^\circ\,,$$

as the unique maximal torus of $\mathbf{G}_\ell^\circ$ that contains $\rho_\ell(F_v)$. Here $\sim_{\mathrm{GL}_{n,\mathbb{Q}_\ell}}$ means the conjugacy under $\mathrm{GL}_{n,\mathbb{Q}_\ell}$-action.

(2) The compatibility conditions only depend on the semisimple part of the elements $\rho_\ell(F_v)$, therefore we may take the semisimplification of an arbitrary representation, then its ℓ-adic algebraic monodromy group $\mathbf{G}_\ell^{\mathrm{red}}$ is the quotient of the Zariski closure of $\rho_\ell(G_K)$ by its unipotent radical R_ℓ. There is an exact sequence

$$1 \longrightarrow R_\ell \longrightarrow \mathbf{G}_\ell \longrightarrow \mathbf{G}_\ell^{\mathrm{red}} \longrightarrow 1\,,$$

which is split. Up to conjugation under the action of R_ℓ, the reductive quotient $\mathbf{G}_\ell^{\mathrm{red}}$ can be viewed as a Levi subgroup of $\mathbf{G}_\ell$. Then applying the construction in (1) for $\mathbf{G}_\ell^{\mathrm{red}} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$, we will obtain the required torus T_v similarly. □

Lemma 4.1.2. *If $v \in \Sigma_{\max}$ for $\mathfrak{h}(X)$, then v is also a maximal place for $\mathfrak{h}^i(X)$ for any $0 \le i \le 2\dim X$.*

Proof. We can assume that $K = K^{\mathrm{conn}}$. As $v \in \Sigma_{\max}$, $T \coloneqq Z_{\mathbf{G}_\ell(X)}(\rho_\ell(F_v)^{\mathrm{ss}})$ is a maximal torus of $\mathbf{G}_\ell(X)$. Recall that the natural projection $\pi_{\ell,i}\colon \mathbf{G}_\ell(X) \to \mathbf{G}_{\ell,i}(X)$ is surjective for any prime ℓ. This implies that the image $\pi_{\ell,i}(T)$ is also a maximal torus of $\mathbf{G}_{\ell,i}(X)$ (see [Mil17, Proposition 17.20]). On the other hand, we have $\pi_{\ell,i}(\rho_\ell(F_v)^{\mathrm{ss}}) = \rho_{\ell,i}(F_v)^{\mathrm{ss}}$. Thus $\pi_{\ell,i}(T)$ is a maximal torus such that $\rho_{\ell,i}(F_v)^{\mathrm{ss}} \in \pi_{\ell,i}(T)^{\mathrm{ss}}(\mathbb{Q}_\ell)$, and

$$\pi_{\ell,i}\,(T) \subseteq Z_{\mathbf{G}_{\ell,i}(X)}(\rho_{\ell,i}(F_v)^{\mathrm{ss}})\,.$$

This implies that $\rho_{\ell,i}(F_v)^{\mathrm{ss}}$ is regular in $\pi_{\ell,i}(T)$ for all prime ℓ, and in particular v is a maximal place for $\mathfrak{h}^i(X)$ by the construction. □

4.1.3. In the construction of the Frobenius torus T_v, we have a conjugacy

$$T_{v,\mathbb{Q}_\ell} \sim_{\mathrm{GL}_{n,\mathbb{Q}_\ell}} Z_{\mathbf{G}_\ell^\circ}(\rho_\ell(F_v)) \subseteq \mathbf{G}_\ell^\circ\,, \tag{4.1.1}$$

The conjugacy (4.1.1) identifies the groups of cocharacters $X_*(T_{v,\mathbb{Q}_\ell})$ as that of a maximal torus of $\mathbf{G}_\ell^\circ$ for any prime ℓ. As $T_{v,\mathbb{Q}_\ell}$ is the base extension from a $\mathbb{Q}$-torus T_v, it induces an $\mathrm{Aut}(\mathbb{Q}_\ell/\mathbb{Q})$-action on $Z_{\mathbf{G}_\ell^\circ}(\rho_\ell(F_v)) \subseteq \mathbf{G}_\ell^\circ$. Moreover, since all cocharacters of

$T_{v,\mathbb{Q}_\ell}$ are defined over a finite extension of $\mathbb{Q}$, the $\mathrm{Aut}(\mathbb{Q}_\ell/\mathbb{Q})$-action factors through a finite subgroup $G \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Suppose that the $\mathbb{Q}$-compatible system $\rho_\bullet$ comes from geometry, i.e., as in the context of 2.3.2. In the construction of T_v in Theorem 4.1.1, for any prime ℓ, we can fix the $\mathbb{Q}$-structure of $ch\colon \mathrm{GL}_{n,\mathbb{Q}_\ell} = \mathrm{GL}_{\mathbb{Q}_\ell}(\mathrm{H}_\ell(\mathcal{M})) \to (\mathbb{G}_m \times \mathbb{A}^{n-1})_{\mathbb{Q}_\ell}$ obtained from the Artin comparison $\mathrm{H}_B(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathrm{H}_\ell(\mathcal{M})$. Then we may assume that

$$T_v \subseteq \mathrm{GL}_{\mathbb{Q}}(\mathrm{H}_B(\mathcal{M}))$$

as a $\mathbb{Q}$-torus. In particular, we may assume that the inclusion of groups of cocharacters

$$X_*(T_{v,\mathbb{Q}_\ell}) \hookrightarrow X_*(\mathrm{GL}_{\mathbb{Q}}(\mathrm{H}_B(\mathcal{M}))_{\mathbb{Q}_\ell})$$

is G-equivariant. The torus T_v is uniquely determined up to the conjugation by $\mathrm{GL}_{\mathbb{Q}}(\mathrm{H}_B(\mathcal{M}))$ in terms of the characteristic polynomial $ch(\rho_\bullet(F_v)) \in \mathbb{Q}[x]$ (see [LP92, (4,7.1)]).

4.1.4. As observed in [Pin98, §3], the rank of the global algebraic monodromy group $\mathbf{G}_\ell$ is determined by the local Galois representation at some good places. Let

- $\mathbf{G}_{\ell,v}$ be the ℓ-adic algebraic monodromy group of V, viewed as a local Galois representation of $G_{K_v} \subset G_K$,
- T_v be the Frobenius torus of $\mathbf{G}_\ell$ at v.
- $\Sigma_{\max}$ be the set of non-archimedean places of K as in Theorem 4.1.1.

Proposition 4.1.5. *Assume that $K = K^{\mathrm{conn}}$. For any place $v \in \Sigma_{\max}$ with residue characteristic p,*

(a) $\mathbf{G}_{p,v}$ is connected; and

(b) there is an element $g \in \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that the conjugation

$$^{g}T_{v,\overline{\mathbb{Q}}_p} \coloneqq g\,T_{v,\overline{\mathbb{Q}}_p}\,g^{-1} \subseteq (\mathbf{G}_{p,v})_{\overline{\mathbb{Q}}_p} \subseteq (\mathbf{G}_p)_{\overline{\mathbb{Q}}_p}$$

gives rise to a maximal torus (for both groups $\mathbf{G}_{p,v}$ and $\mathbf{G}_p$).

In particular, one has $\mathrm{rk}\,\mathbf{G}_{p,v} = \mathrm{rk}\,\mathbf{G}_p$.

Proof. See [Pin98, Proposition (3.13)] and the discussion before it. □

In summary, we have the following consequence for Galois representations from geometry.

Corollary 4.1.6. *Let X be a smooth projective variety over a number field K. Consider the $\mathbb{Q}$-compatible system $\rho_\bullet$ of ℓ-adic Galois representations as in 2.3.2. There exists a place v of K with residue characteristic p, such that X has good reduction at v, and $\mathrm{rk}\,\mathbf{G}_\ell = \mathrm{rk}\,\mathbf{G}_{p,v}$ for any prime ℓ.*

Proof. By Serre's Theorem 4.1.1, the rank of $\mathbf{G}_\ell^{\mathrm{red}}(X)$ is independent of the prime ℓ. Thus, in order to obtain the rank of $\mathbf{G}_\ell^{\mathrm{red}}(X)$ for some prime ℓ, we may take a place $v \in \Sigma_{\max}$ for $\mathfrak{h}(X)$ and set $\ell = p$ as the prime above v according to Proposition 4.1.5. By the definition of Σ, X has good reduction at v. □

4.2. **Hodge cocharacters and Sen theory.** In this subsection, we assume that K_v is a non-archimedean local field with a perfect residue field k such that $\mathrm{char}(k) = p > 0$. Let K_v^{ur} be the maximal unramified extension of K_v in $\overline{K}_v$. We will review some general facts about the Hodge–Tate representations and also highlight the key features of those coming from hyper-Kähler varieties (e.g., Theorem 4.2.11).

4.2.1. Let $\mathbb{C}_p \coloneqq \widehat{\overline{K}_v}$ be the p-adic completion of an algebraic closure $K_v \subseteq \overline{K}_v$. Recall that a p-adic Galois representation $\rho\colon G_{K_v} \to \mathrm{GL}_{\mathbb{Q}_p}(V)$ is called *Hodge–Tate* if it admits a decomposition into $\mathbb{C}_p$-vector subspaces:

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_{r \in \mathbb{Z}} V(r)\,, \tag{4.2.1}$$

such that $\sigma|_{V(r)} = (\chi_p)^r(\sigma)$ for any $\sigma \in \mathrm{Gal}(\overline{K}_v/K_v^{\mathrm{ur}})$, where χ_p is the p-adic cyclotomic character of G_{K_v}. The integers r's appear in (4.2.1) with $V(r)$ non-trivial are called the *Hodge–Tate weights* of V. In particular, a (potentially) crystalline representation is Hodge–Tate.

The Hodge–Tate decomposition (4.2.1) determines a cocharacter

$$\mu_{\mathrm{HT}}\colon \mathbb{G}_{m,\mathbb{C}_p} \longrightarrow \mathrm{GL}_{\mathbb{C}_p}(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)$$

up to conjugation, called the *Hodge–Tate cocharacter* of V.

Definition 4.2.2. Let $\rho\colon G_{K_v} \to \mathrm{GL}_{\mathbb{Q}_p}(V)$ be a Hodge–Tate representation of G_{K_v}.

– The $\mathbb{C}_p$-linear operator $\Theta \in \mathrm{End}_{\mathbb{C}_p}(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)$ is defined as

$$\Theta|_{V(q)} = -q\,\mathbb{1}_{V(q)}\,,$$

where $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_q V(q)$ is the Hodge–Tate decomposition, and Θ is called the *Sen operator* of V.

– The *Sen Lie algebra* $\mathfrak{s}(V)$ of V is the *smallest* $\mathbb{Q}_p$-Lie subalgebra of $\mathrm{End}_{\mathbb{Q}_p}(V)$ such that $\Theta \in \mathfrak{s}(V) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$.

Sen's theory shows that the Lie algebra $\mathfrak{s}(V)$ is algebraic[4] and can be described by the inertia representation when V is Hodge–Tate. Consider the short exact sequence

$$1 \to I_v \longrightarrow G_{K_v} \longrightarrow G_{\mathbb{F}_q} \to 1\,,$$

where I_v is the inertia subgroup of G_{K_v}.

Proposition 4.2.3. *Let $\rho\colon G_{K_v} \to \mathrm{GL}_{\mathbb{Q}_p}(V)$ be a Hodge–Tate representation. Denote $\mathfrak{g}_p(V)$ for the Lie algebra of $\rho(G_{K_v})$.*

(1) The Lie algebra of the image $\rho(I_v)$ is algebraic and isomorphic to the Sen Lie algebra $\mathfrak{s}(V)$.

(2) $\mathfrak{s}(V) \subseteq \mathfrak{g}_p(V)$ is an ideal.

Proof. By the identity $I_v = \mathrm{Gal}(\overline{K}_v/K_v^{\mathrm{ur}})$ for the maximal unramified extension K_v^{ur} of K_v, the inertia representation $\rho|_{I_v}$ can be viewed as the $\mathbb{Q}_p$-representation of $G_{K_v^{\mathrm{ur}}}$. Then the first statement follows from Sen [Sen73, Theorem 1] and [Sen81, Theorem 11]. And for the last assertion, it is sufficient to note that the inertia subgroup $I_v \subseteq G_{K_v}$ is normal, and thus the Zariski closure of $\rho(I_v)$ inside $\mathbf{G}_p(V)$ is also a normal subgroup. □

According to Proposition 4.2.3, the Hodge–Tate cocharacter μ_{HT} factors through the p-adic algebraic monodromy group $\mathbf{G}_{p,v}$. As the p-adic algebraic monodromy group $\mathbf{G}_{p,v}$ is algebraic, we can assume that μ_{HT} is defined over a finite field extension of $\mathbb{Q}_p$, i.e., $\mu_{\mathrm{HT}} \in X_*(\mathbf{G}_{p,v})(\overline{\mathbb{Q}}_p)$.

Definition 4.2.4. Let $\mathbf{S} \subseteq \mathbf{G}_{p,v}$ be the *smallest* $\mathbb{Q}_p$-algebraic subgroup such that $\mathbf{S}(\mathbb{C}_p)$ contains the image of the Hodge–Tate cocharacter

$$\mu_{\mathrm{HT}}\colon \mathbb{G}_{m,\mathbb{C}_p} \longrightarrow (\mathbf{G}_{p,v})_{\mathbb{C}_p}\,.$$

[4] Here, a Lie algebra is called *algebraic* if it is isomorphic to the Lie algebra of a connected algebraic group variety.

Remark 4.2.5. According to the definition, $\mathbf{S}$ is a connected algebraic group over $\mathbb{Q}_p$. Moreover, its Lie algebra

$$\mathrm{Lie}(\mathbf{S}) \cong \mathfrak{s}(V)$$

is the Sen Lie algebra and $\mathbf{S} \subseteq \mathbf{G}_{p,v}$ is normal subgroup by Proposition 4.2.3.

4.2.6. Let X be a smooth proper variety over a *number field* K and v be a non-archimedean place of K. We denote X_v for the base change $X \times_K K_v$ where K_v is the local field by taking the completion of K at v. As a G_K-representation, the p-adic étale cohomology $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}_v}, \mathbb{Q}_p)$ is Hodge–Tate (see [Fal88]).

The key input from the geometry of hyper-Kähler varieties for their Hodge–Tate representations is the following.

Proposition 4.2.7. *Suppose that X is a hyper-Kähler variety over K of dimension $2n$. Let $\mathfrak{s}(X_v)$ be the Sen algebra of $\mathfrak{h}(X_v)$. Then, for any integer $0 < k < 4n$, there is a Lie algebra isomorphism*

$$\mathfrak{s}_k(X_v) = \pi_{p,k}(\mathfrak{s}(X_v)),$$

under the projection $\pi_{p,k}$. Moreover, if $b_k(X) \neq 0$, then it induces an isomorphism of Lie algebras $\mathfrak{s}_k(X_v) \cong \mathfrak{s}(X_v)$. In particular, the projection

$$\pi_{p,2} \colon \mathbf{S}(X_v) \twoheadrightarrow \mathbf{S}_2(X_v)$$

is an isogeny.

Proof. It is sufficient to consider the case $b_k(X) \neq 0$. The projection $\pi_{p,k} \colon \mathfrak{s}(X_v) \to \mathfrak{s}_k(X_v)$ is the restriction of the (twisted) LLV representation

$$\rho_k^{\mathrm{tw}} \colon \mathfrak{g}^{\mathrm{tw}}_{X,\mathbb{Q}_p} \longrightarrow \mathrm{End}_{\mathbb{Q}_p}(\mathrm{H}^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)).$$

In the decomposition $\mathfrak{g}^{\mathrm{tw}}_{X,\mathbb{Q}_p} = \overline{\mathfrak{g}}_{X,\mathbb{Q}_p} \oplus \mathbb{Q}_p h^{\deg}$, the reduced part $\overline{\mathfrak{g}}_{X,\mathbb{Q}_p}$ is a simple Lie algebra. For $0 < k < 4n$, the $\overline{\mathfrak{g}}_X$-module $\mathrm{H}^k(X)$ contains non-trivial irreducible factors if $b_k(X) \neq 0$ ([GKLR22, Proposition 2.35]), thus the restriction of ρ_k^{tw} on $\overline{\mathfrak{g}}_{X,\mathbb{Q}_p}$ is injective (after tensoring $\mathbb{Q}_p$). On the other hand, the action of the center $\mathbb{Q}_p h^{\deg}$ is non-trivial if $k \neq 0$. Therefore, this implies that $\pi_{p,k}$ is injective as $0 < k < 4k$ and $b_k(X) \neq 0$.

The injectivity of $\pi_{p,k}$, combined with an argument analogous to the proof of [GKLR22, Proposition 2.38] and invoking [IIK+25, Theorem 6.2], implies that $\mathfrak{s}_k(X_v) = \pi_{p,k}(\mathfrak{s}(X_v))$. In this step, we use that the degree-k Sen Lie algebra $\mathfrak{s}(X_v)$ (or $\mathfrak{s}_k(X_v)$) is the smallest $\mathbb{Q}_p$-Lie subalgebra containing the corresponding Sen operator. □

4.2.8. Another important cocharacter for $\mathbf{G}_{p,v}$ is the *Newton cocharacter*. Here we briefly recall the definition. Under the assumption that $K_v = K_0$ for $v \in \Sigma_{\max}$, we thus obtain a fiber functor

$$\omega_p \colon \mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(G_{K_v}) \longrightarrow \mathbf{MF}_{\varphi,K_v} \xrightarrow{\text{for.}} \mathbf{Vect}_{K_0}$$

by composing with the forgetful functor. Via the Tannakian formalism, we have

$$\mathbf{G}_{p,v} \sim_{\text{inner twist}} \mathrm{Aut}^{\otimes}\left(\omega|_{\langle \mathrm{D}_{\mathrm{crys}}(X_v)\rangle}\right).$$

Thus the conjugacy class of $\Phi_v \in \mathrm{Aut}^{\otimes}(\omega|_{\langle \mathrm{D}_{\mathrm{crys}}(X_v)\rangle})(K_0)$ corresponds to a conjugacy class in the p-adic monodromy group $\mathbf{G}_{p,v}(\overline{\mathbb{Q}}_p)$ under the inner twist, which is unique up to $\mathbf{G}_{p,v}(\overline{\mathbb{Q}}_p)$-conjugacy. We denote the conjugacy class by $[\Phi_v]$.

Fix a representative Φ_v in the $\mathbf{G}_{p,v}(\overline{\mathbb{Q}}_p)$-conjugacy class $[\Phi_v]$. Since X has good reduction at v, $\mathrm{D}_{\mathrm{crys}}(\mathcal{M}_v)$ is the same as the crystalline cohomology of the special fiber of a good reduction of $\mathcal{M}_v$ at v, as a filtered φ-module over K_v. Then, as a consequence of the Weil conjecture, the eigenvalues of Φ_v are algebraic integers and are independent of the Weil cohomology realization (see [KM74]). Let $\Phi_v = \Phi_v^u \circ \Phi_v^{\mathrm{ss}}$ be the

Jordan decomposition of Φ_v in $\mathbf{G}_{p,v}$. By the construction of the Frobenius torus T_v in Theorem 4.1.1, the semisimple part Φ_v^{ss} is $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$-conjugate to a semisimple element

$$t_v \in T_v(\overline{\mathbb{Q}}_p)$$

which generates a Zariski dense subgroup of T_v. This corresponds to a quasi-cocharacter

$$\mu_N \in X_*(T_v)(\overline{\mathbb{Q}}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

such that $\mu_N(\chi) = \frac{1}{m}\,\mathrm{ord}_v(\chi(t_v))$ for any character $\chi \in X^*(T_v)(\overline{\mathbb{Q}}_p)$, where $\mathrm{ord}_v\colon \overline{\mathbb{Q}}_p \to \mathbb{Q} \cup \{\infty\}$ is the normalized valuation at v. This quasi-cocharacter is the Newton cocharacter of T_v.

4.2.9. Suppose that $v \in \Sigma_{\max}$. By Proposition 4.1.5, we may assume that ${}^gT_{v,\overline{\mathbb{Q}}_p} \subseteq \mathbf{G}_{p,v,\overline{\mathbb{Q}}_p}$ is a maximal torus after a $\mathrm{GL}_{n,\overline{\mathbb{Q}}_p}$-conjugation. For simplicity of notation, we omit the superscript g in the following.

After $\mathbf{G}_{p,v}$-conjugation, we fix a $\mathbb{Q}_p$-form $T_{v,\mathbb{Q}_p} \subseteq \mathbf{G}_{p,v}$ of $T_{v,\overline{\mathbb{Q}}_p}$. For any (quasi-)cocharacter $\mu \in X_*(T_{v,\mathbb{Q}_p})(\overline{\mathbb{Q}}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$, we consider the subset of its conjugates

$$S_\mu \coloneqq \left\{ {}^\sigma\mu\colon \widehat{\mathbb{G}}_{m,\overline{\mathbb{Q}}_p} \to \mathbf{G}_{p,v,\overline{\mathbb{Q}}_p} \;\middle|\; \begin{array}{c} \sigma \in \mathbf{G}_{p,v}(\overline{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\,, \\ \mathrm{Im}({}^\sigma\mu) \subseteq T_{v,\overline{\mathbb{Q}}_p} \end{array} \right\}.$$

Recall that the essential image of $\mathrm{D}_{\mathrm{crys}}$ is the set of those *weakly admissible* filtered φ-modules by Colmez–Fontaine [CF00, Théorème A], i.e., the Newton polygon lies above the Hodge polygon and they have the same endpoint. The weak admissibility of the filtered φ-module $\mathrm{D}_{\mathrm{crys}}(\mathcal{M}_v)$ implies that

$$\mu_N \in \mathrm{Conv}(S_{\mu_{\mathrm{HT}}})^\circ \subseteq X_*(T_{v,\mathbb{Q}_p})(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{R}\,, \tag{4.2.2}$$

where $\mathrm{Conv}(S_{\mu_{\mathrm{HT}}})^\circ$ is the interior of the convex hull of $S_{\mu_{\mathrm{HT}}}$ (see [Pin98, Theorem (2.3)]).

As a connected normal $\mathbb{Q}_p$-subgroup $\mathbf{S}(\mathcal{M}_v) \lhd \mathbf{G}_{p,v}$ (see Remark 4.2.5), take

$$T_{\mathrm{HT},v} \coloneqq \mathbf{S}(\mathcal{M}_v) \cap T_{v,\mathbb{Q}_p}\,,$$

which is a maximal torus of $\mathbf{S}(\mathcal{M}_v)$ such that $T_{\mathrm{HT},v} \subseteq T_{v,\mathbb{Q}_p}$. This induces an injective homomorphism $X_*(T_{\mathrm{HT},v}) \hookrightarrow X_*(T_{v,\mathbb{Q}_p})$. Together with the abundance of Newton cocharacters, we can conclude that the Frobenius torus T_v is generated by Hodge–Tate cocharacters by the Galois action.

Lemma 4.2.10. *Under the injective homomorphism $X_*(T_{\mathrm{HT},v}) \hookrightarrow X_*(T_{v,\mathbb{Q}_p})$, the lattice $X_*(T_{v,\mathbb{Q}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by the* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-orbits of $X_*(T_{\mathrm{HT},v})$.*

Proof. Denote $V = X_*\left(T_{\mathrm{HT},v,\overline{\mathbb{Q}}_p}\right)$ as the group of cocharacters (over $\overline{\mathbb{Q}}_p$) for simplicity. As noted in Remark 4.2.5, the Hodge–Tate cocharacter $\mu_{\mathrm{HT}} \in V$ and $\mathbf{S}(\mathcal{M}_v)$ is a normal subgroup of $\mathbf{G}_{p,v}$. Thus the $\mathbf{G}_{p,v}(\overline{\mathbb{Q}}_p)$-conjugates of μ_{HT} factors through $\mathbf{S}(\mathcal{M}_v)_{\overline{\mathbb{Q}}_p}$ and all $\mu \in S_{\mu_{\mathrm{HT}}}$ are cocharacters of $\mathbf{S}(\mathcal{M}_v)_{\overline{\mathbb{Q}}_p}$. Moreover, as μ factors through $T_{v,\overline{\mathbb{Q}}_p}$, we have $\mu \in V$. For this reason, the subset $S_{\mu_{\mathrm{HT}}} \subset V$, and also the convex hull

$$\mathrm{Conv}\,(S_{\mu_{\mathrm{HT}}}) \subseteq V \otimes_{\mathbb{Z}} \mathbb{R}\,.$$

The Galois orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \mu_N$ of the Newton cocharacter generates $X_*(T_{v,\mathbb{Q}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [Ser86, p.9., Théorème] or [Pin98, Proposition (3.5)]). Therefore, the $\mathbb{Q}$-vector space $X_*(T_{v,\mathbb{Q}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits of $X_*(T_{\mathrm{HT},v})$ by the inclusion (4.2.2). ☐

Theorem 4.2.11. *Suppose that X is a hyper-Kähler variety. Fix a place $v \in \Sigma_{\max}$.*

(1) *The projection* $\pi_{p,2}: \mathbf{G}_p(X_v) \twoheadrightarrow \mathbf{G}_{p,2}(X_v)$ *induces a bijective homomorphism between the groups of cocharacters*

$$X_*\left(T_{v,\overline{\mathbb{Q}}_p}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} X_*\left(T_{v,2,\overline{\mathbb{Q}}_p}\right) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

(2) *Let* $\mathbf{G}^{\mathrm{red}}_{p,v} \subseteq \mathbf{G}_p(X_v)$ *be a Levi subgroup. Under the projection* $\pi_{p,2}$, $\mathbf{G}^{\mathrm{red}}_{p,v}$ *is isogenous onto* $\mathbf{G}_{p,2}(X_v)$.

Proof. By Proposition 4.1.5 (a), $\mathbf{G}_{p,2}(X_v)$ and $\mathbf{G}_p(X_v)$ are connected. According to Lemma 4.1.2 (and its proof), the projection $\pi_{p,2}$ induces a surjective homomorphism of Frobenius tori

$$T_{v,\mathbb{Q}_p} \longrightarrow T_{v,2,\mathbb{Q}_p}\,,$$

which is in fact defined over $\mathbb{Q}$. Consequently, we obtain a surjective group homomorphism

$$X_*\left(T_{v,\overline{\mathbb{Q}}_p}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow X_*\left(T_{v,2,\overline{\mathbb{Q}}_p}\right) \otimes_{\mathbb{Z}} \mathbb{Q}\,, \tag{4.2.3}$$

which is $\mathrm{Aut}(\mathbb{Q}_p/\mathbb{Q})$-equivariant. It remains to show that (4.2.3) is injective. We split the proof of (1) into two steps.

Step 1. We firstly claim that for any $\mu \in V \coloneqq X_*(T_{v,\overline{\mathbb{Q}}_p})$, there is a lift

$$\widetilde{\mu} \in \widetilde{V}_{\mathbb{Q}} \coloneqq X_*\left(\mathrm{GSpin}_{\overline{\mathbb{Q}}_p}\right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

such that $\mu = \rho^{\mathrm{tw}} \circ \widetilde{\mu}$. Here we may view V as a subgroup of $X_*(\mathrm{GL}_{\mathbb{Q}}(\mathrm{H}_B(X)))(\overline{\mathbb{Q}}_p)$.

Also note that Lemma 4.2.10 implies that $V_{\mathbb{Q}}$ is generated by cocharacters in

$$X_*(T_{\mathrm{HT},v}) \subseteq X_*(\mathbf{S}(X_v))$$

under the $\mathrm{Aut}(\mathbb{Q}_p/\mathbb{Q})$-action as in 4.1.3.

If $\mu \in X_*(\mathbf{S}(X_v))$, then the required lift $\widetilde{\mu}$ follows from [IIK+25, Theorem 6.2]. In general, a quasi-cocharacter in $V_{\mathbb{Q}}$ is of the form

$$\mu = \sum_i a_i({}^{\sigma_i}\mu_i)\,, \quad a_i \in \mathbb{Q}$$

for some $\mu_i \in X_*(\mathbf{S}(X_v))$ and $\sigma_i \in \mathrm{Aut}(\mathbb{Q}_p/\mathbb{Q})$. In this case, we can take $\widetilde{\mu} = \sum_i a_i({}^{\sigma_i}\widetilde{\mu}_i)$. Since the twisted LLV representation ρ^{tw} is defined over $\mathbb{Q}$, we have $\rho^{\mathrm{tw}} \circ \widetilde{\mu} = \mu$ as required.

Step 2. Let $\mu \in V$ and $\widetilde{\mu}: \widehat{\mathbb{G}}_{m,\overline{\mathbb{Q}}_p} \to \mathrm{GSpin}_{\overline{\mathbb{Q}}_p}$ be the quasi-cocharacter which lifts μ as in **Step 1**.

Assume that $\pi_{p,2} \circ \mu = 0$, then we have $\rho_2^{\mathrm{tw}} \circ \widetilde{\mu} = 0$. Then it is clear that μ and $\widetilde{\mu}$ are not given by the weight cocharacter under the assumption. Thus $\widetilde{\mu}$ factors through $\mathrm{Spin}_{\overline{\mathbb{Q}}_p}$. By the construction of ρ^{tw} (see 3.2.3), the restriction $\rho_2^{\mathrm{tw}}|_{\mathrm{Spin}}$ is the composition of

$$\mathrm{Spin} \to \mathrm{SO} \hookrightarrow \mathrm{GL}(\mathrm{H}^2(X))\,,$$

where the first arrow is the universal covering of the special orthogonal group, and the second one is the standard representation of the special orthogonal group. So $\rho_2^{\mathrm{tw}}|_{\mathrm{Spin}}$ induces an injective homomorphism $\widetilde{V}_{\mathbb{Q}} \hookrightarrow X_*(\mathrm{GL}(\mathrm{H}^2(X))_{\overline{\mathbb{Q}}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore $\widetilde{\mu} = 0$ in $\widetilde{V}_{\mathbb{Q}}$, and in particular $\mu = 0$ in $V_{\mathbb{Q}}$.

(2) We can assume that $T_{v,\mathbb{Q}_p}$ (resp. $T_{v,2,\mathbb{Q}_p}$) is a maximal torus of the connected reductive group $\mathbf{G}^{\mathrm{red}}_{p,v}$ (resp. $\mathbf{G}_{p,2v}$). Therefore, we can conclude that $\pi_{p,2}$ is an isogeny, which is multiplicative as $\mathbf{G}^{\mathrm{red}}_{p,v}$ is reductive. ☐

4.3. **Rank bounds for hyper-Kähler varieties.** Recall that the *rank* of a connected algebraic group G over a field F is the *dimension of a maximal subtorus* $T_{\overline{F}} \subseteq G_{\overline{F}}$.

Lemma 4.3.1. *Let $f\colon G \to H$ be a surjective homomorphism between connected algebraic groups. Then $\operatorname{rk} G \geq \operatorname{rk} H$. Moreover, for a short exact sequence $1 \to Z \to G \xrightarrow{f} H \to 1$, we have*

$$\operatorname{rk} G = \operatorname{rk} Z^\circ + \operatorname{rk} H\,;$$

and in particular, if f is an isogeny, then $\operatorname{rk} G = \operatorname{rk} H$.

Proof. For these statements, we may assume $F = \overline{F}$. For any maximal torus $T \subset G$, the image $f(T) \subseteq H$ is a maximal torus of H (see [Mil17, Proposition 17.20]). Thus $\operatorname{rk} G \geq \operatorname{rk} H$.

Let $T_0 \subset Z^\circ$ be a maximal torus. We may assume that a maximal torus T of G is chosen such that $T_0 \subseteq T$. In this case, T_0 is a maximal torus of the multiplicative subgroup $\ker(f|_T) \subset Z$. In particular, $\dim T_0 = \dim \ker(f|_T)$, which implies

$$\dim T_0 + \dim f(T) = \dim T\,. \qquad \square$$

Theorem 4.3.2. *Let X be a hyper-Kähler variety over a finitely generated field over $\mathbb{Q}$ with $b_2 \geq 4$. For any rational prime ℓ, we have*

$$\operatorname{rk} \mathbf{G}_\ell(X) = \operatorname{rk} \mathbf{G}_{\ell,+}(X) = \operatorname{rk} \mathbf{G}_{\ell,2}(X)\,.$$

Moreover, the identity component of the even ℓ-adic defect group P_ℓ (resp. P_ℓ^+) is the unipotent radical of $\mathbf{G}_\ell(X)^\circ$ (resp. $\mathbf{G}_{\ell,+}(X)^\circ$).

Proof. As before, we may assume that these three groups are connected by Theorem 2.3.5 after a finite field extension.

(1) First, we treat the case that K is a number field. By Proposition 4.1.5, there exist a prime ℓ and a place $v \mid \ell$ such that $\operatorname{rk} \mathbf{G}_\ell(X) = \operatorname{rk} \mathbf{G}_\ell(X_v)$. Thus we have the following (in)equalities of ranks

$$\begin{aligned}\operatorname{rk} \mathbf{G}_\ell(X) &= \operatorname{rk} \mathbf{G}_\ell(X_v)\\ &= \operatorname{rk} \mathbf{G}_{\ell,2}(X_v)\\ &\leq \operatorname{rk} \mathbf{G}_{\ell,2}(X)\,,\end{aligned}$$

where the second equality is obtained from Theorem 4.2.11 and Lemma 4.3.1. On the other hand, applying Lemma 4.3.1 on the short exact sequence

$$1 \longrightarrow P_\ell \longrightarrow \mathbf{G}_\ell(X) \longrightarrow \mathbf{G}_{\ell,2}(X) \longrightarrow 1\,,$$

we have $\operatorname{rk} \mathbf{G}_\ell(X) = \operatorname{rk} \mathbf{G}_{\ell,2}(X) + \operatorname{rk} P_\ell$. Therefore, combining these two relations, we have

$$\operatorname{rk} \mathbf{G}_\ell(X) = \operatorname{rk} \mathbf{G}_{\ell,2}(X)\,.$$

This also implies that $\operatorname{rk} P_\ell = 0$. Similarly, we also have $\operatorname{rk} P_{\ell,+} = 0$.

Therefore the identity components P_ℓ° and $P_{\ell,+}^\circ$ are connected unipotent groups by [Mil17, Theorem 20.1 (a)]. The reductivity of $\mathbf{G}_{\ell,2}(X)$ implies that $P_{\ell,+}^\circ = R_u(\mathbf{G}_{\ell,+}(X))$ and $P_\ell^\circ = R_u(\mathbf{G}_\ell(X))$.

(2) If $K/\mathbb{Q}$ is not a finite extension, then we choose a normal model B of K as follows: Take a polarization $\mathcal{L}$ on X. The pair $(X, \mathcal{L})$ defines a K-rational point $x \in M$, where M is an irreducible component of some moduli space [5] of polarized hyper-Kähler varieties that contains $(X, \mathcal{L})$. Let B be the normalization of the Zariski closure $\overline{\{x\}} \subseteq M$.

Consider the projective family $\mathfrak{X} \to B$ of hyper-Kähler varieties, obtained by pulling back the universal family along $B \to \overline{\{x\}} \subseteq M$. From the construction, we have that the

[5]Strictly speaking, M is the étale atlas of a moduli stack, where a universal family of hyper-Kähler varieties exists.

generic fiber satisfies $\mathfrak{X}_\eta \cong X$, with $\eta \in B$ the generic point. As discussed in 2.3.8, we may take a closed point $s \in B^{\mathrm{cl}}$, so that $\mathbf{G}_\ell(X)^\circ \cong \mathbf{G}_\ell(\mathfrak{X}_s)^\circ$ under cospecialization. This implies

$$\begin{aligned}\operatorname{rk}\mathbf{G}_\ell(X) = \operatorname{rk}\mathbf{G}_\ell(\mathfrak{X}_s) &= \operatorname{rk}\mathbf{G}_{\ell,2}(\mathfrak{X}_s)\\ &\leq \operatorname{rk}\mathbf{G}_{\ell,2}(X)\,.\end{aligned}$$

Here the second equality follows from (1). Therefore, $\operatorname{rk}\mathbf{G}_\ell(X) = \operatorname{rk}\mathbf{G}_{\ell,2}(X)$. □

Corollary 4.3.3. *The projection $\pi_{\ell,2}\colon \mathbf{G}_\ell(X) \twoheadrightarrow \mathbf{G}_{\ell,2}(X)$ induces isogenies of $\mathbb{Q}_\ell$-algebraic groups for any Levi subgroup $\mathbf{G}_\ell^{\mathrm{red}}(X) \subseteq \mathbf{G}_\ell(X)$:*

(1) $\mathbf{G}_\ell^{\mathrm{red}}(X) \twoheadrightarrow \mathbf{G}_{\ell,2}(X)$*;*

(2) $Z(\mathbf{G}_\ell^{\mathrm{red}}(X))^\circ \twoheadrightarrow Z(\mathbf{G}_{\ell,2}(X))^\circ$*;*

(3) $\mathbf{G}_\ell^{\mathrm{red}}(X)^{\mathrm{der}} \twoheadrightarrow \mathbf{G}_{\ell,2}(X)^{\mathrm{der}}$*.*

The same statement also holds for the projection $\mathbf{G}_{\ell,+}(X) \twoheadrightarrow \mathbf{G}_{\ell,2}(X)$.

Proof. The restriction of projections on any Levi subgroup is a surjective homomorphism between reductive groups with the same rank by Theorem 4.3.2, and thus is an isogeny. □

5. Mumford–Tate conjecture for hyper-Kähler varieties

Let X be a hyper-Kähler variety over a finitely generated field K with $b_2 \geq 4$. In this section, we will focus on the proof of our main results for X. As before, we fix a field embedding $K \hookrightarrow \mathbb{C}$. We assume that $K = K^{\mathrm{conn}}$ for $\mathfrak{h}(X)$ throughout this section.

5.1. **Monodromy group of family of hyper-Kähler varieties.** A fruitful approach, inspired by [And96a] and [Moo17], to address the Mumford–Tate conjecture is to reduce it to a simpler situation using a smooth family with "large monodromy." In the following, we recall several basic properties of the monodromy group associated with families of hyper-Kähler varieties.

5.1.1. Let B be a geometrically connected smooth variety over K. Fix a field embedding $K \subset \mathbb{C}$, and a $\mathbb{Q}$VHS $\mathbb{V}$ on $B_{\mathbb{C}} = B \times_K \mathbb{C}$. Recall that a complex point $s \in B(\mathbb{C})$ is *Hodge generic* (with respect to $\mathbb{V}$) if the Mumford–Tate group $\mathbf{MT}(\mathbb{V}_s)$ is maximal under monodromy conjugation. As a $\mathbb{Q}$-local system, the *algebraic monodromy group* of $\mathbb{V}$ (with a base point $s \in B(\mathbb{C})$) is the identity component of the Zariski closure of

$$\rho\colon \pi_1(B_{\mathbb{C}}, s) \longrightarrow \mathrm{GL}_{\mathbb{Q}}(\mathbb{V}_s)\,,$$

denoted by $\mathbf{M}(\mathbb{V})$. If one chooses a different base point, the monodromy representation changes by conjugation via parallel transport, and therefore the algebraic monodromy group stays isomorphic. In what follows, we sometimes suppress the base point when it causes no ambiguity.

5.1.2. Let us recall some general features of $\mathbf{M}(\mathbb{V})$:

- It is a normal subgroup $\mathbf{M}(\mathbb{V}) \triangleleft \mathbf{MT}(\mathbb{V}_s)^{\mathrm{der}}$ if s is Hodge generic in $B(\mathbb{C})$. See [And92, Theorem 1]. If there is a point $s \in B(\mathbb{C})$ such that $\mathbf{MT}(\mathbb{V}_s)$ is abelian, then $\mathbf{M}(\mathbb{V}) = \mathbf{MT}(\mathbb{V}_s)^{\mathrm{der}}$.
- If there is a $\mathbb{Q}_\ell$-local system $\mathbb{V}_\ell$ on S such that $\mathbb{V} \otimes \mathbb{Q}_\ell \cong \mathbb{V}_\ell^{\mathrm{an}}$, then it induces an isomorphism of $\mathbb{Q}_\ell$-algebraic groups:
$$\mathbf{M}(\mathbb{V})_{\mathbb{Q}_\ell} \cong \mathbf{M}(\mathbb{V}_\ell)\,. \tag{5.1.1}$$
See [Urb22, Lemma 3.3.] for details. Here $\mathbf{M}(\mathbb{V}_\ell)$ is the identity component of the Zariski closure of the image of
$$\pi_1^{\text{ét}}(B_{\overline{K}}, s) \to \mathrm{GL}_{\mathbb{Q}_\ell}(\mathbb{V}_s \otimes \mathbb{Q}_\ell)\,.$$

Remark 5.1.3. Motivic Galois groups of André motives in a family share similar features of those of Mumford–Tate groups. Let S be a geometrically connected smooth variety over $K \subseteq \mathbb{C}$ and $\mathcal{M}/S$ be a family of motives over S. Assume that the generic motivic Galois group $\mathbf{G}_{\mathrm{mot}}(\mathcal{M}/S)$ is connected. Then the algebraic monodromy group of its Betti realization satisfies

$$\mathbf{M}(\mathrm{H}_B(\mathcal{M}/S)) \lhd \mathbf{G}_{\mathrm{mot}}(\mathcal{M}/S)^{\mathrm{der}}$$

as a normal subgroup. If there is a point $s \in S(\mathbb{C})$ such that $\mathbf{G}_{\mathrm{mot}}(\mathcal{M}_s)$ is abelian, then $\mathbf{M}(\mathrm{H}_B(\mathcal{M}/S)) = \mathbf{G}_{\mathrm{mot}}(\mathcal{M}/S)^{\mathrm{der}}$. See [And96b, Theorem 0.6.4].

5.1.4. Let $\mathbb{V}_\ell$ be an ℓ-adic local system on B. Consider the subset of points of B whose ℓ-adic algebraic monodromy group is strictly smaller than that of the generic fiber under the specialization $\eta \rightsquigarrow s$, i.e.,

$$\mathrm{Exc}_\ell \coloneqq \left\{ s \in B \;\middle|\; \mathbf{G}_{\ell,s} \subsetneq \mathbf{G}_{\ell,\eta} \right\}.$$

The subset Exc_ℓ is called the (ℓ-adic) exceptional locus of $\mathbb{V}_\ell$; and any point $s \in S \smallsetminus \mathrm{Exc}_\ell$ is called a *(ℓ-adic) Galois generic point.* Clearly, the generic point $\eta \in B$ is Galois generic with respect to any ℓ-adic local system on B by the definition.

Example 5.1.5. Let $f\colon \mathfrak{X} \to B$ be a smooth projective family of hyper-Kähler varieties (with $b_2 \geq 4$). For the ℓ-adic local system $\mathbb{V}_{\ell,2} \coloneqq R^2 f_* \mathbb{Q}_\ell$, a point $s \in B$ is Galois generic with respect to $\mathbb{V}_{\ell,2}$ if and only if $s_{\mathbb{C}} \in B(\mathbb{C})$ is Hodge generic with respect to $\mathbb{V}_2 \coloneqq R^2 f_{\mathbb{C},*}\mathbb{Q}$ for any field embedding $k(s) \hookrightarrow \mathbb{C}$, by ($\mathrm{MTC}_2$) (= Theorem 1.2.3) for hyper-Kähler varieties.

Remark 5.1.6. Our definition of the exceptional locus coincides with the traditional one based on the dimension of ℓ-adic monodromy groups. While the latter is standard in the literature, Bogomolov's theorem guarantees that for the geometric local systems considered here, the images are Zariski-dense open subgroups, rendering the two approaches equivalent.

5.1.7. In a general set-up, the reductivity of $\mathbf{G}_{\ell,s}$ is missing. It is convenient to consider the following two variants of the exceptional loci for an ℓ-adic local system

- $\mathrm{Exc}_\ell^{\mathrm{ss}} = \left\{ s \in B \;\middle|\; \mathrm{ss\,rk}\,\mathbf{G}_{\ell,s} < \mathrm{ss\,rk}\,\mathbf{G}_{\ell,\eta} \right\}$;
- $\mathrm{Exc}_\ell^{\mathrm{rk}} = \left\{ s \in B \;\middle|\; \mathrm{rk}\,\mathbf{G}_{\ell,s} < \mathrm{rk}\,\mathbf{G}_{\ell,\eta} \right\}$,

which take the *semisimple rank* and *rank* into account respectively. Clearly, $\mathrm{Exc}_\ell^{\mathrm{ss}} \subseteq \mathrm{Exc}_\ell$ and $\mathrm{Exc}_\ell^{\mathrm{rk}} \subseteq \mathrm{Exc}_\ell$.

Lemma 5.1.8. *Suppose $\mathbb{V}_\ell^{\mathrm{an}} \cong \mathbb{V} \otimes \mathbb{Q}_\ell$ for some $\mathbb{Q}$-local system $\mathbb{V}$ that supports a polarizable $\mathbb{Q}$VHS. If $s \in S \smallsetminus (\mathrm{Exc}_\ell^{\mathrm{ss}} \cup \mathrm{Exc}_\ell^{\mathrm{rk}})$, then there is a Levi subgroup $\mathbf{G}_{\ell,s}^{\mathrm{red}}$ and a natural injective homomorphism $i_s\colon \mathbf{M}(\mathbb{V}_\ell) \to \mathbf{G}_{\ell,s}^{\mathrm{red,der}}$ whose image is a normal subgroup.*

Proof. There is a short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\text{ét}}(S_{\overline{K}}) & \longrightarrow & \pi_1^{\text{ét}}(S) & \longrightarrow & G_K & \longrightarrow & 1\,. \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & G_{k(\eta)} & & \end{array}$$

The Galois action on $\mathbb{V}_{\overline{\eta}}$ factors through the surjection $G_{k(\eta)} \twoheadrightarrow \pi_1^{\text{ét}}(S)$, and therefore

$$\mathbf{G}_{\ell,\eta} = \mathrm{Im}\left(\pi_1^{\text{ét}}(S) \to \mathrm{GL}_{\mathbb{Q}_\ell}(\mathbb{V}_{\overline{\eta}})\right)^{\mathrm{Zar},\circ},$$

It induces an injective homomorphism $i_\eta\colon \mathbf{M}(\mathbb{V}_\ell) \longrightarrow \mathbf{G}_{\ell,\eta}$ whose image is a normal subgroup.

As $\mathbb{V}$ supports a $\mathbb{Q}$VHS, the Artin comparison implies that $\mathbf{M}(\mathbb{V}_\ell) \cong \mathbf{M}(\mathbb{V})_{\mathbb{Q}_\ell}$ is semisimple, and thus $\mathbf{M}(\mathbb{V}_\ell) \subseteq \mathbf{G}_{\ell,\eta}^{\mathrm{red,der}}$ for some Levi subgroup $\mathbf{G}_{\ell,\eta}^{\mathrm{red}} \subseteq \mathbf{G}_{\ell,\eta}$. The condition $s \in S \smallsetminus (\mathrm{Exc}_\ell^{\mathrm{ss}} \cup \mathrm{Exc}_\ell^{\mathrm{rk}})$ implies that $\mathrm{rk}\, \mathbf{G}_{\ell,s} = \mathrm{rk}\, \mathbf{G}_{\ell,\eta}$, and $\mathbf{G}_{\ell,s}^{\mathrm{red,der}} = \mathbf{G}_{\ell,\eta}^{\mathrm{red,der}}$ by Borel-de Siebenthal theorem for some Levi subgroup $\mathbf{G}_{\ell,s}^{\mathrm{red}} \subseteq \mathbf{G}_{\ell,s}$. Hence $\mathrm{sp}_{\eta,s,*}(\mathbf{M}(\mathbb{V}_\ell)) \subseteq \mathbf{G}_{\ell,s}^{\mathrm{red,der}}$. We can thus take $i_s \coloneqq \mathrm{sp}_{\eta,s,*} \circ i_\eta$. ☐

Lemma 5.1.9. *Let $\mathfrak{X} \to B$ be a smooth projective family of hyper-Kähler varieties with $b_2 \geq 4$. If $s \in S$ is ℓ-adic Galois generic with respect to $\mathbb{V}_{\ell,2}$, then $s \in S \smallsetminus (\mathrm{Exc}_\ell^{\mathrm{ss}} \cup \mathrm{Exc}_\ell^{\mathrm{rk}})$ with respect to $\mathbb{V}_\ell = Rf_*\mathbb{Q}_\ell$.*

Proof. The cospecialization along $\eta \rightsquigarrow s$ provides us a commutative diagram

$$\begin{array}{ccc} \mathbf{G}_{\ell,s} & \hookrightarrow & \mathbf{G}_{\ell,\eta} \\ \downarrow & & \downarrow \\ \mathbf{G}_{\ell,2,s} & \xrightarrow{\simeq} & \mathbf{G}_{\ell,2,\eta} \end{array}$$

where the vertical arrows are the restriction of $\pi_{\ell,2}$ on derived subgroups, and are both surjective. The bottom arrow is an isomorphism as $s \in S \smallsetminus \mathrm{Exc}_\ell$ for $\mathbb{V}_{\ell,2}$. The projections $\pi_{\ell,2}$ are isogeny at both s and η after being restricted on Levi subgroups and the their derived subgroups by Corollary 4.3.3. Hence

$$\mathrm{ss}\,\mathrm{rk}\, \mathbf{G}_{\ell,s} = \mathrm{ss}\,\mathrm{rk}\, \mathbf{G}_{\ell,\eta} \quad \mathrm{rk}\, \mathbf{G}_{\ell,s} = \mathrm{rk}\, \mathbf{G}_{\ell,\eta}.$$ ☐

5.2. **Families of maximal monodromy.** Let $\mathbb{V}$ be a $\mathbb{Q}$VHS on S. If the algebraic monodromy group $\mathbf{M}(\mathbb{V}) = \mathbf{MT}(\mathbb{V}_s)^{\mathrm{der}}$ for Hodge generic points s, then we say $\mathbb{V}$ has *maximal monodromy.*

If $\mathbb{V}$ arises from a family of hyper-Kähler varieties in degree 2, then it is a $\mathbb{Q}$VHS of K3-type. According to Zarhin's results [Zar83], $\mathbf{MT}(\mathbb{V}_{2,s})^{\mathrm{der}}$ is simple over $\mathbb{Q}$ when $\dim \mathbb{V}_2$ is sufficiently large. Let $E = \mathrm{End}_{\mathbb{Q}\,\mathrm{VHS}}(\mathbb{V}_2)$ the endomorphism algebra.

Lemma 5.2.1. [Moo17, Proposition 6.4 (iii)] *Let $\mathfrak{X} \to S$ be a non-isotrivial family of hyper-Kähler varieties. Then $\mathbb{V}_2$ has maximal monodromy if* $\mathrm{rk}_E \mathbb{V}_2 \neq 4$ *or E is a CM field.*

Remark 5.2.2. In the case that E is totally real and $\dim_E \mathbb{V}_2 = 4$, if the discriminant of BBF form q on $\mathbb{V}_2$ is not a square in E, then the special orthogonal group $\mathrm{SO}_{E/\mathbb{Q}}(\mathbb{V}_s, q_s)$ is still $\mathbb{Q}$-(almost) simple, so $\mathbb{V}_2$ also has maximal monodromy.

5.2.3. As recalled in 5.1.2, in Hodge theory, the derived subgroup of the generic Mumford–Tate group of a (polarized) $\mathbb{Q}$-VHS contains the algebraic monodromy group as a normal subgroup, which follows from Deligne's theorem of fixed part. For a family of hyper-Kähler varieties, this makes it possible to determine the derived subgroup of the ℓ-adic monodromy group in degree 2 for a Galois generic fiber, assuming the Mumford–Tate conjecture holds in degree 2.

With the estimation of ranks for ℓ-adic algebraic monodromy groups, we have seen the derived subgroups $\mathbf{G}_\ell^{\mathrm{red,der}}$ and $\mathbf{G}_{\ell,+}^{\mathrm{red,der}}$ are both isogenous to the derived subgroup in degree 2 (Corollary 4.3.3). Here we give a more precise description for the isogenies at the Galois generic fibers.

Proposition 5.2.4. *Let $f\colon \mathfrak{X} \to B$ be a smooth projective family of hyper-Kähler varieties over a smooth geometrically connected variety B over K which is of maximal monodromy in degree 2. Let $\mathbf{G}_{\ell,\square}^{\mathrm{red,der}} \coloneqq \mathbf{G}_{\ell,\square}^{\mathrm{red}}(\mathfrak{X}_s)^{\mathrm{der}}$ be the derived group of a Levi subgroup of the ℓ-adic algebraic monodromy group of the fiber $\mathfrak{X}_s$ at some point $s \in S \smallsetminus \mathrm{Exc}_{\ell,2}$. Then*

$$\mathbf{MT}_\square(\mathfrak{X}_{s_{\mathbb{C}}})^{\mathrm{der}} = \mathbf{M}_\square(\mathbb{V}) = \mathbf{G}_{\ell,\square}^{\mathrm{red,der}}$$

in $\mathbf{G}_{\mathrm{mot},\square}(\mathfrak{X}_s)$. *Moreover,*

(1) The projection $\pi_{\ell,2}\colon \mathbf{G}_{\ell,+}^{\mathrm{red,der}} \longrightarrow \mathbf{G}_{\ell,2}^{\mathrm{der}}$ *is an isomorphism.*

(2) Suppose that the fibers have non-zero odd degree cohomologies. Let $\delta \in \mathbf{G}_{\ell,\eta}$ *be the involution as in Lemma 3.3.5. The projection*

$$\pi_{\ell,2}\colon \mathbf{G}_{\ell}^{\mathrm{red,der}} \longrightarrow \mathbf{G}_{\ell,2}^{\mathrm{der}}$$

is an isogeny of degree 2*, whose kernel is generated by* δ.

Proof. Let $\mathbb{V}_{\ell,2} \coloneqq R^2_{\mathrm{pr}} f_* \mathbb{Q}_\ell$ be the ℓ-adic local system of primitive cohomologies on the family f, and similarly for the even part. Let $\mathbf{M}(\mathbb{V}_{\ell,2})$ and $\mathbf{M}(\mathbb{V}_{\ell,+})$ be the geometric ℓ-adic algebraic monodromy groups of $\mathbb{V}_{\ell,+}$ and $\mathbb{V}_{\ell,2}$ respectively (with $\overline{\eta}$ the base point). By Lemma 5.1.8, there is an injective homomorphism $i_2\colon \mathbf{M}(\mathbb{V}_{\ell,2}) \longrightarrow \mathbf{G}_{\ell,2}$ whose image is a normal subgroup. Under the isomorphisn $\mathbf{G}_{\ell,2} \xrightarrow{\sim} \mathbf{G}_{\mathrm{mot},2,\mathbb{Q}_\ell}$, the image of $\mathbf{M}(\mathbb{V}_{\ell,2})$ is equal to the derived group of $\mathbf{MT}_{2,\mathbb{Q}_\ell}$ under the maximal monodromy assumption on $\mathbb{V}_2$.

(1). The projection of algebraic monodromy group $\mathbf{M}(\mathbb{V}_+) \twoheadrightarrow \mathbf{M}(\mathbb{V}_2)$ is an isomorphism since it is the restriction of the isomorphism $\pi_2\colon \mathbf{MT}_+ \xrightarrow{\sim} \mathbf{MT}_2$. Then the Artin comparison (5.1.1) implies that

$$\pi_{\ell,2}\colon \mathbf{M}(\mathbb{V}_{\ell,+}) \xrightarrow{\sim} \mathbf{M}(\mathbb{V}_{\ell,2}),$$

The following diagram commutes

$$\begin{array}{ccccc} \mathbf{M}(\mathbb{V}_{\ell,+}) & \xrightarrow{i_+} & \mathbf{G}_{\ell,+}^{\mathrm{red,der}} & \longrightarrow & \mathbf{G}_{\mathrm{mot},+,\mathbb{Q}_\ell} \\ \downarrow{\scriptstyle \simeq} & & \downarrow & & \downarrow \uparrow{\scriptstyle \sigma_{\mathbb{Q}_\ell}} \\ \mathbf{M}(\mathbb{V}_{\ell,2}) & \xrightarrow[i_2]{\simeq} & \mathbf{G}_{\ell,2}^{\mathrm{der}} & \longrightarrow & \mathbf{G}_{\mathrm{mot},2,\mathbb{Q}_\ell}. \end{array} \tag{5.2.1}$$

Here i_2 exists by Lemma 5.1.8 as s is ℓ-adic Galois generic (with respect to $\mathbb{V}_{\ell,2}$); and i_+ exists by Lemma 5.1.9. We set $\sigma_\ell \coloneqq i_+ \circ \pi_{\ell,2}^{-1} \circ i_2^{-1}$, which is the splitting required.

(2). Again, by Lemma 5.1.8 and Lemma 5.1.9, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{M}(\mathbb{V}_{\ell}) & \xhookrightarrow{i} & \mathbf{G}_{\ell}^{\mathrm{red,der}} \\ \downarrow{\scriptstyle 2:1} & & \downarrow \\ \mathbf{M}(\mathbb{V}_{\ell,2}) & \xrightarrow[\sim]{i_2} & \mathbf{G}_{\ell,2}^{\mathrm{der}}. \end{array} \tag{5.2.2}$$

The left vertical two-to-one isogeny is from the isogeny $\pi_2\colon \mathbf{MT}(X_{\mathbb{C}}) \twoheadrightarrow \mathbf{MT}_2(X_{\mathbb{C}})$ in Lemma 3.2.5. In fact, the equality $\mathbf{M}(\mathbb{V}_2) = \mathbf{MT}_2(X_{\mathbb{C}})^{\mathrm{der}}$ from the maximal monodromy assumption forces the equality $\mathbf{M}(\mathbb{V}) = \mathbf{MT}(X_{\mathbb{C}})^{\mathrm{der}}$ which provides us the isogeny by the Artin comparison.

In both (1) and (2), since $\dim \mathbf{G}_{\ell}^{\mathrm{red,der}} = \dim \mathbf{G}_{\ell,2}^{\mathrm{der}}$ by Theorem 4.3.2, i and i_+ are actually isomorphisms. In particular,

$$\mathbf{MT}_{\square}(X_{\mathbb{C}})^{\mathrm{der}} = \mathbf{M}_{\square}(\mathbb{V}) = \mathbf{G}_{\ell,\square}^{\mathrm{red,der}}.$$

under the comparison isomorphism. The rest statements then follow from the situation for Mumford–Tate groups, as in Lemma 3.2.5. □

Combining with the (local) Torelli theorem of hyper-Kähler varieties and the descent results given in [And96b], we can deduce the following result on the existence of a family of maximal monodromy.

Proposition 5.2.5. *Let X be a hyper-Kähler variety over K with $b_2 \geq 4$. There is a smooth projective family $\mathfrak{X} \to B$ of hyper-Kähler varieties with maximal monodromy over a smooth geometrically connected variety B such that $\mathfrak{X}_b \cong X$ for some $b \in B \smallsetminus \mathrm{Exc}_{\ell,2}$.*

Proof. Fix a field embedding $K \subseteq \mathbb{C}$. Fix a polarization $\mathcal{L}$ on X and a polarized universal family $(f\colon \mathfrak{X} \to B, \underline{\mathcal{L}})$ such that $f^{-1}(b) \cong X$ and $\underline{\mathcal{L}}_b$ for some point $b \in B(K)$. The period map

$$\Phi_2\colon B_{\mathbb{C}} \longrightarrow \Gamma\backslash D_\Lambda\,,$$

has open image and is finite étale onto its image, by the local Torelli theorem (see [Bea83, §8]).

If X is of CM-type, i.e., $\mathbf{MT}_2(X)$ is abelian, then its derived subgroup is trivial and the requirements trivially hold for $X \to \mathrm{Spec}(K)$. So we may further assume that $\mathbf{MT}_2(X)$ is not abelian. Under the assumption, the derived subgroup $G_2 \coloneqq \overline{\mathbf{MT}}_2(X)^{\mathrm{der}}$ is non-trivial as a subgroup of $\mathrm{SO}(\mathcal{L}_b^{\perp})$. Let $U \subseteq B$ be the connected component of the preimage of the special subvariety $\Gamma'\backslash D_{G_2} \subseteq \Gamma\backslash D_\Lambda$ such that $b_{\mathbb{C}} \in U(\mathbb{C})$, along Φ_2. Then $b \in U$ is a point such that $b_{\mathbb{C}} \in U(\mathbb{C})$ is Hodge generic with respect to the $\mathbb{Q}$VHS $\underline{\mathcal{L}}^{\perp}|_U$. Clearly, U is smooth as Φ_2 is étale onto its image. On the other hand, since $\Gamma'\backslash D_{G_2}$ is a connected component of a Shimura variety, $U(\mathbb{C})$ contains points whose image in $\Gamma'\backslash D_{G_2}$ are special points (=Hodge structures of CM-type).

Moreover, Φ_2 is defined over a finite extension of K by the descent results in [And96b]. Therefore, after a finite extension of K, we may assume that U is defined over K and b is a K-rational point. By Example 5.1.5, we can see that b is ℓ-adic Galois generic with respect to $\underline{\mathcal{L}}^{\perp}_{\mathbb{Q}_\ell}|_U$. □

Corollary 5.2.6. *For a hyper-Kähler variety over K with $b_2 \geq 4$, the derived subgroup* $\mathbf{G}_\ell(X)^{\mathrm{red,der}} = \mathbf{MT}(X_{\mathbb{C}})^{\mathrm{der}} \subseteq \mathbf{G}_{\mathrm{mot}}(X)$

Proof. This is a combination of Proposition 5.2.5 and Proposition 5.2.4. □

5.3. Hyper-Kähler varieties with real multiplication.

Let $T(X) \subseteq \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q})$ be the transcendental part, i.e., the *smallest* $\mathbb{Q}$-sub Hodge structure such that $T(X) \otimes_{\mathbb{Q}} \mathbb{C}$ contains the 1-dimensional $\mathbb{C}$-vector space $\mathrm{H}^{2,0}(X_{\mathbb{C}})$. Then the *endomorphism field*

$$E \coloneqq \mathrm{End}_{\mathrm{Hdg}}(T(X)) \subset \mathbb{C}$$

is a number field by [Zar83, Theorem 1.6], and is equipped with an involution $*$ on E such that

$$q(ex, y) = q(x, e^* y), \quad \forall e \in E \text{ and } x, y \in T(X)\,.$$

Definition 5.3.1. A hyper-Kähler variety X has *real multiplication* if the endomorphism field E of $X_{\mathbb{C}}$ is *totally real.*

Remark 5.3.2. In the literature, the condition that X has real multiplication typically also includes the requirement that $E \neq \mathbb{Q}$. For the sake of simplicity, we will omit this assumption here.

5.3.3. Let $E_\ell \coloneqq E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. The ℓ-adic algebraic monodromy group $\mathbf{G}_{\ell,2}(X)$ can be classified into two cases, by Zarhin's results in *loc. cit.* and MTC_2 for X (or Theorem 2.6 in [Moo17]).

(SO) If the involution $*$ on E is trivial, i.e., E is totally real, then we have the almost direct product

$$\mathbf{G}_{\ell,2}(X)^{\circ} \cong \mathbb{G}_{m,\mathbb{Q}_\ell} \cdot \mathrm{SO}_{E_\ell/\mathbb{Q}_\ell}(T(X), \widetilde{q}_X)$$

with $T(X)$ viewed as an E-vector space and $\widetilde{q}$ the E-bilinear form on $T(X)$ extending the BBF-form q_X, and $\mathrm{SO}_{E_\ell/\mathbb{Q}_\ell}(T(X), \widetilde{q}_X)$ is viewed as a $\mathbb{Q}_\ell$-algebraic group by taking the Weil restriction.

(U) If the involution $*$ on E is non-trivial, then we have the almost direct product

$$\mathbf{G}_{\ell,2}(X)^\circ \cong \mathbb{G}_{m,\mathbb{Q}_\ell} \cdot \mathrm{U}_{E_\ell/\mathbb{Q}_\ell}(T(X), \widetilde{q}_X)$$

with $\widetilde{q}_X$ the E-Hermitian form extending the BBF-form q_X.

In both cases, the subgroup $\mathbb{G}_{m,\mathbb{Q}_\ell} \subseteq \mathbf{G}_{\ell,2}(X)^\circ$ is the group of homotheties. Consider the decomposition

$$\mathbf{G}_\ell^{\mathrm{red}}(X) = Z(\mathbf{G}_\ell^{\mathrm{red}}(X)) \cdot \mathbf{G}_\ell^{\mathrm{red}}(X)^{\mathrm{der}}.$$

In the case (SO), the corresponding decomposition for $\mathbf{G}_{\ell,2}(X)^\circ$ is

$$\mathbb{G}_{m,\mathbb{Q}_\ell} \cdot \mathrm{SO}_{E_\ell/\mathbb{Q}_\ell}(T(X), \widetilde{q}_X).$$

In the case (U), let $E_0 = E^*$ the invariant subfield of E under the involution. The corresponding decomposition for $\mathbf{G}_{\ell,2}(X)^\circ$ is

$$\left(\mathbb{G}_{m,\mathbb{Q}_\ell} \cdot \mathrm{Res}^{(1)}_{E/E_0}(\mathbb{G}_m)_{\mathbb{Q}_\ell}\right) \cdot \mathrm{SU}_{E_\ell}(T(X), \widetilde{q}_X),$$

where the 1-dimensional E_0-torus

$$\mathrm{Res}^{(1)}_{E/E_0}(\mathbb{G}_m) \coloneqq \ker\left(\mathrm{Res}_{E/E_0}(\mathbb{G}_m) \xrightarrow{N_{E/E_0}} \mathbb{G}_m\right)$$

is the center of $\mathrm{U}_{E/\mathbb{Q}}(T(X), \widetilde{q}_X)$.

As an example, we can consider a universal family of polarized hyper-Kähler varieties, i.e., a projective smooth family $\mathfrak{X} \to S$, which is the base change along a finite étale covering $S \to M$, onto some connected component M of the moduli space of polarized hyper-Kähler varieties.

Proposition 5.3.4. *Let $\mathfrak{X} \to S$ be a universal family of polarized hyper-Kähler varieties with $b_2 \geq 4$.*

(1) The generic fiber $\mathfrak{X}_\eta$ has endomorphism field $E = \mathbb{Q}$. In particular, it is of real multiplication.

(2) We have $\mathbf{G}^{\mathrm{red,der}}_{\ell,\eta} = \overline{\mathbf{MT}}(\mathfrak{X}_{\eta_\mathbb{C}})_{\mathbb{Q}_\ell}$ for some Levi subgroup $\mathbf{G}^{\mathrm{red}}_{\ell,\eta}$, under the comparison isomorphism.

Proof. (1) Let $\Lambda_h = h^\perp$ be the sublattice of $\mathrm{H}^2(\mathfrak{X}_{\eta_\mathbb{C}}, \mathbb{Z})$ obtained by the orthogonal complement. The local Torelli theorem of hyper-Kähler varieties implies the monodromy representation

$$\pi_1(S_\mathbb{C}) \to \mathrm{GL}_\mathbb{Q}\left(\mathrm{H}^2(\mathfrak{X}_{\eta_\mathbb{C}}, \mathbb{Q})\right)$$

has dense image in $\mathrm{O}(\Lambda_h)_\mathbb{Q}$ or $\mathrm{SO}(\Lambda_h)_\mathbb{Q}$ (see [And96b, Corollary 3.3.3.]). Therefore, the algebraic monodromy group $\mathbf{M}(\mathbb{V}_2) = \mathrm{SO}(\Lambda_h)_\mathbb{Q}$. The geometric generic point $\eta_\mathbb{C}$ is Hodge generic as $\mathbb{C}$-point, and thus $\mathbf{M}(\mathbb{V}_2) \lhd \overline{\mathbf{MT}}_2(\mathfrak{X}_{\eta_\mathbb{C}})$. On the other hand, the special Mumford–Tate group $\overline{\mathbf{MT}}_2(\mathfrak{X}_{\eta_\mathbb{C}}) \subseteq \mathrm{SO}(\Lambda_h)_\mathbb{Q}$, which forces an equality $\overline{\mathbf{MT}}_2(\mathfrak{X}_{\eta_\mathbb{C}}) = \mathrm{SO}(\Lambda_h)_\mathbb{Q}$ as a $\mathbb{Q}$-algebraic subgroup. Compare its dimension with Zarhin's computation, we can see $T(\mathfrak{X}_{\eta_\mathbb{C}}) \cong \Lambda_h$, and $E = \mathbb{Q}$.

(2) We may assume $\mathrm{H}^-(X) \neq 0$. Notice that $\mathbf{M}(\mathbb{V}_\ell) \cong \mathbf{M}(\mathbb{V})_{\mathbb{Q}_\ell} \cong \mathrm{Spin}(\Lambda_h)_{\mathbb{Q}_\ell}$. On the other hand, we have $\delta \in \overline{\mathbf{MT}}(\mathfrak{X}_{\eta_\mathbb{C}}) \cong \mathbf{M}(\mathbb{V})$. Hence $\delta \in \mathbf{M}(\mathbb{V}_\ell)$ under the Artin comparison. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}(\Lambda_h)_{\mathbb{Q}_\ell} \cong \mathbf{M}(\mathbb{V}_\ell) & \hookrightarrow & \mathbf{G}^{\mathrm{red,der}}_{\ell,\eta} \\ \downarrow{\scriptstyle 2:1} & & \downarrow \\ \mathrm{SO}(\Lambda_h)_{\mathbb{Q}_\ell} \cong \mathbf{M}(\mathbb{V}_{2,\ell}) & \xrightarrow{\simeq} & \mathbf{G}^{\mathrm{der}}_{\ell,2,\eta}. \end{array}$$

Therefore, $\mathbf{G}_{\ell,\eta}^{\mathrm{red,der}} \twoheadrightarrow \mathbf{G}_{\ell,2,\eta}^{\mathrm{der}}$ is the two-to-one covering of a special orthogonal group, and $\delta \in \mathbf{G}_{\ell,\eta}^{\mathrm{der}}$ is the generator of the kernel. □

Recall that there is an isogeny

$$Z(\mathbf{G}_{\ell,+}(X)) \longrightarrow Z(\mathbf{G}_{\ell,2}(X)) \tag{5.3.1}$$

by Corollary 4.3.3. According to Zarhin's computation, the center of ℓ-adic algebraic monodromy group of a hyper-Kähler variety with real multiplication is relatively easily to be described.

Lemma 5.3.5. *Suppose that X has real multiplication. Then*

$$Z(\mathbf{G}_{\ell}^{\mathrm{red}}(X)) \cong Z(\mathbf{G}_{\ell,+}^{\mathrm{red}}(X))^{\circ} \cong \mathbb{G}_{m,\mathbb{Q}_\ell}.$$

Proof. Since X has real multiplication, we have $Z(\mathbf{G}_{\ell,2}(X)) = \mathbb{G}_{m,\mathbb{Q}_\ell}$ as the subgroup of homotheties by Zarhin's results. This also implies that $\dim Z(\mathbf{G}_\ell(X)) = 1$ since the centers are isogenous. As the group of homotheties $\mathbb{G}_{m,\mathbb{Q}_\ell} \subseteq Z(\mathbf{G}_\ell(X))$, we can conclude that the connected component $Z(\mathbf{G}_\ell(X))^{\circ} = \mathbb{G}_{m,\mathbb{Q}_\ell}$. For $\mathbf{G}_{\ell,+}(X)$ the argument is the same. □

Proposition 5.3.6. *Suppose that X has real multiplication. There is an injective homomorphism between the centers*

$$Z(\mathbf{G}_{\ell,2}(X)) \longrightarrow Z(\mathbf{G}_{\ell,+}(X)),$$

which splits (5.3.1) *with the image equal to the connected component $Z(\mathbf{G}_{\ell,+}(X))^{\circ}$.*

Proof. If X has real multiplication, then $Z(\mathbf{G}_{\ell,2}(X)) = \mathbb{G}_{m,\mathbb{Q}_\ell}$ is given by the group of homotheties on $\mathrm{H}^2_\ell(X)$. Then the required statement follows from Lemma 2.2.11, i.e., the image of weight cocharacter $\mathbb{G}_{m,\mathbb{Q}_\ell} \subset Z\left(\mathbf{G}_{\ell,+}(X)\right)$. Moreover, as the restriction of the projection $\pi_{\ell,2}$ on $\mathbb{G}_{m,\mathbb{Q}_\ell} \subset Z\left(\mathbf{G}_{\ell,+}(X)\right)$ is given as

$$(1, z^2, \cdots, \underbrace{z^{2k}}_{\mathrm{H}^{2k}_\ell(X)}, \cdots, z^{4n}) \mapsto (z^2, \cdots, z^2),$$

which is injective. Therefore, we can define $Z(\mathbf{G}_{\ell,2}(X)) \cong \mathbb{G}_{m,\mathbb{Q}_\ell} \hookrightarrow Z(\mathbf{G}_{\ell,+}(X))$ as

$$(t, \cdots, t) \mapsto (1, t, \cdots, t^k, \cdots, t^{2n}).$$

We can directly check that this is the injective homomorphism as we required. The image is a connected component by Lemma 5.3.5. □

5.3.7. We show that the action of the Levi subgroup $\mathbf{G}_\ell^{\mathrm{red}}$ factors through the twisted LLV representation. For general hyper-Kähler varieties, this is a direct consequence of Proposition 5.3.4. In addition, we employ a specialization argument similar to that in previous sections.

Consider a universal family $\mathfrak{X} \to S$ (over K) of hyper-Kähler varieties which contains X as a fiber at a K-rational point $s \in S(K)$. By Proposition 5.3.4, the generic fiber $\mathfrak{X}_\eta$ admits real multiplication. The specialization isomorphism $\mathrm{H}_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}_{\eta,s}} \mathrm{H}_{\text{ét}}(\mathfrak{X}_{\overline{s}}, \mathbb{Q}_\ell)$ on ℓ-adic étale cohomology can be chosen to be motivated (see [Cad13, Lemma 4.6]). Therefore, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{G}_{\ell,s}^{\mathrm{red}} & \xhookrightarrow{\mathrm{sp}^*_{\eta,s}} & \mathbf{G}_{\ell,\eta}^{\mathrm{red}} \\ \big\downarrow & & \big\downarrow \\ (\mathbf{G}_{\mathrm{mot},s})_{\mathbb{Q}_\ell} & \xhookrightarrow{\mathrm{sp}^*_{\eta,s}} & (\mathbf{G}_{\mathrm{mot},\eta})_{\mathbb{Q}_\ell}\,. \end{array} \tag{5.3.2}$$

Proposition 5.3.8. *Keep the notations same as in Notation 3.2.2. For a hyper-Kähler variety X over K with $b_2 \geq 4$, we have*

$$\mathbf{G}_\ell^{\mathrm{red}}(X) \subseteq \rho^{\mathrm{tw}}(\mathrm{GSpin}_\ell)\,,$$

for some Levi subgroup $\mathbf{G}_\ell^{\mathrm{red}}(X) \subseteq \mathbf{G}_\ell(X)$.

Proof. Consider the generic fiber $\mathfrak{X}_\eta$. Proposition 5.3.4 implies that $\mathfrak{X}_\eta$ has real multiplication, and

$$\mathbf{G}_{\ell,\eta}^{\mathrm{red}} = \mathbb{G}_{m,\ell} \cdot \mathbf{G}_{\ell,\eta}^{\mathrm{red,der}} = \left(\mathbb{G}_m \cdot \overline{\mathbf{MT}}_\eta\right)_{\mathbb{Q}_\ell}\,,$$

by Lemma 5.3.5 and Proposition 5.3.4. Therefore, $\mathbf{G}_{\ell,\eta}^{\mathrm{red}} \subseteq \rho^{\mathrm{tw}}(\mathrm{GSpin}_\ell)$ by Remark 3.2.4. Now we can deduce the inclusion for hyper-Kähler variety $X = \mathfrak{X}_s$ by applying Lemma 3.2.7 on (5.3.2). □

Corollary 5.3.9. *There is a unique Levi subgroup $\mathbf{G}_\ell^{\mathrm{red}}(X) \subseteq \mathbf{G}_\ell(X)$.*

Proof. By [Hoc81, Theorem 4.3], all Levi subgroups of $\mathbf{G}_\ell(X)$ are P_ℓ°-conjugate to $\mathbf{G}_\ell^{\mathrm{red}}(X)$ in Proposition 5.3.8. However, Lemma 3.3.2 implies the conjugate action of P_ℓ on $\mathbf{G}_\ell(X)$ is trivial as $\mathbf{G}_\ell^{\mathrm{red}}(X) \subseteq \rho^{\mathrm{tw}}(\mathrm{GSpin}_\ell)$. □

Remark 5.3.10. Corollary 5.3.9 implies the Levi subgroups chosen in Proposition 5.2.4 and Proposition 5.3.4 are canonical.

5.4. **Connectivity of $P_{\ell,+}$.** Consider the short exact sequence

$$1 \longrightarrow P_{\ell,+} \longrightarrow \mathbf{G}_{\ell,+}(X) \xrightarrow{\pi_{\ell,2}} \mathbf{G}_{\ell,2}(X) \longrightarrow 1\,. \tag{5.4.1}$$

The kernel $P_{\ell,+}$ is called the *ℓ-adic even defect group* of X. Since $P_{\ell,+}^\circ$ is the unipotent radical of the connected algebraic group $\mathbf{G}_{\ell,+}(X)$ by Theorem 4.3.2, we have a semi-direct product

$$\mathbf{G}_{\ell,+}(X) = \mathbf{G}_{\ell,+}^{\mathrm{red}}(X) \rtimes P_{\ell,+}^\circ$$

which is unique up to $P_{\ell,+}^\circ$-conjugation (see [Hoc81, Theorem 4.3]). The projection $\mathbf{G}_{\ell,+}^{\mathrm{red}}(X) \twoheadrightarrow \mathbf{G}_{\ell,2}(X)$ is an isogeny with the kernel equal to the group of connected components $\pi_0(P_{\ell,+})$.

In this subsection, we will prove that $P_{\ell,+}$ is connected and that the semi-direct product is actually a product, which is compatible with the decomposition (3.3.3).

5.4.1. As a reductive group, the second degree ℓ-adic monodromy group $\mathbf{G}_{\ell,2}$ admits a decomposition

$$\mathbf{G}_{\ell,2} = Z(\mathbf{G}_{\ell,2}) \cdot \mathbf{G}_{\ell,2}^{\mathrm{der}}\,,$$

where $Z(G_{\ell,2})$ is the center and $\mathbf{G}_{\ell,2}^{\mathrm{der}}$ is the derived subgroup of $\mathbf{G}_{\ell,2}$. The projection $\pi_{\ell,2}$ induces surjective homomorphisms

$$Z(\mathbf{G}_{\ell,+}) \longrightarrow Z(\mathbf{G}_{\ell,2}) \quad \text{and} \quad \mathbf{G}_{\ell,+}^{\mathrm{der}} \longrightarrow \mathbf{G}_{\ell,2}^{\mathrm{der}}\,.$$

To obtain a splitting of $\pi_{\ell,2}$, it is sufficient to construct splittings for these two homomorphisms separately.

Firstly, we deal with the case that X has real multiplication.

Theorem 5.4.2. *Let X be a hyper-Kähler variety over a finitely generated field K with $b_2(X) \geq 4$. Suppose that X has real multiplication. There is a direct product*

$$\mathbf{G}_{\ell,+}(X) = \mathbf{G}_{\ell,+}^{\mathrm{red}} \times P_{\ell,+}$$

with $\mathbf{G}_{\ell,+}^{\mathrm{red}} \cong \mathbf{G}_{\ell,2}(X)^\circ$, such that

$$\mathbf{G}_{\ell,+}^{\mathrm{red}} = \sigma_{\mathbb{Q}_\ell}\left(\mathbf{G}_{\mathrm{mot},2}(X)\right) = \mathbf{MT}_+(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$$

as a subgroup of $\mathbf{G}_{\mathrm{mot},+}(X)$ by the inclusion $\mathbf{G}_{\ell,+}(X) \hookrightarrow \mathbf{G}_{\mathrm{mot},+}(X)$.

Proof. Corollary 5.2.6 together with Proposition 5.3.6 provide an injective homomorphism
$$\sigma_\ell\colon \mathbf{G}_{\ell,2}(X) \to \mathbf{G}_{\ell,+}(X)$$
such that $\pi_{\ell,2} \circ \sigma_\ell = \mathbb{1}_{\mathbf{G}_{\ell,2}}$ and $\sigma_\ell(\mathbf{G}_{\ell,2}(X)) = \mathbf{MT}_+(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$. By Lemma 3.3.2 and Remark 3.2.4 (or Proposition 5.3.8), we can see that $P_{\ell,+} \subseteq P_{+,\mathbb{Q}_\ell}$ commutes with $\sigma_\ell(\mathbf{G}_{\ell,2}(X))$. Therefore, σ_ℓ induces the direct product decomposition as required. □

5.4.3. If E is a CM field (Case (U)), the center of $\mathbf{G}_{\ell,2}(X)$ receives a non-trivial contribution from the special Mumford–Tate group. Consequently, constructing the splitting σ_ℓ for the center is not as straightforward a priori as in Proposition 5.3.6. However, the connectivity of $P_{\ell,+}$ can be established by reducing to the case of real multiplication.

Theorem 5.4.4. *Let X be a hyper-Kähler variety over a finitely generated field K with $b_2 \geq 4$. For any prime ℓ, the ℓ-adic defect group $P_{\ell,+}$ is connected. In particular, $P_{\ell,+}$ is the unipotent radical of $\mathbf{G}_{\ell,+}(X)$ and $\mathbf{G}^{\mathrm{red}}_{\ell,+}(X) \cong \mathbf{G}_{\ell,2}(X)$.*

Proof. Let $\mathfrak{X} \to S$ be a universal family of polarized hyper-Kähler varieties such that $\mathfrak{X}_s \cong X$ for some K-rational point $s\colon \mathrm{Spec}(K) \to S$, where S is a geometrically connected smooth variety over a number field. The generic fiber $\mathfrak{X}_\eta$ is a hyper-Kähler variety over the finitely generated field $k(\eta)$ with real multiplication by Proposition 5.3.4.

For a point $t \in S$, we denote $P_{+,\ell,t}$ for the ℓ-adic even defect group of the fiber $\mathfrak{X}_t$. There is a decomposition
$$\mathbf{G}_{\ell,+,\eta} = \mathbf{G}_{\ell,2,\eta} \times P_{\ell,+,\eta}$$
by Theorem 5.4.2. In this case, $P_{\ell,+,\eta}$ is connected, and thus the unipotent radical of $\mathbf{G}_{\ell,\eta}$ by Theorem 4.3.2. Therefore, $\mathbf{G}_{\ell,2,\eta}$ is isomorphic to the maximal reductive quotient $\mathbf{G}^{\mathrm{red}}_{\ell,+,\eta}$, and the splitting $\mathbf{G}_{\ell,2,\eta} \hookrightarrow \mathbf{G}_{\ell,+,\eta}$ is unique.

At the K-rational point s, take a decomposition $\mathbf{G}_{\ell,+,s} \cong \mathbf{G}^{\mathrm{red}}_{\ell,+,s} \rtimes P^\circ_{\ell,+,s}$. The specialization $\eta \rightsquigarrow s$ provides a commutative diagram
$$\begin{array}{ccccc} \mathbf{G}^{\mathrm{red}}_{\ell,+,s} & \hookrightarrow & \mathbf{G}_{\ell,+,s} & \xrightarrow{\mathrm{sp}^*_{\eta,s}} & \mathbf{G}_{\ell,+,\eta} \\ & \searrow^{\pi'_{\ell,2}} & \downarrow{\scriptstyle \pi_{\ell,2}} & & \downarrow{\scriptstyle \pi_{\ell,2}} \\ & & \mathbf{G}_{\ell,2,s} & \xrightarrow{\mathrm{sp}^*_{\eta,s}} & \mathbf{G}_{\ell,2,\eta}\,. \end{array}$$
Since $\mathbf{G}^{\mathrm{red}}_{\ell,+,\eta} \subseteq \mathbf{G}_{\ell,+,\eta}$ is the unique Levi subgroup, the image $\mathrm{sp}^*_{\eta,s}\left(\mathbf{G}^{\mathrm{red}}_{\ell,+,s}\right) \subseteq \mathbf{G}^{\mathrm{red}}_{\ell,+,\eta}$. Therefore, the surjective homomorphism $\pi'_{\ell,2}$ is injective, which also implies that $P_{\ell,+,s}$ is connected since $\mathbf{G}_{\ell,+,s}$ is connected. Hence $P_{\ell,+,s}$ is the unipotent radical of $\mathbf{G}_{\ell,+,s}$ by Theorem 4.3.2 again. □

Corollary 5.4.5. *Suppose X is a hyper-Kähler variety over K with $b_2(X) \geq 4$. The exact sequence*
$$1 \longrightarrow P_{\ell,+} \longrightarrow \mathbf{G}_{\ell,+}(X) \longrightarrow \mathbf{G}_{\ell,2}(X) \longrightarrow 1$$
admits a unique splitting $\sigma_\ell\colon \mathbf{G}_{\ell,2}(X) \to \mathbf{G}_{\ell,+}(X)$, which induces a direct product decomposition $\mathbf{G}_{\ell,+}(X) = \sigma_\ell\left(\mathbf{G}_{\ell,2}(X)\right) \times P_{\ell,+}$.

Proof. Since $P_{\ell,+}$ is the unipotent radical of $\mathbf{G}_{\ell,+}(X)$, there is a semi-direct product decomposition
$$\mathbf{G}_{\ell,+}(X) \cong P_{\ell,+} \ltimes \mathbf{G}_{\ell,2}(X)\,, \tag{5.4.2}$$
and the inclusion $\mathbf{G}_{\ell,2}(X) \hookrightarrow \mathbf{G}_{\ell,+}(X)$ is unique up to conjugation under the $P_{\ell,+}$-action (see [Hoc81, Theorem 4.3]). However, $P_{\ell,+} \subseteq P_{+,\mathbb{Q}_\ell}$ commutes with the $\sigma_\ell\left(\mathbf{G}_{\ell,2}(X)\right) = \mathbf{G}^{\mathrm{red}}_{\ell,+}(X)$ by Lemma 3.3.2 and Proposition 5.3.8, which implies the semi-direct product (5.4.2) is actually a direct product and the splitting is unique. □

Remark 5.4.6. The uniqueness of the splitting implies that σ_ℓ coincides with the one constructed in Theorem 5.4.2 when X has real multiplication, and thus is compatible with the splitting σ for motivic Galois groups.

5.5. **Semisimple Mumford–Tate conjecture and semisimplicity.** In this subsection, we will split the usual Mumford–Tate conjecture into two parts.

5.5.1. The Mumford–Tate group $\mathbf{MT}_i(X_{\mathbb{C}})$ of a smooth projective complex variety $X_{\mathbb{C}}$ is a reductive group over $\mathbb{Q}$ by the existence of Hodge–Riemann relations. It follows that the ℓ-adic algebraic monodromy group $\mathbf{G}_{\ell,i}(X)^\circ$ is also reductive, assuming (MTC_i).

The reductivity of $\mathbf{G}_{\ell,i}(X)^\circ$ implies that the Galois representation $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is semisimple. As we already mentioned in the introduction, the semisimplicity of the Galois representation $V_{\ell,i}$ is still widely open, except for the Galois representations arising from $\mathcal{M} \in \mathbf{Mot}_K(\mathscr{A})$ (see [FWG+92, VI, §3, Theorem 1 (a)] for the case of abelian varieties). It is also well-known to be a consequence of Tate's conjecture, i.e., the cycle class maps

$$\mathrm{cl}^i_\ell\colon \mathrm{CH}^i(Y_{\overline{F}}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow \mathrm{H}^{2i}_{\text{ét}}(Y_{\overline{F}}, \mathbb{Q}_\ell(i))$$

are surjective onto the subgroups of Tate classes for all smooth projective varieties over a finitely generated field F over $\mathbb{Q}$, by [Moo19, Theorem 1].

5.5.2. To simplify the problem into only the cases of reductive groups, consider the following process.

Let ρ^{ss} be the semisimplification of a Galois representation $\rho\colon G_K \to \mathrm{GL}_{\mathbb{Q}_\ell}(V)$, i.e., a semisimple representation of G_K (on V) such that

$$\operatorname{Tr} \rho^{\mathrm{ss}}(g) = \operatorname{Tr} \rho(g) \quad \forall g \in G_K\,.$$

In the following we also denote V^{ss} for the G_K-module structure on V given by ρ^{ss}. Let $\mathbf{G}^{\mathrm{red}}_{\ell,i}(X)$ be the connected component of the Zariski closure of the image of

$$\rho^{\mathrm{ss}}_{\ell,i}\colon G_K \longrightarrow \mathrm{GL}_{\mathbb{Q}_\ell}(V_{\ell,i})\,.$$

The connected algebraic group $\mathbf{G}^{\mathrm{red}}_{\ell,i}(X)$ is reductive, as the faithful representation $V^{\mathrm{ss}}_{\ell,i}$ is completely reducible. The reductive group $\mathbf{G}^{\mathrm{red}}_{\ell,i}(X)$ is a Levi subgroup of $\mathbf{G}^\circ_{\ell,i}(X)$. In particular, there is a natural surjective homomorphism $\mathbf{G}^\circ_{\ell,i}(X) \twoheadrightarrow \mathbf{G}^{\mathrm{red}}_{\ell,i}(X)$ whose kernel is unipotent. The following is well known to experts. We record it here for the sake of completeness.

Lemma 5.5.3. *Let X be a smooth projective variety over K, and $0 < i < 2\dim X$ be a positive integer. The Mumford–Tate conjecture for X in degree i holds if and only if*

(SS) *$V_{\ell,i}$ is semisimple as a G_K-module, and*

(SMTC) $\mathbf{G}^{\mathrm{red}}_{\ell,i}(X) \cong \mathbf{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$.

Proof. As discussed in 5.5.1, the Mumford–Tate conjecture for $\mathfrak{h}^i(X)$ implies (SS) and (SMTC).

Now, assume the condition (SMTC), then it is sufficient to see that the condition (SS) would imply that the identity component $\mathbf{G}_{\ell,i}(X)^\circ$ of $\mathbf{G}_{\ell,i}(X)$ is reductive. Since the image $\rho(G_K)$ is Zariski dense in $\mathbf{G}_{\ell,i}(X)(\mathbb{Q}_\ell)$, $V_{\ell,i}$ is semisimple as a G_K-module if and only if it is semisimple as a $\mathbf{G}_{\ell,i}(X)^\circ$-module. If the unipotent radical R_ℓ of $\mathbf{G}_{\ell,i}(X)^\circ$ is non-trivial, then $V^{R_\ell}_{\ell,i} \neq 0$. As $V_{\ell,i}$ is assumed semisimple by (SS), we may take the decomposition

$$V_{\ell,i} = \bigoplus_k V_k$$

into irreducible representations. For all V_k, the invariant space $V_k^{R_\ell} \neq 0$ as R_ℓ is unipotent, and is stable under $\mathbf{G}_{\ell,i}(X)^\circ$-action since R_ℓ is a normal subgroup. This implies that $V_k = V_k^{R_\ell}$. Therefore R_ℓ lies in the kernel of $\rho\colon \mathbf{G}_{\ell,i}(X)^\circ \to \mathrm{GL}(V_{\ell,i})$, which implies that $R_\ell = 0$. □

If the condition (SMTC) in Lemma 5.5.3 holds for X and a positive integer $0 < i < 2\dim X$, then we say that the *semisimple Mumford–Tate conjecture* holds for X in degree i.

Lemma 5.5.4. *Let X be a hyper-Kähler variety over a finitely generated field K. Then the semisimplicity* (SS) *holds for X in degree $2k$ if and only if the action of $P_{\ell,+}$ on $\mathrm{H}^{2k}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is trivial.*

Proof. Consider the surjective homomorphism $\mathbf{G}_{\ell,+} \twoheadrightarrow \mathbf{G}_{\ell,2k}$. As $P_{\ell,+} \subseteq \mathbf{G}_{\ell,+}$ is the unipotent radical, the image of $P_{\ell,+}$ is also the unipotent radical of $\mathbf{G}_{\ell,2k}$ (see [Hoc81, Chapter VIII, Theorem 4.4]). By the definition, the action of $P_{\ell,+}$ on $\mathrm{H}^{2k}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is trivial if and only if the image of $P_{\ell,+}$ is trivial in $\mathbf{G}_{\ell,2k}$ is trivial, which is also equivalent to the reductivity of $\mathbf{G}_{\ell,2k}$. So we can conclude. □

5.5.5. It is also common to consider the following condition:

(FSS) Almost all Frobenius elements $F_v \in G_K$ act semisimply on $V_{\ell,i}$.

Clearly, for any Galois representation, (FSS) implies (SS) by the Chebotarev density theorem when K is a finitely generated field over $\mathbb{Q}$. Conversely, it is not clear whether $V_{\ell,i}^{\mathrm{ss}}$ is (FSS) or not in general. Fortunately, the opposite direction is true for hyper-Kähler varieties.

Proposition 5.5.6. *Let X be a hyper-Kähler variety over a finitely generated field K, with $b_2 \geq 4$. The semisimplification $\mathrm{H}^{2k}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\mathrm{ss}}$ is Frobenius-semisimple* (FSS) *for any $0 \leq k \leq \dim X$.*

Proof. With the Tannaka duality, we can see $\mathrm{H}^{2k}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\mathrm{ss}} \in \langle \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)\rangle$ as G_K-representations by Theorem 5.4.4. Then the G_K-representation $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is (FSS) by the Kuga–Satake construction in [And96a]. Therefore, $\mathrm{H}^{2k}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\mathrm{ss}}$ is (FSS) as being (FSS) is stable under tensor product and subquotient. □

5.5.7. Applying the results of Cadoret [Cad15], we can see the deformation invariance of the semisimplicity in a family.

Theorem 5.5.8. *Let $f\colon \mathfrak{X} \to S$ be a smooth projective family of hyper-Kähler varieties over a smooth and geometrically connected variety S. Let ℓ be a prime. If there exists a finite-type point $s_0 \in S$ such that $\mathbf{G}_\ell(\mathfrak{X}_{s_0})^\circ$ is reductive, then $\mathbf{G}_\ell(\mathfrak{X}_s)^\circ$ is reductive for any point $s \in S$.*

Proof. For the ℓ-adic local system $\mathbb{V}_\ell \coloneqq Rf_*\mathbb{Q}_\ell$, denote $\mathrm{Exc}_\ell \subset S$ for the exceptional locus of $\mathbb{V}_\ell$. Since $\mathbb{V}_\ell$ is abstractly motivic by [Cad15, Theorem 3.5], Theorem 1.2 (3) in *loc. cit.* implies that $\mathbf{G}_\ell(\mathfrak{X}_s)^\circ$ is reductive for all $s \in S \smallsetminus \mathrm{Exc}_\ell$, in particular for the generic point $\eta \in S$.

Suppose that $s \in \mathrm{Exc}_\ell$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{\ell,s} & \longrightarrow & \mathbf{G}_\ell(\mathfrak{X}_s) & \longrightarrow & \mathbf{G}_{\ell,2}(\mathfrak{X}_s) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P_{\ell,\eta} & \longrightarrow & \mathbf{G}_\ell(\mathfrak{X}_\eta) & \longrightarrow & \mathbf{G}_{\ell,2}(\mathfrak{X}_\eta) & \longrightarrow & 1\,. \end{array}$$

Since the ℓ-adic defect group $P^\circ_{\ell,\eta}$ is the unipotent radical of $\mathbf{G}_\ell(X_\eta)^\circ$ by Theorem 4.3.2, the reductivity of $\mathbf{G}_\ell(\mathfrak{X}_\eta)^\circ$ implies that the group $P_{\ell,\eta}$ is finite. Therefore $P_{\ell,s}$ is also finite, which implies that $\mathbf{G}_\ell(\mathfrak{X}_s)^\circ$ is reductive by applying Theorem 4.3.2 again. □

5.6. **Proof of main results.** In this section, we will focus on proving Theorem B and Theorem C. As before, we consider a hyper-Kähler variety X over a finitely generated field K. We also assume that $K = K^{\mathrm{conn}}$ for $\mathfrak{h}(X)$.

5.6.1. Theorem 5.4.4 together with Lemma 3.2.5 implies that there are abstract isomorphisms of $\mathbb{Q}_\ell$-algebraic groups

$$\mathbf{G}^{\mathrm{red}}_{\ell,+}(X) \cong \mathbf{MT}_+(X_\mathbb{C})_{\mathbb{Q}_\ell}$$

via the second degree projection. Then to verify the semisimple Mumford–Tate conjecture for $\mathfrak{h}^+(X)$, it remains to see that these two $\mathbb{Q}_\ell$-algebraic subgroups actually coincide as subgroups in the motivic Galois group $\mathbf{G}_{\mathrm{mot},+}(X)_{\mathbb{Q}_\ell}$. The case with real multiplication can be deduced from the discussions in Section 5.4.

5.6.2. As a first step toward the remaining cases, we can verify the semisimple Mumford–Tate conjecture for hyper-Kähler varieties over CM-type via LLV representation.

Proposition 5.6.3. *Let X over K be a hyper-Kähler variety with $b_2(X) \geq 4$. Then we have*

$$Z\left(\mathbf{G}^{\mathrm{red}}_\ell(X)\right)^\circ = Z\left(\mathbf{MT}(X_\mathbb{C})_{\mathbb{Q}_\ell}\right)^\circ$$

inside $\mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell}$. In particular, if X is of CM-type, i.e., $\mathbf{MT}_2(X_\mathbb{C})$ is abelian, then the semisimple Mumford–Tate conjecture holds.

Proof. Let $Z_\ell \coloneqq Z\left(\mathbf{G}^{\mathrm{red}}_\ell(X)\right)^\circ$ and $Z = Z\left(\mathbf{MT}(X_\mathbb{C})\right)^\circ$. Use the notations in Notation 3.2.2. Let T_{Spin} be a maximal torus of the spin group Spin and T_{SO} its image under the universal covering Spin → SO.

Firstly, we show that $Z_\ell \subseteq Z_{\mathbb{Q}_\ell}$. Let $\mu \in X_*(Z_\ell)$ be any cocharacter. It is sufficient to see that the image of μ is contained in $Z_{\mathbb{Q}_\ell}$. Since the image of the weight cocharacter is contained in $\mathbf{MT}(X_\mathbb{C})_{\mathbb{Q}_\ell} \cap \mathbf{G}^{\mathrm{red}}_\ell(X)$, we may assume that μ is not the weight cocharacter. Under this assumption, the image of μ lies in $\rho^{\mathrm{LLV}}(T_{\mathrm{Spin},\mathbb{Q}_\ell})$ by Proposition 5.3.8. Take μ_2 to be the composition

$$\mathbb{G}_{m,\mathbb{Q}_\ell} \xrightarrow{\mu} Z_\ell \subseteq \mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell} \xrightarrow{\pi_2} Z_{\ell,2} \subseteq \mathbf{G}_{\mathrm{mot},2}(X)_{\mathbb{Q}_\ell}.$$

Under the assumption that μ is not the weight cocharacter, the image of μ_2 lies inside $T_{\mathrm{SO},\mathbb{Q}_\ell}$; and μ is the unique cocharacter of Spin_ℓ that lifts $\mu_2 \colon \mathbb{G}_{m,\mathbb{Q}_\ell} \to \mathrm{SO}_\ell$.

Let $\widetilde{\mu} \colon \widehat{\mathbb{G}}_{m,\mathbb{Q}_\ell} \to Z_{\mathbb{Q}_\ell}$ be the unique quasi-cocharacter that lifts μ_2 along the isogeny

$$Z_{\mathbb{Q}_\ell} \twoheadrightarrow Z_{2,\mathbb{Q}_\ell} \subseteq \mathbf{G}_{\mathrm{mot},2}(X_\mathbb{C}).$$

We have $\mathbf{MT}(X_\mathbb{C})_{\mathbb{Q}_\ell} \subseteq \rho^{\mathrm{tw}}(\mathrm{GSpin}_\ell)$ as in 3.2.4. Thus $\widetilde{\mu} = \mu \in X_*(T_{\mathrm{Spin},\mathbb{Q}_\ell})_\mathbb{Q}$ by the uniqueness. Here we view μ as a quasi-cocharacter. This implies that the image of μ is contained inside $Z_{\mathbb{Q}_\ell}$ as required.

For the inclusion $Z_{\mathbb{Q}_\ell} \subseteq Z_\ell$, the argument is the same, just switching the roles of $\mathbf{MT}(X_\mathbb{C})_{\mathbb{Q}_\ell}$ and $\mathbf{G}^{\mathrm{red}}_\ell(X)$. □

Now, we are ready to prove the semisimple Mumford–Tate conjecture for general hyper-Kähler varieties.

Theorem 5.6.4. *For a hyper-Kähler variety X over a finitely generated field K over prime field $\mathbb{Q}$, with $b_2(X) \geq 4$, the semisimple Mumford–Tate conjecture holds for X. Namely, for any integer $0 \leq i \leq 2\dim X$, the comparison isomorphism induces an identification of algebraic groups*

$$\mathbf{G}_{\ell,i}^{\mathrm{red}}(X) = \mathbf{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_\ell}.$$

for any prime ℓ and $0 \leq i \leq 2\dim X$.

Proof. Let $\mathbf{G}_\ell^{\mathrm{der}}$ be the derived subgroup of $\mathbf{G}_\ell^{\mathrm{red}}(X)$. Consider the decomposition

$$\mathbf{G}_\ell^{\mathrm{red}}(X) = Z\left(\mathbf{G}_\ell^{\mathrm{red}}(X)\right)^\circ \cdot \mathbf{G}_\ell^{\mathrm{der}}.$$

- Corollary 5.2.6 implies $\mathbf{G}_\ell^{\mathrm{der}} = \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}^{\mathrm{der}}$ in $\mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell}$.
- Proposition 5.6.3 implies that

$$Z\left(\mathbf{G}_\ell^{\mathrm{red}}(X)\right)^\circ = Z\left(\mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}\right)^\circ \subseteq \mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell}.$$

Hence $\mathbf{G}_\ell^{\mathrm{red}}(X) = \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ in $\mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell}$, i.e., the semisimple Mumford–Tate conjecture holds for $\mathfrak{h}(X)$, which in particular implies the Mumford–Tate conjecture for each even degree i: Under the projection $\pi_i\colon \mathbf{G}_{\mathrm{mot}}(X)_{\mathbb{Q}_\ell} \to \mathbf{G}_{\mathrm{mot},i}(X)_{\mathbb{Q}_\ell}$, the image of $\mathbf{G}_\ell^{\mathrm{red}}(X)$ is a Levi subgroup $\mathbf{G}_{\ell,i}^{\mathrm{red}}(X)$, which is the same as the Mumford–Tate group $\mathbf{MT}_i(X_{\mathbb{C}})$. □

Remark 5.6.5. We can actually obtain a natural isomorphism

$$\mathbf{G}_{\ell,+}^{\mathrm{red}}(X) \cong \mathbf{G}_{\ell,2k}^{\mathrm{red}}(X) \quad \forall 1 \leq k \leq 2n-1$$

under the projection $\pi_{\ell,2k}$ and its image $\pi_{\ell,2k}\left(P_{\ell,+}\right)$ is the unipotent radical of $\mathbf{G}_{\ell,2k}(X)^\circ$.

Combing Theorem 5.6.4 and the invariance of reductivity in families (Theorem 5.5.8), we can see that it is sufficient to prove the Mumford–Tate conjecture for a single member, especially the part (SS), for the Mumford–Tate conjecture of the deformation type.

Corollary 5.6.6. *Let $f\colon \mathfrak{X} \to S$ be a smooth projective family of hyper-Kähler varieties over a smooth and geometrically connected variety S. If there is a point of finite-type $s_0 \in S$ such that* (MTC$_i$) *holds for $\mathfrak{X}_{s_0}$, then* (MTC$_i$) *holds for all fibers of $\mathfrak{X}_s$, $s \in S$.*

Proof. By Theorem 5.6.4, it is sufficient to prove that $\mathbf{G}_{\ell,i}(\mathfrak{X}_s)^\circ$ is reductive for any prime ℓ. As (MTC$_i$) holds for $\mathfrak{X}_{s_0}$, $\mathbf{G}_{\ell,i}(\mathfrak{X}_{s_0})^\circ$ is reductive for any prime ℓ. Therefore, $\mathbf{G}_{\ell,2k}(\mathfrak{X}_s)^\circ$ is reductive for any prime ℓ by Theorem 5.5.8 as required. □

5.6.7. *Proof of Theorem C.*

- (a) $\Longrightarrow$ (b) and (c) trivially.
- (b) $\Longrightarrow$ (a) by (SMTC$_i$), which is verified in Theorem 5.6.4.
- (a) $\Longleftrightarrow$ (c) by Corollary 5.6.6. □

5.7. **Semisimplicity for known examples.** For hyper-Kähler varieties belonging to the four established deformation types, semisimplicity can be deduced just from their cohomological structure, without using the full power of the information of their motives.

Using Theorem 5.5.8, we may therefore restrict our attention to any concrete construction representing each of these deformation types.

5.7.1. Let S be a smooth projective surface over a finitely generated field K. By classical results from Nakajima [Nak97] and Grojnowski [Gro96] (see also [Nak99, Chapter 8]), the direct sum of the ℓ-adic étale cohomologies:

$$\bigoplus_{n\geq 0} \mathrm{H}^{4n-\bullet}_{\text{ét}}(S^{[n]}_{\overline{K}}, \mathbb{Q}_\ell)$$

is an irreducible representation of the Heisenberg superalgebra generated by $\mathrm{H}^{4-\bullet}_{\text{ét}}\left(S_{\overline{K}}, \mathbb{Q}_\ell\right)$, the (co)homology ring of the surface S. Moreover, the highest weight vector is given by the class $[\mathrm{Spec}(K)]$ of $\mathrm{H}^0_{\text{ét}}(S^{[0]}_{\overline{K}}, \mathbb{Q}_\ell)$. The action of the Heisenberg superalgebra is given by some classes

$$P_\alpha[i] = \varpi_*\left(\Pi^*\alpha \cap [P[i]]\right) \in \mathrm{H}_{\text{ét}}((S^{[n-i]} \times S^{[n]})_{\overline{K}}, \mathbb{Q}_\ell),$$

where $\alpha \in \mathrm{H}^\bullet_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_\ell)$, $i \in \mathbb{Z}$, $[P[i]]$ is an algebraic $(2n-i+1)$-cycle on $S^{[n-i]} \times S^{[n]} \times S$, and

$$S^{[n-i]} \times S^{[n]} \xleftarrow{\ \varpi\ } S^{[n-i]} \times S^{[n]} \times S \xrightarrow{\ \Pi\ } S$$

are the projections respectively. From this, we can deduce that, for any $0 \leq k \leq 4n$,

$$\mathrm{H}^k_{\text{ét}}(S^{[n]}_{\overline{K}}, \mathbb{Q}_\ell) \in \langle \mathrm{H}^\bullet_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_\ell)\rangle^{\otimes}$$

as a G_K-representation after a finite field extension.

If S is a K3 or abelian surface, then it is well-known that the G_K-representation $\mathrm{H}^\bullet_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_\ell)$ is semisimple by [Del72], which implies the semisimplicity of $\mathrm{H}^k_{\text{ét}}(S^{[n]}_{\overline{K}}, \mathbb{Q}_\ell)$ for all $n \geq 1$ in this case.

Remark 5.7.2. In fact, we have motivic decompositions for Hilbert schemes of points on a smooth algebraic surface [dCM02], which shows that the motive $\mathfrak{h}(S^{[n]})$ is generated by $\mathfrak{h}(S)$ by taking a direct sum of subquotients. Then we can see the semisimplicity of Galois representations of $S^{[n]}$ from that of S. This motivic decomposition was also used in [Sol22] and [FFZ21] to establish the abelianicity of the André motives of $\mathrm{K3}^{[n]}$-type varieties and Kum_n-varieties.

Remark 5.7.3. We thank Floccari for pointing out that for the Hilbert scheme of n-points $X = S^{[n]}$ on a K3 surface S, the $\mathfrak{g}_0$-equivariant decomposition defined by Markman in [Mar08, §4.2] can be used to establish the semisimplicity of X (see also [Flo22b, Theorem 6.2]). In fact, the full cohomology $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is generated as a $\mathbb{Q}_\ell$-algebra by a subspace

$$C^\bullet_\ell \coloneqq \bigoplus_{i\geq 1} C^i_\ell,$$

where $C^i_\ell \subseteq \mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is a $\mathfrak{g}_{0,\ell}$-subrepresentation. For $X = S^{[n]}$, the odd cohomology vanishes, and the irreducible factors in each C^{2k}_ℓ (as $\mathrm{SO}(\mathrm{H}^2)$-module) appear with multiplicity at most one. Arguments similar to those in the proof of Proposition 5.7.10 imply that the Galois action on $C^\bullet_\ell$ is semisimple. Therefore, X satisfies the semisimplicity conjecture.

5.7.4. If $S = A$ is an abelian surface, then the semisimplicity of $\mathrm{H}^\bullet_{\text{ét}}(K_n(A)_{\overline{K}}, \mathbb{Q}_\ell)$ can be deduced from that of $\mathrm{H}^\bullet_{\text{ét}}(A^{[n+1]}_{\overline{K}}, \mathbb{Q}_\ell)$ by viewing $K_n(A)$ as a fiber of the isotrivial fibration $A^{[n+1]} \to A$ given by taking the sum of points.

5.7.5. As in [Sol22, §4.2], for a hyper-Kähler variety of OG 6-type, semisimplicity follows from the corresponding result for varieties of $\mathrm{K3}^{[3]}$-type, using a dominant generically finite rational map from a $\mathrm{K3}^{[3]}$-type variety constructed in [MRS18].

5.7.6. We can also use the LLV representation to study the semisimplicity of Galois representations for hyper-Kähler varieties, especially for those of OG 10-type.

Since the LLV algebra $\mathfrak{g} \cong \mathfrak{so}(\widetilde{\mathrm{H}}(X))$ is semisimple, one may consider the decomposition of the $\mathfrak{g}$-module $\mathrm{H}^*(X)$ into irreducible $\mathfrak{g}$-modules

$$\mathrm{H}^*(X) \otimes_E \overline{E} = \bigoplus_{\mu\in\Delta_r^+} V_\mu^{\oplus m_\mu} . \tag{5.7.1}$$

where $r = \lfloor b_2(X)/2 \rfloor$, Δ_r^+ is the set of dominant weights of $\mathfrak{g}$ and $\mu = \sum_{i=0}^r \mu_i \epsilon_i \in \Delta_r^+$ is the highest weight of the irreducible representation V_μ. Here $\{\pm\epsilon_i\}$ is the set of all the weights of the standard representation $\widetilde{\mathrm{H}}(X)$, and the decomposition (5.7.1) is called the *LLV decomposition* of X.

Example 5.7.7. According to the Weyl construction, $V_{(n)}$ is the "largest" irreducible subrepresentation of $\mathrm{Sym}^n V$. Moreover, Verbitsky has shown that for a hyper-Kähler variety X of dimension $2n$, the subalgebra $\mathrm{SH}(X) \subset \mathrm{H}^*(X)$ generated by $\mathrm{H}^2(X)$ is isomorphic to the irreducible $\mathfrak{g}$-module $V_{(n)} \subset \mathrm{Sym}^n V$ of the highest weight $\mu = (n)$. Then $\mathrm{SH}(X) \cong V_{(n)}$ is called the *Verbitsky component* of $\mathrm{H}(X)$. Actually, according to the construction, $\mathrm{SH}(X) \subseteq \mathrm{H}^+(X)$.

Since $\mathfrak{g}$ is semisimple, the cohomology admits a $\mathfrak{g}$-module decomposition

$$\mathrm{H}(X) = V_{(n)} \oplus V' . \tag{5.7.2}$$

For arbitrary hyper-Kähler manifold X, the multiplicity of $V_{(n)}$ in $\mathrm{H}^*(X)$ is one, that is, V' does not contain an irreducible $\mathfrak{g}$-module of highest weight $\mu = (n)$, and moreover, actually any μ that appears in V' satisfies $|\mu| \le n-1$, see [GKLR22, Proposition 2.34]. In *loc. cit.*, the authors compute the LLV decomposition for $\mathrm{K3}^{[n]}$, Kum_n, OG 6 and OG 10-varieties.

Example 5.7.8. According to [GKLR22, Theorem 3.26], the LLV decomposition of OG 10 is

$$\mathrm{H}(X) = V_{(5)} \oplus V_{(2,2)} ,$$

i.e., $V' = V_{(2,2)}$ is irreducible. For OG 6-type, we have

$$V' = V_{(1,1,1)} \oplus V^{\oplus 135} \oplus E^{\oplus 240}$$

by Theorem 3.39 in *loc. cit.*.

Lemma 5.7.9. *For any $L \in \mathfrak{g}_\ell$, we have $gLg^{-1} = L$ for all $g \in P_\ell$.*

Proof. For any $x \in \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ satisfying the Hard Lefschetz property, the Lefschetz operator L_x commutes with $\rho_\ell(g), \forall g \in P_\ell$, since g acts trivially on x and is compatible with the cup product on $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ according to the definition. Moreover, Λ_x also commutes with those g since the Galois action preserves the shifted degree operator h and (L_x, h, Λ_x) is the unique $\mathfrak{sl}_2$-triple that extends L_x. □

Proposition 5.7.10. *If all irreducible factors in the LLV decomposition* (5.7.1) *of X have multiplicity ≤ 1, then the Galois representations of $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is semisimple.*

Proof. As P_ℓ° is the unipotent radical of $\mathbf{G}_\ell(X)$ by Theorem 4.3.2, it is sufficient to see that $P_\ell(\overline{\mathbb{Q}}_\ell) = \pi_0(P_\ell)$ is finite. For any irreducible factor V_μ and $g \in P_\ell(\overline{\mathbb{Q}}_\ell)$, we can see $g(V_\mu) \subseteq \mathrm{H}(X)_{\overline{\mathbb{Q}}_\ell}$ is also a $\mathfrak{g}_{\mathbb{Q}_\ell}$-representation by Lemma 5.7.9, which is isomorphic to V_μ. Under the assumption that $m_\mu = 1$, we have $g(V_\mu) = V_\mu \subseteq \mathrm{H}(X)_{\overline{\mathbb{Q}}}$. Thus the action of any $g \in P_\ell(\overline{\mathbb{Q}}_\ell)$ induces an $\overline{\mathbb{Q}}_\ell$-linear automorphism of V_μ as a $\mathfrak{g}_{\mathbb{Q}_\ell}$-representation. By Schur lemma, we can see the action $g \circlearrowright V_\mu$ is given via the multiplication by some $c_\mu \neq 0 \in \overline{\mathbb{Q}}_\ell$. However, as the identity component P_ℓ° is unipotent, the action $g \circlearrowright \mathrm{H}(X)_{\overline{\mathbb{Q}}}$

is quasi-unipotent by Theorem 4.3.2, which forces $c_\mu^k = 1$ for all V_μ, with $k = |\pi_0(P_\ell)|$. Thus any $g \in P_\ell(\overline{\mathbb{Q}}_\ell)$ is torsion of order k, and P_ℓ is thus a finite group scheme. □

As we have seen in Example 5.7.8, the factors in LLV decomposition of an OG 10-variety have multiplicity ≤ 1. Thus, we can conclude the Mumford–Tate conjecture for OG 10-varieties.

Corollary 5.7.11. *Let X be an* OG 10*-variety over K. Then the Galois representation* $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ *is semisimple and thus Mumford–Tate conjecture holds for X.*

Remark 5.7.12. Actually, the arguments in Proposition 5.7.10 are similar to those in [Flo22b, Proposition 6.1], for the (motivic) Mumford–Tate conjecture of OG 10-varieties. For this reason, Corollary 5.7.11 doesn't give an essentially new proof for Mumford–Tate conjecture for OG 10-varieties.

6. Applicatons

6.1. **Arithmetic Nagai conjecture.** Recall that a hyper-Kähler variety X over a p-adic local field K_v is of *Type I reduction* if

$$\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_{\ell'})$$

is potentially unramified for *some* prime $\ell' \neq p$ (and thus for all primes $\neq p$ by Corollary 5.3 in [IIK+25]).

The semisimple Mumford–Tate conjecture enables us to extend [IIK+25, Corollary 4.3], which was originally established for the four known deformation types, to any hyper-Kähler variety with $b_2 \geq 4$. Consequently, we can establish the *arithmetic Nagai conjecture* (Conjectures 1.3 and 1.4 in *loc. cit.*) in this general framework.

Theorem 6.1.1. *Let X be a hyper-Kähler variety over a p-adic local field K_v with $b_2(X) \geq 4$ and Type I reduction. Then $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is potentially unramified for any $0 \leq i \leq 4n$ and prime $\ell \neq p$.*

Firstly, we note that Theorem 5.6.4 implies the ℓ-adic local Galois representation of a hyper-Kähler variety (after Frobenius semisimplification) is also partially controlled by the Mumford–Tate group.

Proposition 6.1.2. *Let X be a hyper-Kähler variety over a p-adic* local field *K_v with $b_2(X) \geq 4$. Then any Levi subgroup $\mathbf{G}_\ell^{\text{red}}(X)$ of the ℓ-adic algebraic monodromy group $\mathbf{G}_\ell(X)$ satisfies*

$$\mathbf{G}_\ell^{\text{red}}(X) \subseteq \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$$

for any field embedding $K_v \hookrightarrow \mathbb{C}$.

Proof. Fix a field embedding $K_v \hookrightarrow \mathbb{C}$. The standard spreading out argument shows that there is a smooth projective variety X_0 over a finitely generated subfield $K \subseteq K_v$ such that $X_0 \times_K K_v \cong X$ as K_v-varieties. It is clear that X_0 is a hyper-Kähler variety over K, thus $\mathbf{G}_\ell^{\text{red}}(X_0) \cong \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ by Theorem 5.6.4 for the field embedding $K \hookrightarrow K_v \hookrightarrow \mathbb{C}$. On the other hand, we have the natural injective homomorphism $\mathbf{G}_\ell^{\text{red}}(X) \hookrightarrow \mathbf{G}_\ell^{\text{red}}(X_0)$ along the field extension $K \subseteq K_v$. This implies that $\mathbf{G}_\ell^{\text{red}}(X) \subseteq \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$. □

Remark 6.1.3. In contrast to the global field case, the ℓ-adic monodromy group $\mathbf{G}_\ell(X)$ ($\ell \neq p$) for X over a local field K_v is not expected to be reductive. This is because the inertia representation $\rho_\ell|_{I_v}$ could probably contribute a normal subgroup of $\mathbf{G}_\ell(X)$ via the short exact sequence

$$1 \longrightarrow I_v \longrightarrow G_{K_v} \longrightarrow G_{\mathbb{F}_q} \longrightarrow 1$$

when $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}_v}, \mathbb{Q}_\ell)$ is not (potentially) unramified, which is quasi-unipotent by Grothendieck's monodromy theorem.

For this reason, the Levi subgroup $\mathbf{G}_\ell^{\mathrm{red}}(X)$ is not the ℓ-adic algebraic monodromy group of the Frobenius-semisimplification in general.

Proof of Theorem 6.1.1. Keep the notation same as in the proof of Proposition 6.1.2. Let $N_\ell \in \mathfrak{g}_\ell(X)$ be the ℓ-adic monodromy operator of $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{K}_v}, \mathbb{Q}_\ell)$ for a prime $\ell \neq p$. Under the assumption that X has Type I reduction, we have $\pi_{\ell,2}(N_\ell) = N_{\ell,2} = 0$. In other words, $N_\ell \in \mathrm{Lie}(P_\ell)$, where P_ℓ is the ℓ-adic defect group of X_0 via the inclusion $\mathbf{G}_\ell(X) \hookrightarrow \mathbf{G}_\ell(X_0)$. Proposition 6.1.2 implies the semisimplification

$$\rho_\ell(\mathrm{Frob}_v)^{\mathrm{ss}} \in \mathbf{G}_\ell^{\mathrm{red}}(X) \subseteq \mathbf{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$$

of a lift of Frobenius $\mathrm{Frob}_v \in G_{K_v}$, whose adjoint action on $\mathrm{Lie}(P_\ell) \subseteq \mathrm{Lie}(P)_{\mathbb{Q}_\ell}$ is trivial by Proposition 5.3.8 and Lemma 3.3.2. However, we have the equality

$$p \cdot \rho_\ell(\mathrm{Frob}_v)^{\mathrm{ss}} \circ N_\ell = N_\ell \circ \rho_\ell(\mathrm{Frob}_v)^{\mathrm{ss}}$$

from the fundamental relation between Frobenius action and ℓ-adic monodromy operator, which implies that $N_\ell = 0$. ☐

6.2. Maximality of Galois action.

In this final subsection, we prove Conjecture 1.5.5 for hyper-Kähler varieties. We retain the notation introduced in 1.5.4.

Let $\mathbf{G}_\ell(X)^{\mathrm{sc}}$ denote the simply connected cover of the adjoint group of $\mathbf{G}_\ell(X)$. Recall that $\Gamma_\ell^{\mathrm{sc}}$ is defined as the preimage of $\rho^{\mathrm{ss}}(G_K) \subset \mathbf{G}_\ell^{\mathrm{red}}(X)(\mathbb{Q}_\ell)$ under the composition of the simply connected covering map $\mathbf{G}_\ell(X)^{\mathrm{sc}} \to \mathbf{G}_\ell^{\mathrm{red,der}}(X)$ and the natural isogeny $\mathbf{G}_\ell^{\mathrm{red,der}}(X) \to \mathbf{G}_\ell(X)^{\mathrm{ad}}$.

Theorem 6.2.1. *Let X be a hyper-Kähler variety over a finitely generated field K with $b_2(X) \geq 4$. Then the ℓ-adic Lie group $\Gamma_\ell^{\mathrm{sc}} \subseteq \mathbf{G}_\ell(X)^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is a hyperspecial maximal compact subgroup for all sufficiently large primes ℓ.*

Proof. Corollary 4.3.3 implies that the projection $\mathbf{G}_\ell(X) \longrightarrow \mathbf{G}_{\ell,2}(X)$ induces an isomorphism of simply connected $\mathbb{Q}_\ell$-algebraic groups

$$\pi_2^{\mathrm{sc}} \colon \mathbf{G}_\ell(X)^{\mathrm{sc}} \xrightarrow{\sim} \mathbf{G}_{\ell,2}(X)^{\mathrm{sc}}.$$

According to [HL20, Theorem 1.3 (b)], the compact subgroup $\Gamma_{\ell,2}^{\mathrm{sc}}$ is hyperspecial and maximal in $\mathbf{G}_{\ell,2}(X)^{\mathrm{sc}}(\mathbb{Q}_\ell)$. Therefore, its preimage $\Gamma_\ell^{\mathrm{sc}} = (\pi_2^{\mathrm{sc}})^{-1}(\Gamma_{\ell,2}^{\mathrm{sc}})$ is also a hyperspecial maximal compact subgroup in $\mathbf{G}_\ell(X)^{\mathrm{sc}}(\mathbb{Q}_\ell)$ via this isomorphism. ☐

Remark 6.2.2. In [HL20], the authors also address the case where $b_2(X) = 3$. If the base field K is a number field, the assumption $b_2 \geq 4$ in the proof of Theorem 4.3.2 (and also Corollary 4.3.3) can be removed. Consequently, the conclusion of Theorem 6.2.1 remains valid for $b_2 = 3$ over number fields. Unfortunately, we are unable to establish this case here in full generality.

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