ARITHMETIC MONODROMY OF HYPER-KÄHLER VARIETIES OVER p-ADIC FIELDS

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Abstract. In this paper, we study the p-adic and ℓ-adic monodromy operators associated with hyper-Kähler varieties over p-adic fields, in connection with Looijenga–Lunts–Verbitsky Lie algebras. We investigate a conjectural relation between the nilpotency indices of these monodromy operators on higher-degree cohomology groups and on the second cohomology, which may be viewed as an arithmetic analogue of Nagai's conjecture for degenerations of hyper-Kähler manifolds over a disk. We verify this arithmetic version of Nagai's conjecture for hyper-Kähler varieties over p-adic fields, assuming they belong to one of the four known deformation types. As part of our approach, we introduce a new method to analyze the p-adic cohomology of hyper-Kähler varieties via Sen's theory.

Contents

1. Introduction

We fix a prime number p throughout the paper. Let K be a complete discretely valued field of mixed characteristic $(0,p)$ with perfect residue field k. Let ℓ be a prime number. Let X be a smooth proper variety over K. Then we obtain Galois representations

$$\phi_{\ell,i}\colon \operatorname{Gal}(\overline{K}/K) \to \operatorname{GL}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$$

for integers $0 \leq i \leq 2\dim X$. (Here $\overline{K}$ is an algebraic closure of K and $X_{\overline{K}} \coloneqq X \otimes_K \overline{K}$.) If $\ell \neq p$, Grothendieck's ℓ-adic monodromy theorem in [SGA7-I] implies that there is an open subgroup I of the inertia subgroup $I_K \subset \operatorname{Gal}(\overline{K}/K)$ such that the restriction of $\phi_{\ell,i}$ on I is *unipotent*, i.e., there is an integer $\nu \geq 0$ such that $(\phi_{\ell,i}(g) - 1)^{\nu+1} = 0$ for all $g \in I$. We call

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the minimal integer ν with this property the (ℓ-adic) monodromy nilpotency index in degree i. Denote it by $\nu_{\ell,i}$. If $\ell = p$, a similar monodromy nilpotency index $\nu_{p,i}$ can be defined via Fontaine's formalism on potentially semistable p-adic Galois representations.

Problem 1.1. How can one determine the monodromy nilpotency index $\nu_{\ell,i}$ for a prime ℓ (including $\ell = p$) in a degree $0 \leq i \leq 2\dim X$?

In this paper, we will focus on Problem 1.1 in the case that X is a *hyper-Kähler variety.* In complex geometry, hyper-Kähler varieties are also known as irreducible holomorphic symplectic (IHS) varieties, which are higher-dimensional analogues of K3 surfaces and form one of the three building blocks of compact Kähler manifolds with trivial first Chern class. As with K3 surfaces, the notion of hyper-Kähler variety extends naturally to general base fields, at least in characteristic zero. We will see that the rich geometric structure would lead to an ingredient for the Problem 1.1 for hyper-Kähler varieties over K.

1.1. **Nagai's conjecture and arithmetic analogues.** In studying a projective degeneration of a complex hyper-Kähler variety, Nagai proposed the following conjecture on the nilpotency indices of local monodromy operators:

Conjecture 1.2 ([Nag08, Conjecture 5.1])**.** *Denote $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ for the open disk. Let $\pi\colon \mathcal{X} \to \Delta$ be a degeneration of hyper-Kähler varieties of dimension $2n$ over $\Delta^* = \Delta \setminus \{0\}$. Consider the (log-)monodromy operator N_i on the Betti cohomology $H^i(X, \mathbb{Q})$ of a smooth fiber $X = \pi^{-1}(t)$ at $t \in \Delta^*$. The monodromy operators satisfy*

$$\nu(N_{2i}) = i\nu(N_2)$$

for every $0 \leq i \leq n$. Here $\nu(N)$ is the nilpotency index *of a nilpotent linear operator $N \in \mathrm{End}(V)$ on a finite-dimensional vector space over a field, i.e.,*

$$\nu(N) := \min\{m \in \mathbb{Z}_{\geq 0} \mid N^{m+1} = 0\}.$$

The local monodromy theorem implies that $\nu(N_2) \in \{0, 1, 2\}$ (cf. [Lan73, (A.2)]), and we say that the projective degeneration $\mathcal{X}/\Delta$ is of Type I, II, or III respectively. Conjecture 1.2 posits a deep relation between these three degeneration types and monodromy nilpotency indices in higher degrees, which has been studied by many people.

- If $\mathcal{X}/\Delta$ is of Type I ($\nu(N_2) = 0$) or Type III ($\nu(N_2) = 2$), then Nagai's conjecture is confirmed by Kollár–Laza–Saccà–Voisin (cf. [Kol+18, Theorem 1.7, Proposition 7.14 (2)]). Soldatenkov [Sol20, Corollary 3.6] gives a proof for possibly *non-projective* degenerations of Type I.
- Nagai's conjecture is still widely open for Type II ($\nu(N_2) = 1$) degenerations in general. Recently, Green–Kim–Laza–Robles [Gre+22] proved Conjecture 1.2 when X is in one of the four known deformation classes (i.e., $K3^{[n]}$-type, Kum_n-type, OG_6-type, and OG_{10}-type). Their method involves a detailed study on the action of the *Looijenga–Lunts–Verbitsky (LLV) Lie algebra* on the cohomology.
- In [HM23], Huybrechts–Mauri provide a different viewpoint for the Type II degenerations by the perverse filtration attached to an isotropic class.

It is generally believed that projective models over the ring of integers $\mathcal{O}_K$ of K are arithmetic analogues of "degenerating families over the unit disk Δ"; see [Del71, Table 9.1] for example. Thus, it is natural to ask an arithmetic analogue of Conjecture 1.2. More precisely, we formulate the following two conjectures.

If $\ell \neq p$, Grothendieck's ℓ-adic monodromy theorem we mentioned before actually asserts that there is a nilpotent operator $N_{\ell,i} \in \mathrm{End}_{\mathbb{Q}_\ell}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ such that $\phi_{\ell,i}(\sigma) = \exp(t_\ell(\sigma)N_{\ell,i})$ for any $\sigma \in I \subset I_K$, where $t_\ell \colon I_K \to \mathbb{Z}_\ell$ is the composition of ℓ-primary tame quotient of the inertia group with a fixed trivialization $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$.

Conjecture 1.3 (ℓ-adic analogue of Nagai's conjecture)**.** *For a hyper-Kähler variety X over K of dimension $2n$, the monodromy operators satisfy $\nu(N_{\ell,2i}) = i\nu(N_{\ell,2})$ for every $0 \leq i \leq n$.*

If $\ell = p$, by applying Fontaine's functor $D_{\mathrm{pst}}(-)$ for potentially semistable representations, we obtain a K_0^{ur}-vector space $D_{\mathrm{pst}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ with a nilpotent operator $N_{p,i}$, where K_0^{ur} is the maximal unramified extension of the fraction field K_0 of the ring of Witt vectors $W(k)$. See Section 2.3 below.

Conjecture 1.4 (p-adic analogue of Nagai's conjecture)**.** *For a hyper-Kähler variety X over K of dimension $2n$, the monodromy operators satisfy $\nu(N_{p,2i}) = i\nu(N_{p,2})$ for every $0 \leq i \leq n$.*

We explore these two conjectures and, in particular, confirm their validity for hyper-Kähler varieties of four known deformation types.

Theorem 1.5. *Let X be a hyper-Kähler variety over K of dimension $2n$. Assume that X is one of four types: $K3^{[n]}$-type, Kum_n-type, OG_6-type, or OG_{10}-type (see Definition 2.3). Then Conjecture 1.3 and Conjecture 1.4 hold true for X.*

Furthermore, we have a geometric description for the monodromy nilpotency index $\nu(N_{\ell,2})$, which does not depend on ℓ (including $\ell = p$), in terms of the reduction type of *Kuga–Satake abelian varieties* for any hyper-Kähler variety X over K with $b_2(X) \geq 4$ (see Theorem 5.2). We thus obtain the notion of reduction type for hyper-Kähler varieties X over K; we say that X has *Type I, II, or III reduction* if $\nu(N_{\ell,2}) = 0, 1$ or 2 respectively. By definition, we have $\nu_{\ell,i} = \nu(N_{\ell,i})$. We actually obtain a satisfactory answer to Problem 1.1 for hyper-Kähler varieties in these four types combining with Theorem 1.5. A direct consequence of our discussion, together with Poincaré duality, is the following:

Corollary 1.6. *Assume X satisfies conditions in Theorem 1.5. We have $\nu(N_{p,2i}) = \nu(N_{\ell,2i})$ for any prime ℓ and any $0 \leq i \leq 2n$.*

1.2. **General remarks on arithmetic Nagai's conjectures.** As in [Gre+22], the theory of LLV Lie algebras plays a crucial role in our proof. Verbitsky's description of the graded algebra generated by the second cohomology inside the full cohomology (often referred to as the Verbitsky component) implies that for any hyper-Kähler variety X over K of dimension $2n$, the inequality $\nu(N_{\ell,2i}) \geq i\nu(N_{\ell,2})$ holds for every $0 \leq i \leq n$ and ℓ. Since we have $\nu(N_{\ell,2i}) \leq 2i$ in general, this implies Conjecture 1.3 and Conjecture 1.4 when X has Type III reduction; see Theorem 7.5.

A crucial step in the study of the case where X has Type I or Type II reduction is to show that the ℓ-adic and p-adic monodromy operators in all (even) degrees are contained in the LLV Lie algebra. In fact, this inclusion is enough to prove the conjectures in the Type I case; see Theorem 7.6.

For the four known deformation types, this can be deduced from the Mumford–Tate conjecture for the full cohomology; see Section 4. On the other hand, for p-adic monodromy operators, we provide an alternative approach using Sen's theory as a new input. This enables us to prove that p-adic monodromy operators lie in the LLV Lie algebra without using the Mumford–Tate conjecture, and even without assuming that our hyper-Kähler varieties belong to one of the four known deformation types. See Section 6 for details.

Remark 1.7. Along the way, we also present a Kuga–Satake type construction for higher-degree p-adic cohomology groups of hyper-Kähler varieties. As another application of our calculations, we will see that for any hyper-Kähler variety X over K with $b_2(X) \geq 4$, the monodromy operator on $D_{\mathrm{pst}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ satisfies *Griffiths transversality* for all i. See Section 6.2 for details.

Remark 1.8. Unlike the p-adic version, we can prove ℓ-adic Nagai's conjecture (Conjecture 1.3) of Type I only for the four known deformation types. It is unclear how to show that the ℓ-adic monodromy operators lie in the LLV Lie algebra in general.

In the case of Type II reduction, we have to investigate the irreducible decomposition of the full cohomology with respect to the action of the LLV Lie algebra. This decomposition is called the *LLV decomposition*. In [Gre+22], Green–Kim–Laza–Robles gave a criterion for the validity of Nagai's conjecture in terms of a representation-theoretic condition on the LLV decomposition. Then they proved Nagai's conjecture for the four known deformation types by verifying this representation-theoretic condition. We employ the same strategy to establish the arithmetic analogues; see Theorem 7.7.

Finally, we remark that we can also compute nilpotency indices in odd degrees in some cases by a similar strategy; see Section 7.3. For example, we can prove the following result, as observed by Soldatenkov in the topological case (cf. [Sol20, Theorem 3.8]).

Theorem 1.9 (Theorem 7.9)**.** *Suppose X is a hyper-Kähler variety over K of dimension $2n$ with $b_2(X) \geq 4$ and $b_3(X) \neq 0$. Then we have $\nu(N_{p,2i+1}) \leq 2i-1$ for any $1 \leq i \leq n-1$. If X has Type III reduction, then the equality holds.*

1.3. **Organization of the paper.** In Section 2, we recall some basic definitions to fix our notation. In Section 3, we develop the theory of LLV Lie algebras for both p-adic and ℓ-adic cohomological realizations, based on results known in the topological case. In Section 4, we study the relation between Galois representations associated with étale cohomology of hyper-Kähler varieties and the LLV Lie algebra, and explain that ℓ-adic monodromy operators belong to the LLV Lie algebra for the four known deformation types. We also review the Kuga–Satake construction and its relation to the LLV Lie algebra. In Section 5, we give a geometric description for the monodromy nilpotency index $\nu(N_{\ell,2})$ in terms of Kuga–Satake abelian varieties. In Section 6, we study the p-adic LLV Lie algebra using Sen's theory, and prove that p-adic monodromy operators belong to the LLV Lie algebras. Finally, in Section 7, we prove the main results.

2. Preliminaries

2.1. **Hyper-Kähler varieties.**

We first recall the definition and basic properties of hyper-Kähler varieties. See, for example, [Huy03], [Bin21], and [Fu+22] for more details.

Let X be a smooth projective variety over a field L of characteristic zero. We will always assume varieties are geometrically connected throughout this paper. We say X is a *hyper-Kähler variety* if $\pi_1^{\text{ét}}(X_{\overline{L}}) = 1$ and $H^0(X, \Omega_X^2) = L \cdot \omega$ for some 2-form $\omega\colon \mathcal{O}_X \to \Omega_X^2$ which is nowhere degenerate (i.e., the adjunction $T_X = \Omega_X^{1,\vee} \to \Omega_X^1$ is an isomorphism). The second condition implies the dimension of X is even. If there is a field embedding $\sigma\colon L \hookrightarrow \mathbb{C}$, then X is a hyper-Kähler variety if and only if the complex manifold $X_{\sigma,\mathbb{C}}(\mathbb{C})$ is a hyper-Kähler manifold in the sense of [Huy03]; see [Bin21, Lemma 3.1.3] and [Fu+22, Section 2]. Here we set $X_{\sigma,\mathbb{C}} := X \otimes_{L,\sigma} \mathbb{C}$.

Remark 2.1. Up to deformation equivalence, there exist the following four known types of complex hyper-Kähler manifolds.

(1) **($K3^{[n]}$-type):** Let S be a complex algebraic K3 surface, and n a positive integer. Then the Hilbert scheme of n-points $S^{[n]}$ is a hyper-Kähler manifold of dimension $2n$. A hyper-Kähler manifold X is called $K3^{[n]}$-*type* if X is deformation equivalent to some $S^{[n]}$. We have $b_2(X) = 23$ if $n \geq 2$ and $b_2(X) = 22$ if $n = 1$. Moreover, $b_{2i+1}(X) = 0$.

(2) **(Kum_n-type):** Let A be a complex algebraic abelian surface, and $n \geq 2$ an integer. Let $A^{[n+1]}$ be the Hilbert scheme of n-points on A, and $K_n(A)$ the fiber of the summation morphism $A^{[n+1]} \to A$ over $0 \in A$. Then $K_n(A)$ is a hyper-Kähler manifold of dimension $2n$. A hyper-Kähler manifold X is called Kum_n-*type* (or *generalized Kummer type*) if X is deformation equivalent to some $K_n(A)$. We have $b_2(X) = 7$ (as $n \geq 2$).

(3) **(OG_6-type and OG_{10}-type):** There exists a 6-dimensional (resp. 10-dimensional) exceptional hyper-Kähler manifold constructed by O'Grady ([OGr03] (resp. [OGr99])). A hyper-Kähler manifold X is called OG_6-*type* (resp. OG_{10}-*type*) if X is deformation equivalent to the O'Grady's example. We have $b_2(X) = 8$ (resp. $b_2(X) = 24$). Moreover, $b_{2i+1}(X) = 0$ for these two deformation types.

Lemma 2.2. *Let X be a hyper-Kähler variety over L. If $X_{\sigma,\mathbb{C}}$ is of*

$$K3^{[n]},\ \mathrm{Kum}_n,\ \mathrm{OG}_6,\ \textit{or}\ \mathrm{OG}_{10} \tag{$\star$}$$

types for some embedding $\sigma\colon L \hookrightarrow \mathbb{C}$, then so is $X_{\sigma',\mathbb{C}}$ for any embedding $\sigma'\colon L \hookrightarrow \mathbb{C}$.

Proof. See [Fu+22, Proposition 2.3]. □

Definition 2.3. Let X be a hyper-Kähler variety over L. Let $L' \subset L$ be a subfield which is finitely generated over $\mathbb{Q}$ such that X has a model X' over L'. We say that X is *of a known deformation type* ($\star$) if there exists an embedding $\sigma\colon L' \hookrightarrow \mathbb{C}$ such that $X'_{\sigma,\mathbb{C}}$ is of a known deformation type as in ($\star$). By Lemma 2.2, this condition is independent of the choices of L', X' and σ.

Proposition 2.4. *Let X be a hyper-Kähler variety over L. Then there exists a unique quadratic form*

$$q\colon H^2_{\text{ét}}(X_{\overline{L}}, \widehat{\mathbb{Z}}(1)) \to \widehat{\mathbb{Z}}$$

that is primitive *(i.e., if there is another $\widehat{\mathbb{Z}}$-quadratic form q' such that $q = cq'$ with $c \in \widehat{\mathbb{Z}}$, then $c \in \widehat{\mathbb{Z}}^\times$) and is a $\mathbb{Q}$-multiple of the quadratic form*

$$\alpha \mapsto \int \sqrt{\mathrm{td}_{X_{\overline{L}}}}\alpha^2,$$

such that for any ample line bundle $\mathscr{L}$ on X, we have $q(c_1(\mathscr{L})) \in \mathbb{Z}_{>0}$. Moreover q is compatible with the action of the absolute Galois group $G_L := \mathrm{Gal}(\overline{L}/L)$. Furthermore, if $L = \mathbb{C}$, q comes from a primitive quadratic form $q\colon H^2(X(\mathbb{C}), \mathbb{Z}) \to \mathbb{Z}$. We call q the Beauville–Bogomolov–Fujiki (BBF) form *of X.*

Proof. See [Bin21, Lemma 4.2.1] (and also [Bin21, Theorem 4.2.4]). □

2.2. **ℓ-adic monodromy operators.** In Section 2.2 and Section 2.3, we recall the definition of ℓ-adic and p-adic monodromy operators, respectively.

For $\ell \neq p$, we fix an isomorphism $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$. Let $t_\ell\colon I_K \to \mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ be the ℓ-primary tame quotient of I_K, i.e., the homomorphism defined by $\sigma \mapsto (\sigma(\varpi^{1/\ell^m})/\varpi^{1/\ell^m})_m$ for a uniformizer ϖ of K.

Let X be a smooth proper scheme over K of dimension d, and i a non-negative integer. Grothendieck's ℓ-adic monodromy theorem (see [SGA7-I, Exposé I, Variante 1.3] and also [ST68, Appendix]) implies that there exists a unique nilpotent endomorphism

$$N_{\ell,i}\colon H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$$

such that for some open subgroup $I \subset I_K$, any $\sigma \in I$ acts on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ as $\exp(t_\ell(\sigma)N_\ell)$. We call $N_{\ell,i}$ the *(ℓ-adic) monodromy operator*.

We will need the following fact.

Lemma 2.5. *We have $\nu(N_{\ell,i}) \leq i$ for any integer $0 \leq i \leq d$.*

Proof. Using de Jong's alteration [Jon96, Theorem 6.5], we reduce to the case where X has a strictly semistable model over $\mathcal{O}_K$. In this case, the assertion follows from [SGA7-I, Exposé I, Corollaire 3.4]. □

2.3. **Potentially semistable representations and p-adic monodromy operators.** We recall Fontaine's formalism of linear-algebraic data for potentially semistable representations. Throughout this paper, we will use the following notation. Let K_0 be the fraction field of the ring of Witt vectors $W(k)$. We may regard K as a finite totally ramified extension of K_0. Let $K_0^{\mathrm{ur}} \subset \overline{K}$ be the maximal unramified extension of K_0. Let B_{dR}, B_{cris} and B_{st} be Fontaine's period rings of K defined in [Fon94a].

Remark 2.6. As usual, we fix a valuation on $\overline{K}$ and an extended usual p-adic logarithm $\log\colon \overline{K}^\times \to \overline{K}$. Following the constructions in [Fon94a], we have the corresponding B_{cris}-derivation $N\colon B_{\mathrm{st}} \to B_{\mathrm{st}}$, and an embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ over B_{cris}.

Let V be a p-adic G_K-representation, that is, a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous action of G_K. (Here $G_K := \mathrm{Gal}(\overline{K}/K)$.) As in [Fon94b], we define

$$D_{\mathrm{pst}}(V) := \varinjlim_{L/K} (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_L},$$

where L runs over all finite extensions of K contained in $\overline{K}$. Then $D_{\mathrm{pst}}(V)$ is a finite-dimensional K_0^{ur}-vector space with $\dim_{K_0^{\mathrm{ur}}} D_{\mathrm{pst}}(V) \leq \dim_{\mathbb{Q}_p} V$. This is equipped with the Frobenius map $\varphi\colon D_{\mathrm{pst}}(V) \to D_{\mathrm{pst}}(V)$, the monodromy operator $N\colon D_{\mathrm{pst}}(V) \to D_{\mathrm{pst}}(V)$ (which is a K_0^{ur}-linear map such that $N\varphi = p\varphi N$), and the Hodge filtration $F^\bullet$ (which is a separated and exhaustive decreasing filtration on $D_{\mathrm{pst}}(V) \otimes_{K_0^{\mathrm{ur}}} \overline{K}$). We say that V is a *potentially semistable* representation if $\dim_{K_0^{\mathrm{ur}}} D_{\mathrm{pst}}(V) = \dim_{\mathbb{Q}_p} V$. In this case, we have a natural isomorphism

$$D_{\mathrm{pst}}(V) \otimes_{K_0^{\mathrm{ur}}} B_{\mathrm{st}} \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$$

which is compatible with Frobenius maps and monodromy operators (in the usual sense). If V is potentially semistable, it is *de Rham* (by [Fon94b, Théorème in §5.6.7]) and we have

$$D_{\mathrm{pst}}(V) \otimes_{K_0^{\mathrm{ur}}} \overline{K} \xrightarrow{\sim} D_{\mathrm{dR}}(V) \otimes_K \overline{K}$$

that is filtered. Here $D_{\mathrm{dR}}(V) \coloneqq (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$ is endowed with the natural filtration from B_{dR}.

Let X be a smooth proper scheme over K of dimension d. Then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is potentially semistable by the validity of the C_{st}-conjecture (see [Tsu99, Theorem 0.2]) and de Jong's alteration [Jon96, Theorem 6.5]. Moreover, the j-th successive quotient F^j/F^{j+1} of the Hodge filtration $F^\bullet$ of $D_{\mathrm{pst}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ is isomorphic to $H^{i-j}(X, \Omega^j_X) \otimes_K \overline{K}$. The monodromy operator N on $D_{\mathrm{pst}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ will be denoted by $N_{p,i}$ and called the *p-adic monodromy operator*.

As in Lemma 2.5, we have an upper bound of the nilpotency index of $N_{p,i}$. In fact, in the p-adic case, the following slightly more precise statement holds.

Lemma 2.7. *Let $0 \leq i \leq d$. Let j_1 (resp. j_2) denote the largest (resp. smallest) integer j such that $H^{i-j}(X, \Omega^j_X) \neq 0$. Then $\nu(N_{p,i}) \leq j_1 - j_2$. In particular, we have $\nu(N_{p,i}) \leq i$.*

Proof. We may assume that the residue field k is algebraically closed. In this case, the underlying F-isocrystal $D_{\mathrm{pst}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ admits the slope decomposition $\bigoplus_{\alpha \in \mathbb{Q}} D(\alpha)$ by Dieudonné–Manin's classification, where $D(\alpha)$ has a single slope α. We have $j_2 \leq \alpha \leq j_1$ for $D(\alpha) \neq 0$, by [Fon94b, Proposition 5.4.2, Remark 4.4.6]. The p-adic monodromy operator satisfies $N_{p,i}(D(\alpha)) \subset D(\alpha - 1)$ since $N_{p,i}\varphi = p\varphi N_{p,i}$, which implies $N_{p,i}^{j_1 - j_2 + 1} = 0$. □

2.4. **Representations of orthogonal Lie algebras.** Here we recall some facts and fix the notation on representations of orthogonal Lie algebras. Our main references are [FH91] and [Gre+22, Appendix A].

Let F be an algebraically closed field of characteristic zero. Let V be a non-degenerate quadratic space over F of rank $N+2$ with $N > 0$. We consider the orthogonal Lie algebra $\mathfrak{so}(V)$ over F and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)$. Let us denote the non-trivial $\mathfrak{h}$-weights of the natural action of $\mathfrak{so}(V)$ on V by

$$\{\pm\epsilon_0, \pm\epsilon_1, \ldots, \pm\epsilon_r\} \subset \mathfrak{h}^* \setminus \{0\}$$

where $r = \lfloor \frac{N}{2} \rfloor$. Then we consider the following positive system of roots R^+ and the corresponding set of dominant integral weights Λ^+:

- (Type B_{r+1}) If $N = 2r+1$, then $R^+ := \{\epsilon_i + \epsilon_j\}_{i<j} \cup \{\epsilon_i - \epsilon_j\}_{i<j} \cup \{\epsilon_i\}_i$ and the set of dominant integral weights is

$$\Lambda^+ = \left\{ \mu = \sum_{i=0}^{r} \mu_i \epsilon_i \;\middle|\; \mu_0 \geq \mu_1 \geq \cdots \geq \mu_r \geq 0,\, \mu_i \in \frac{1}{2}\mathbb{Z},\, \mu_i - \mu_j \in \mathbb{Z} \right\}. \tag{2.1}$$

- (Type D_{r+1}) If $N = 2r$, then $R^+ := \{\epsilon_i + \epsilon_j\}_{i<j} \cup \{\epsilon_i - \epsilon_j\}_{i<j}$ and the set of dominant integral weights is

$$\Lambda^+ := \left\{ \mu = \sum_{i=0}^{r} \mu_i \epsilon_i \;\middle|\; \mu_0 \geq \mu_1 \geq \cdots \geq \mu_{r-1} \geq |\mu_r|,\, \mu_i \in \frac{1}{2}\mathbb{Z},\, \mu_i - \mu_j \in \mathbb{Z} \right\}. \tag{2.2}$$

For a dominant integral weight $\mu \in \Lambda^+$, let V_μ denote the irreducible $\mathfrak{so}(V)$-module with highest weight μ. We will identify Λ^+ with the set S of sequences $(\mu_0, \mu_1, \ldots, \mu_r)$ satisfying the conditions as in (2.1) or (2.2) depending on the parity of N. Then for a sequence $(\mu_0, \mu_1, \ldots, \mu_r) \in S$, we have the corresponding irreducible $\mathfrak{so}(V)$-module

$$V_\mu = V_{(\mu_0, \mu_1, \ldots, \mu_r)}.$$

Remark 2.8. We discuss how the construction

$$(\mu_0, \mu_1, \ldots, \mu_r) \mapsto V_{(\mu_0, \mu_1, \ldots, \mu_r)}$$

depends on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)$ and the choices of the numbering and the sign for $\{\pm\epsilon_i\}_{0 \leq i \leq r}$. Let $V'_{(\mu_0, \mu_1, \ldots, \mu_r)}$ be the irreducible $\mathfrak{so}(V)$-module corresponding to a sequence $(\mu_0, \mu_1, \ldots, \mu_r) \in S$ with respect to another choice of $\mathfrak{h}'$ and $\{\pm\epsilon'_i\}_{0 \leq i \leq r}$.

- If N is odd, then
$$V_{(\mu_0, \mu_1, \ldots, \mu_r)} \simeq V'_{(\mu_0, \mu_1, \ldots, \mu_r)}$$
for any $(\mu_0, \mu_1, \ldots, \mu_r) \in S$.
- If N is even, then for any $(\mu_0, \mu_1, \ldots, \mu_r) \in S$, we have either
$$V_{(\mu_0, \mu_1, \ldots, \mu_r)} \simeq V'_{(\mu_0, \mu_1, \ldots, \mu_r)} \quad \text{or} \quad V_{(\mu_0, \mu_1, \ldots, \mu_r)} \simeq V'_{(\mu_0, \mu_1, \ldots, -\mu_r)}.$$

To see this, since all Cartan subalgebras of $\mathfrak{so}(V)$ are conjugate under inner automorphisms, we may assume that $\mathfrak{h} = \mathfrak{h}'$. If N is odd, the result follows from the fact that $\{\pm\epsilon_i\}_{0 \leq i \leq r}$ and $\{\pm\epsilon'_i\}_{0 \leq i \leq r}$ are conjugate under the action of the Weyl group. If N is even, then $\{\pm\epsilon'_i\}_{0 \leq i \leq r}$ is conjugate to $\{\pm\epsilon_0, \ldots, \pm\epsilon_{r-1}, \pm\epsilon_r\}$ or $\{\pm\epsilon_0, \ldots, \pm\epsilon_{r-1}, \mp\epsilon_r\}$ under the action of the Weyl group, which will give the same positive system R^+. These observations now lead to the result.

Remark 2.9. Let $F \subset F'$ be an extension of algebraically closed fields. The construction $V_{(\mu_0, \mu_1, \cdots, \mu_r)} \mapsto V_{(\mu_0, \mu_1, \cdots, \mu_r)} \otimes_F F'$ induces a bijection between the set of irreducible representations of $\mathfrak{so}(V)$ over F and that of irreducible representations of $\mathfrak{so}(V)_{F'}$ over F'. Indeed, by working with the base changes of $\mathfrak{h}$ and $\{\pm\epsilon_i\}_{0 \leq i \leq r}$ to F', we can parametrize irreducible representations of $\mathfrak{so}(V)_{F'}$ over F' by the same set S, and the irreducible representation of $\mathfrak{so}(V)_{F'}$ corresponding to $\mu \in S$ agrees with $V_{(\mu_0, \mu_1, \cdots, \mu_r)} \otimes_F F'$.

3. Looijenga–Lunts–Verbitsky Lie algebras of varieties over p-adic fields

In this section, we develop the theory of the Looijenga–Lunts–Verbitsky Lie algebra (LLV Lie algebra) for ℓ-adic and p-adic cohomology realizations.

3.1. **Looijenga–Lunts–Verbitsky algebras.** Let E be a field of characteristic zero. Let $d \geq 0$ be an integer. Let $A^\bullet = \bigoplus_{0 \leq i \leq 2d} A^i$ be a graded E-algebra such that $A^\bullet$ is finite-dimensional as an E-vector space and $\dim_E A^{2d} = 1$. We assume that $A^\bullet$ is a (graded) *Frobenius algebra* (of degree $2d$), i.e., the bilinear form $A^\bullet \times A^\bullet \to A^{2d} \simeq E$ defined by $(x, y) \mapsto (x \cdot y)_{2d}$ is non-degenerate.

For any $x \in A^2$, let $e_x\colon A^\bullet \to A^{\bullet+2}$ be the map defined by $a \mapsto x \cdot a$. An element $x \in A^2$ satisfies the *Hard Lefschetz (HL) property* if $(e_x)^i\colon A^{d-i} \to A^{d+i}$ is bijective for any integer $0 \leq i \leq d$. On the other hand, we consider the shifted degree map

$$h\colon A^\bullet \to A^\bullet \quad \text{such that} \quad h|_{A^i} = (i-d)\,\mathrm{id}_{A^i}$$

for all i. By the Jacobson–Morozov theorem, $x \in A^2$ satisfies the HL property if and only if (e_x, h) extends to an $\mathfrak{sl}_2$-triple, i.e., there is a map $f_x \in \mathrm{End}_E(A^\bullet)$ such that

$$[e_x, h] = -2e_x, \quad [f_x, h] = 2f_x, \quad [e_x, f_x] = h.$$

Let $\mathfrak{a}_A \subset A^2$ be the set of all elements satisfying the HL property, which is a Zariski open subset. We say that $A^\bullet$ is *Lefschetz–Frobenius* if $\mathfrak{a}_A \neq 0$. In this case, we define the *total Lie algebra* of $A^\bullet$ as the Lie subalgebra

$$\mathfrak{g}^{\mathrm{tot}}(A^\bullet) \subset \mathrm{End}_E(A^\bullet)$$

generated by all $\mathfrak{sl}_2$-triples (e_x, h, f_x) for all $x \in \mathfrak{a}_A$.

Lemma 3.1. *Let $\gamma\colon A^\bullet \xrightarrow{\sim} B^\bullet$ be an isomorphism of Lefschetz–Frobenius E-algebras. The isomorphism $\mathrm{End}_E(A^\bullet) \xrightarrow{\sim} \mathrm{End}_E(B^\bullet)$ induced by γ maps $\mathfrak{g}^{\mathrm{tot}}(A^\bullet)$ isomorphically onto $\mathfrak{g}^{\mathrm{tot}}(B^\bullet)$.*

Proof. This is clear from the definition. □

Proposition 3.2. *Let $E \subset F$ be an inclusion of fields of characteristic zero. Let $A^\bullet$ be a Lefschetz–Frobenius E-algebra and $A_F^\bullet := A^\bullet \otimes_E F$. Then $\mathfrak{g}^{\mathrm{tot}}(A^\bullet) \otimes_E F = \mathfrak{g}^{\mathrm{tot}}(A_F^\bullet)$ as Lie subalgebras of $\mathrm{End}_E(A^\bullet) \otimes_E F = \mathrm{End}_F(A_F^\bullet)$.*

Proof. Since E is an infinite field, we see that $A^2 \subset A_F^2$ is Zariski dense. Thus $\mathfrak{a}_A \subset \mathfrak{a}_{A_F}$ is also Zariski dense. Since $\mathfrak{g}^{\mathrm{tot}}(A^\bullet) \otimes_E F$ is Zariski closed in $\mathrm{End}_F(A_F^\bullet)$, it follows that $\mathfrak{g}^{\mathrm{tot}}(A^\bullet) \otimes_E F$ contains e_x and f_x for all $x \in \mathfrak{a}_{A_F}$. This implies the assertion. □

Let L be a field of characteristic zero. Let $H^\bullet(-)$ be a Weil cohomology theory with coefficients in E for smooth projective varieties X over L, such that the hard Lefschetz theorem holds. Then, for any X of dimension d, the cohomology algebra $H^\bullet(X)$ naturally forms a Lefschetz–Frobenius algebra of degree $2d$. The *LLV Lie algebra* of X with respect to the cohomology theory $H^\bullet(-)$ is the total Lie algebra $\mathfrak{g}^{\mathrm{tot}}(H^\bullet(X))$. Let $\rho\colon \mathfrak{g}^{\mathrm{tot}}(H^\bullet(X)) \to \mathrm{End}_E(H^\bullet(X))$ denote the standard Lie algebra representation.

Example 3.3. Assume that $L = \mathbb{C}$. We denote by $\mathfrak{g}_B(X)$ the LLV Lie algebra with respect to the Betti realization $H_B^\bullet(X) = H^\bullet(X(\mathbb{C}), \mathbb{Q})$.

Example 3.4. The LLV Lie algebra of X with respect to the ℓ-adic étale cohomology $H^\bullet_\ell(X) = H^\bullet_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_\ell)$ is denoted by $\mathfrak{g}_\ell(X)$. There is a natural continuous G_L-action on $\mathfrak{g}_\ell(X)$. For an embedding $\overline{L} \hookrightarrow \mathbb{C}$, we have a natural identification of $\mathbb{Q}_\ell$-Lie algebras

$$\mathfrak{g}_B(X \otimes_L \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathfrak{g}_\ell(X). \tag{3.1}$$

by Proposition 3.2.

Let the notation be as in Section 2.3. Let $A^\bullet$ be a Lefschetz–Frobenius algebra over $\mathbb{Q}_p$ with a continuous G_K-action that is compatible its algebra structure. Assume that $A^\bullet$ is potentially semistable. Since $\mathfrak{g}^{\text{tot}}(A^\bullet)$ is a G_K-subrepresentation of $\operatorname{End}_{\mathbb{Q}_p}(A^\bullet)$, we obtain a Lie subalgebra

$$D_{\text{pst}}(\mathfrak{g}^{\text{tot}}(A^\bullet)) \subset D_{\text{pst}}(\operatorname{End}_{\mathbb{Q}_p}(A^\bullet)) = \operatorname{End}_{K_0^{\text{ur}}}(D_{\text{pst}}(A^\bullet)).$$

On the other hand $D_{\text{pst}}(A^\bullet) = \bigoplus_i D_{\text{pst}}(A^i)$ is naturally a Lefschetz–Frobenius algebra over K_0^{ur} and thus we have the total Lie algebra

$$\mathfrak{g}^{\text{tot}}(D_{\text{pst}}(A^\bullet)) \subset \operatorname{End}_{K_0^{\text{ur}}}(D_{\text{pst}}(A^\bullet))$$

of $D_{\text{pst}}(A^\bullet)$.

Proposition 3.5. *If $A^\bullet$ is potentially semistable, then we have an identification*

$$D_{\text{pst}}(\mathfrak{g}^{\text{tot}}(A^\bullet)) = \mathfrak{g}^{\text{tot}}(D_{\text{pst}}(A^\bullet))$$

as Lie subalgebras of $\operatorname{End}_{K_0^{\text{ur}}}(D_{\text{pst}}(A^\bullet))$.

Proof. We have the inclusions $K_0^{\text{ur}} \subset \overline{K} \subset B_{\text{dR}}$ of fields. It suffices to prove the equality after tensoring with B_{dR}. Since $\operatorname{End}_{\mathbb{Q}_p}(A^\bullet)$ is potentially semistable, so is $\mathfrak{g}^{\text{tot}}(A^\bullet)$. Thus we have the following identifications

$$D_{\text{pst}}(\mathfrak{g}^{\text{tot}}(A^\bullet)) \otimes_{K_0^{\text{ur}}} B_{\text{dR}} = \mathfrak{g}^{\text{tot}}(A^\bullet) \otimes_{\mathbb{Q}_p} B_{\text{dR}} = \mathfrak{g}^{\text{tot}}(A^\bullet \otimes_{\mathbb{Q}_p} B_{\text{dR}})$$

where the second equality follows from Proposition 3.2. On the other hand, we have

$$\mathfrak{g}^{\text{tot}}(D_{\text{pst}}(A^\bullet)) \otimes_{K_0^{\text{ur}}} B_{\text{dR}} = \mathfrak{g}^{\text{tot}}(D_{\text{pst}}(A^\bullet) \otimes_{K_0^{\text{ur}}} B_{\text{dR}}) = \mathfrak{g}^{\text{tot}}(A^\bullet \otimes_{\mathbb{Q}_p} B_{\text{dR}})$$

by Proposition 3.2 again. The assertion follows from these equalities. □

Example 3.6. Let X be a smooth projective variety over K. We define

$$\mathfrak{g}_{\text{pst}}(X) := D_{\text{pst}}(\mathfrak{g}_p(X)) = \mathfrak{g}^{\text{tot}}(D_{\text{pst}}(H^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))),$$

which is a Lie algebra over K_0^{ur} with a Frobenius map, a monodromy operator, and a filtration (on $\mathfrak{g}_{\text{pst}}(X) \otimes_{K_0^{\text{ur}}} \overline{K}$). We have a natural isomorphism of Lie algebras over B_{dR}

$$\mathfrak{g}_{\text{pst}}(X) \otimes_{K_0^{\text{ur}}} B_{\text{dR}} \simeq \mathfrak{g}_p(X) \otimes_{\mathbb{Q}_p} B_{\text{dR}}. \tag{3.2}$$

3.2. **LLV Lie algebras of hyper-Kähler varieties.** Here we record some important features of the LLV Lie algebras of hyper-Kähler varieties.

Set-up 3.7. Let us consider one of the following realization functors:

(a) Let X be a $2n$-dimensional hyper-Kähler variety over $\mathbb{C}$. We set

$$H^\bullet(X) := H^\bullet_B(X) = H^\bullet(X(\mathbb{C}), \mathbb{Q}) \quad \text{and} \quad \mathfrak{g}(X) := \mathfrak{g}_B(X)$$

(see Example 3.3). Let $E := \mathbb{Q}$ and $F := \mathbb{C}$.

(b) Let X be a $2n$-dimensional hyper-Kähler variety over a field L of characteristic zero. We set

$$H^\bullet(X) := H^\bullet_\ell(X) = H^\bullet_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_\ell) \quad \text{and} \quad \mathfrak{g}(X) := \mathfrak{g}_\ell(X)$$

(see Example 3.4). Let $E := \mathbb{Q}_\ell$ and $F := \overline{\mathbb{Q}}_\ell$.

(c) Let X be a $2n$-dimensional hyper-Kähler variety over K. We abuse notation slightly and write

$$H^\bullet(X) := H^\bullet_{\text{pst}}(X) := D_{\text{pst}}(H^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \quad \text{and} \quad \mathfrak{g}(X) := \mathfrak{g}_{\text{pst}}(X)$$

(see Example 3.6). Let $E := K_0^{\text{ur}}$ and $F := \overline{K}$. Note that, the Beauville–Bogomolov–Fujiki form q on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ induces a form on $H^2(X)$. We also write this form as q and call it the Beauville–Bogomolov–Fujiki (BBF) form.

In all cases, $\mathfrak{g}(X) \subset \operatorname{End}_E(H^\bullet(X))$ is a Lie subalgebra over E containing the degree operator h.

Remark 3.8. Strictly speaking, the BBF form q should be defined as a quadratic form on $H^2(X)(1) := H^2(X) \otimes_E E(1)$ where $E(1)$ is the Tate twist, though one can choose an isomorphism $E(1) \simeq E$ so that we can regard q as a quadratic form on $H^2(X)$. The resulting quadratic form on $H^2(X)$ is independent of the choice of $E(1) \simeq E$, up to scalar multiplication.

We can describe the LLV Lie algebra $\mathfrak{g}(X)$ using the quadratic space

$$\widetilde{H}^2(X) := H^2(X) \oplus U,$$

where $U = Ev \oplus Ew$ is equipped with the quadratic form $(a, b) \mapsto ab$. The quadratic space $\widetilde{H}^2(X)$ is called the *Mukai completion* of $H^2(X)$.

Remark 3.9. We endow $\widetilde{H}^2(X)$ with the structure of a graded algebra $A^\bullet$ over E with grading given by

$$A^0 = Ev = E, \quad A^2 = H^2(X), \quad A^4 = Ew = E$$

and with multiplication determined by $x \cdot y = q(x, y)w$ for all $x, y \in A^2 = H^2(X)$. Then $A^\bullet$ is a Lefschetz–Frobenius E-algebra of degree 4. By [Ver95, Theorem 9.1], the total Lie algebra of $A^\bullet$ is $\mathfrak{so}(\widetilde{H}^2(X))$.

Theorem 3.10. *In Set-up 3.7 (a), (b) or (c), the following assertions hold.*

(1) The LLV Lie algebra $\mathfrak{g}(X)$ over E is a semisimple Lie algebra.

(2) There exists a decomposition $\mathfrak{g}(X) = \mathfrak{g}(X)_2 \oplus \mathfrak{g}(X)_0 \oplus \mathfrak{g}(X)_{-2}$, where $\operatorname{ad} h$ acts on $\mathfrak{g}(X)_i$ as multiplication by i. If we set $\overline{\mathfrak{g}}(X) := [\mathfrak{g}(X)_0, \mathfrak{g}(X)_0]$, then

$$\mathfrak{g}(X)_0 = \overline{\mathfrak{g}}(X) \oplus Eh.$$

The subspace $\mathfrak{g}(X)_2$ (resp. $\mathfrak{g}(X)_{-2}$) is generated by operators e_x (resp. f_x) for $x \in H^2(X)$ satisfying the HL property.

(3) The action of $\overline{\mathfrak{g}}(X)$ on $H^\bullet(X)$ preserves the grading. For $i \geq 0$, let

$$\rho_i \colon \overline{\mathfrak{g}}(X) \to \operatorname{End}_E(H^i(X))$$

be the natural representation. Then ρ_2 induces an isomorphism of Lie algebras over E

$$\rho_2 \colon \overline{\mathfrak{g}}(X) \simeq \mathfrak{so}(H^2(X)).$$

(4) The isomorphism ρ_2 extends uniquely to an isomorphism

$$\psi\colon\ \mathfrak{g}(X) \simeq \mathfrak{so}(\widetilde{H}^2(X))$$

of Lie algebras over E with the following properties:

- *$\psi(h)$ is zero on $H^2(X)$ and we have $\psi(h)(v) = -2v$ and $\psi(h)(w) = 2w$.*
- *For any $x \in H^2(X)$, we have $\psi(e_x)(v) = x$, $\psi(e_x)(w) = 0$, and $\psi(e_x)(y) = q(x,y)w$ for every $y \in H^2(X)$.*

Proof. For the Betti realization $\mathfrak{g}_B(X)$, the assertions follow from [Ver95], [LL97], [Gre+22, Subsection 2.1.1]. See also [Flo22a, Theorem 2.2].

We shall prove (1)–(3) for realizations $\mathfrak{g}_\ell(X)$ and $\mathfrak{g}_{\mathrm{pst}}(X)$. For $\mathfrak{g}_\ell(X)$, by replacing L by a subfield $L' \subset L$ which is finitely generated over $\mathbb{Q}$ such that X is defined over L', we may assume that there is an embedding $\overline{L} \hookrightarrow \mathbb{C}$. Then the assertions follow from those for $\mathfrak{g}_B(X_{\mathbb{C}})$ and (3.1). For $\mathfrak{g}_{\mathrm{pst}}(X)$, since (1)–(3) can be checked after taking the base change to B_{dR} (which is a field), these assertions follow from those for $\mathfrak{g}_p(X)$ and (3.2).

We shall prove (4). First, the uniqueness of ψ follows from the fact that $\mathfrak{g}(X)_{-2}$ is generated by operators f_x together with the Jacobson–Morozov theorem applying to the graded algebra $\widetilde{H}^2(X)$ (see Remark 3.9). The existence of ψ for $\mathfrak{g}_\ell(X)$ follows easily from the case of $\mathfrak{g}_B(X)$ by the procedure as above. Let us prove the existence of ψ for $\mathfrak{g}_{\mathrm{pst}}(X)$. By the same argument as in the uniqueness part, we observe that if there exists an isomorphism as in the statement over B_{dR}, then it automatically descends to an isomorphism over K_0^{ur}. Thus the existence of ψ follows from the case of $\mathfrak{g}_p(X)$ and (3.2). □

Definition 3.11. The Lie algebra $\overline{\mathfrak{g}}(X)$ as in Theorem 3.10 is called the *reduced LLV Lie algebra* of X and is denoted by

$$\overline{\mathfrak{g}}_B(X) \quad (\text{resp. } \overline{\mathfrak{g}}_\ell(X), \quad \text{resp. } \overline{\mathfrak{g}}_{\mathrm{pst}}(X))$$

in the case (a) (resp. (b), resp. (c)).

Let us discuss the representation-theoretic property of the action of $\mathfrak{g}(X)_F := \mathfrak{g}(X) \otimes_E F$ on $H^\bullet(X)_F := H^\bullet(X) \otimes_E F$. We note that $b_2(X) \geq 3$, and thus we can apply the results of Section 2.4 to $\mathfrak{g}(X)_F$ (and $\overline{\mathfrak{g}}(X)_F$). We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}(X)_F$. Let us denote the non-trivial $\mathfrak{h}$-weights of the natural action of $\mathfrak{g}(X)_F \simeq \mathfrak{so}(\widetilde{H}^2(X)_F)$ on the Mukai completion $\widetilde{H}^2(X)_F$ by

$$\{\pm\epsilon_0, \pm\epsilon_1, \ldots, \pm\epsilon_r\} \subset \mathfrak{h}^* \setminus \{0\},$$

where $r = \lfloor \frac{b_2(X)}{2} \rfloor$. Then we choose a positive system of roots R^+ and the corresponding set of dominant integral weights Λ^+ of $\mathfrak{g}(X)_F$ as in Section 2.4. For a dominant integral weight $\mu = \sum_{i=0}^r \mu_i \epsilon_i \in \Lambda^+$, let $V_{\mu,F}$ denote the irreducible $\mathfrak{g}(X)_F$-module with highest weight μ. We will identify μ with the sequence $(\mu_0, \mu_1, \ldots, \mu_r)$ and write

$$V_{\mu,F} = V_{(\mu_0,\mu_1,\ldots,\mu_r),F}.$$

We will often omit the zero components of μ from the notation. For example, we will write $(j) = (j, 0, \ldots, 0)$ for $j \geq 0$.

Definition 3.12 (LLV decomposition)**.** The decomposition into irreducible $\mathfrak{g}(X)_F$-modules

$$H^\bullet(X)_F = \bigoplus_{\mu \in \Lambda^+} V_{\mu,F}^{\oplus m_\mu}$$

is called the *LLV decomposition*, where $m_\mu \geq 0$.

Remark 3.13. We will also have to consider the decomposition of $H^i(X)_F$ into irreducible representations of the reduced LLV Lie algebra $\overline{\mathfrak{g}}(X)_F$. For this, we will employ the following notation for irreducible $\overline{\mathfrak{g}}(X)_F$-modules. We fix a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}(X)_F$ and denote the non-trivial $\overline{\mathfrak{h}}$-weights of the natural action on $H^2(X)_F$ by $\{\pm\epsilon'_1, \ldots, \pm\epsilon'_r\} \subset \overline{\mathfrak{h}}^* \backslash \{0\}$. Let $\overline{\Lambda}^+$ be the set of sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ satisfying the conditions as in (2.1) or (2.2) depending on whether $b_2(X)$ is odd or even. For each sequence $\lambda \in \overline{\Lambda}^+$, one can associate an irreducible $\overline{\mathfrak{g}}(X)_F$-module

$$\overline{V}_{\lambda,F} = \overline{V}_{(\lambda_1,\lambda_2,\ldots,\lambda_r),F}$$

of highest weight λ.

Let $SH^2(X) \subset H^\bullet(X)$ be the graded E-algebra generated by $H^2(X)$. We call $SH^2(X)$ the *Verbitsky component* of X.

Theorem 3.14. *Recall that the dimension of X is $2n$.*

(1) The Verbitsky component $SH^2(X)$ is a $\mathfrak{g}(X)$-submodule of $H^\bullet(X)$.

(2) Let $0 \leq i \leq n$. The map induced by the cup product

$$\operatorname{Sym}^i H^2(X) \to H^{2i}(X)$$

is injective. In other words, in the Verbitstky component, the degree $2i$ part is isomorphic to $\operatorname{Sym}^i H^2(X)$.

(3) $SH^2(X)_F$ is an irreducible $\mathfrak{g}(X)_F$-module of highest weight (n).

Proof. It is clear that $SH^2(X)$ is a $\overline{\mathfrak{g}}(X)$-submodule of $H^\bullet(X)$. We can check that $SH^2(X)$ is also closed under the action of $h, \mathfrak{g}(X)_2, \mathfrak{g}(X)_{-2}$ by the same argument as in [Gre+22, Theorem 2.15]. This shows (1).

For $SH^2_B(X)$, the assertions (2) and (3) are essentially obtained by Verbitsky as explained in [Gre+22, Theorem 2.15]. For $SH^2_\ell(X)$, as in the proof of Theorem 3.10, we may assume that $\overline{L} \subset \mathbb{C}$. Then the $\mathfrak{g}_\ell(X)$-module $SH^2_\ell(X)$ can be identified with the base change of the $\mathfrak{g}_B(X_\mathbb{C})$-module $SH^2_B(X_\mathbb{C})$. Thus the assertions follow from the case of $SH^2_B(X_\mathbb{C})$ together with Remark 2.8 and Remark 2.9.

For $SH^2_{\mathrm{pst}}(X)$, the assertions can be checked over B_{dR} (or its algebraic closure) by the same remarks as above. By (3.2), after tensoring with B_{dR}, the $\mathfrak{g}_{\mathrm{pst}}(X)$-module $SH^2_{\mathrm{pst}}(X)$ can be identified with the $\mathfrak{g}_p(X)$-module $SH^2_p(X)$. Therefore the results follow from the case of $SH^2_p(X)$. $\square$

Corollary 3.15. *Let $N \in \overline{\mathfrak{g}}(X) \subset \mathrm{End}_E(H^\bullet(X))$ be a nilpotent operator. Denote N_i for the restriction of N on $H^i(X)$. Then we have $\nu(N_{2i}) \geq i\nu(N_2)$ for any $0 \leq i \leq n$.*

Proof. For any $0 \leq i \leq n$, the restriction of N_{2i} on $\mathrm{Sym}^i H^2(X) \subset H^{2i}(X)$ agrees with the action $\mathrm{Sym}^i N_2$ induced by N_2. On the other hand, we have $\nu(\mathrm{Sym}^i N_2) = i\nu(N_2)$ by a direct computation (see [Gre+22, Lemma 5.6]). Thus the required inequality follows. □

Similarly, we obtain the following result for Galois representations of X.

Corollary 3.16. *Let X be a hyper-Kähler variety over K of dimension $2n$. Let $N_{\ell,2i}$ be the ℓ-adic monodromy operator on $H^{2i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ if $\ell \neq p$ and let $N_{p,2i}$ be the p-adic monodromy operator on $D_{\mathrm{pst}}(H^{2i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ if $\ell = p$. Then the inequality $\nu(N_{\ell,2i}) \geq i\nu(N_{\ell,2})$ holds for all $0 \leq i \leq n$ and all ℓ.*

Proof. Since the injection $\mathrm{Sym}^i H^2(X) \to H^{2i}(X)$ is compatible with (ℓ-adic and p-adic) monodromy operators, this follows from the same argument as in Corollary 3.15. □

For X being a hyper-Kähler variety over $\mathbb{C}$ of one of the four known deformation types ($\star$) as in Remark 2.1, Green–Kim–Laza–Robles in [Gre+22] provide a concrete description of the LLV decomposition of $H^\bullet_B(X)_\mathbb{C}$ (or more strongly, the LLV decomposition of $H^\bullet_B(X)_\mathbb{R}$). As a consequence, we have the following result.

Theorem 3.17 (Green–Kim–Laza–Robles)**.** *We assume that X is in one of four known deformation types ($\star$). For any $\mu = \sum_{i=0}^r \mu_i \epsilon_i \in \Lambda^+$ such that $V_{\mu,F}$ appears in $H^\bullet(X)_F$, we have $\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \leq n$. In particular, $\mu_0 + \mu_1 + \mu_2 \leq n$.*

Proof. For $H^\bullet_B(X)$, this result is proved in [Gre+22, Theorem 6.1]. One can deduce the results for $H^\bullet_\ell(X)$ and $H^\bullet_{\mathrm{pst}}(X)$ from the case of $H^\bullet_B(X)$ by the same argument as in Theorem 3.14. (Again, we have used Remark 2.8 and Remark 2.9.) □

We also include the following result. Note that the action of $\mathfrak{g}(X)_F$ preserves the decomposition

$$H^\bullet(X)_F = H^+(X)_F \oplus H^-(X)_F,$$

where $H^+(X)_F$ and $H^-(X)_F$ consist of parts of even degree and odd degree.

Proposition 3.18. *(1) For every irreducible $\mathfrak{g}(X)_F$-module component $V_{\mu,F} \subset H^+(X)_F$, we have $\lambda_j \in \mathbb{Z}$ for any j. If $H^-(X)_F \neq 0$, then for every $V_{\mu,F} \subset H^-(X)_F$, we have $\lambda_j \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ for any j.*

(2) For every irreducible $\overline{\mathfrak{g}}(X)_F$-module component $\overline{V}_{\lambda,F} \subset H^{2i}(X)$, we have $\lambda_j \in \mathbb{Z}$ for any j. If $H^{2i+1}(X)_F \neq 0$, then for every $\overline{V}_{\lambda,F} \subset H^{2i+1}(X)$, we have $\lambda_j \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ for any j.

Proof. For $H^\bullet_B(X)$, these two statements are proved in [Gre+22, Proposition 2.35]. One can deduce them for $H^\bullet_\ell(X)$ and $H^\bullet_{\mathrm{pst}}(X)$ from the case of $H^\bullet_B(X)$, by the same argument as in Theorem 3.14. □

Corollary 3.19. *For $1 \le i \le 4n-1$ such that $H^i(X) \neq 0$, the homomorphism $\rho_i \colon \overline{\mathfrak{g}}(X) \to \mathrm{End}_E(H^i(X))$ is injective. In particular, this is the case when i is even.*

Proof. Note that $\overline{\mathfrak{g}}(X) \simeq \mathfrak{so}(H^2(X))$ is a simple Lie algebra. Thus ρ_i is injective if and only if the representation is non-trivial. If $1 \le i \le 4n-1$ is even, then the latter holds by Theorem 3.14 (for $i \le 2n$) and Poincaré duality (for $2n < i$). If i is odd and $H^i(X) \neq 0$, then the latter holds by Proposition 3.18. □

For our purpose, it will be convenient to consider integrated representations associated with LLV Lie algebras. More precisely, we consider the integration

$$\widetilde{\rho} \colon \mathrm{Spin}(H^2(X)) \to \prod_i \mathrm{GL}_E(H^i(X))$$

of the representation $\rho \colon \overline{\mathfrak{g}}(X) \to \prod_i \mathrm{End}_E(H^i(X))$. Recall that

$$\mathrm{GSpin}(H^2(X)) = \mathrm{Spin}(H^2(X)) \cdot \mathbb{G}_{\mathrm{m}}$$

where $\mathbb{G}_{\mathrm{m}}$ is the center of $\mathrm{GSpin}(H^2(X))$. As discussed in [FFZ21, Section 6.2] and [Flo22a, Section 2.1], we can extend $\widetilde{\rho}$ to a representation

$$R \colon \mathrm{GSpin}(H^2(X)) \to \prod_i \mathrm{GL}_E(H^i(X)) \tag{3.3}$$

such that each $\lambda \in \mathbb{G}_m \subset \mathrm{GSpin}(H^2(X))$ acts on $H^i(X)$ as multiplication by λ^i for all i. This representation R is called the *twisted LLV representation*.

Remark 3.20. The Lie algebra representation induced by R is identified with the faithful representation

$$\mathfrak{g}(X)_0^{\mathrm{tw}} := \overline{\mathfrak{g}}(X) \oplus E h^{\deg} \hookrightarrow \prod_i \mathrm{End}_E(H^i(X)), \tag{3.4}$$

where $h^{\deg}$ is the map acting on each $H^i(X)$ as multiplication by i. Note that we have

$$h^{\deg} = h + 2n.$$

On the other hand, the LLV representation $\rho \colon \mathfrak{g}_0(X) \to \prod_i \mathrm{End}_E(H^i(X))$ can be integrated to a representation $\mathrm{GSpin}(H^2(X)) \to \prod_i \mathrm{GL}_E(H^i(X))$ extending $\widetilde{\rho}$ such that each $\lambda \in \mathbb{G}_m$ acts on $H^i(X)$ as multiplication by λ^{i-2n} for all i.

3.3. **Weil operator.** Here we recall Verbitsky's observation in [Ver90], which reveals that the LLV Lie algebra of a hyper-Kähler variety interestingly incorporates its Hodge structure information.

Let V be a $\mathbb{Q}$-Hodge structure (i.e., a direct sum of finitely many pure $\mathbb{Q}$-Hodge structures of possibly different weights). Let $V_{\mathbb{C}} = \bigoplus_{s,t \in \mathbb{Z}} V^{s,t}$ be the Deligne splitting. The *Weil operator* $W \in \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ is the map defined by

$$V^{s,t} \ni x \mapsto (t-s)\sqrt{-1}x \in V^{s,t}.$$

Theorem 3.21 (Verbitsky)**.** *Let X be a hyper-Kähler variety over $\mathbb{C}$. Then the Weil operator W of the full cohomology $H_B^\bullet(X)$ is contained in $\overline{\mathfrak{g}}_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof. This is observed by Verbitsky in [Ver90, Theorem 1.4]; see also [Gre+22, Proposition 2.24]. □

Remark 3.22. Let $\mathrm{MT}(X)$ be the Mumford–Tate group of the $\mathbb{Q}$-Hodge structure $H_B^\bullet(X)$. Recall the twisted LLV representation $R\colon \mathrm{GSpin}(H_B^2(X)) \to \prod_i \mathrm{GL}_{\mathbb{Q}}(H_B^i(X))$ from (3.3). It follows from Theorem 3.21 that $\mathrm{MT}(X)$ is contained in the image of R; see [FFZ21, Lemma 6.7].

4. Galois representations associated with hyper-Kähler varieties

Let L be a field of characteristic zero and X a hyper-Kähler variety over L. In this section, we recall the relation between the Galois representation

$$\phi_X\colon G_L \to \prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H_\ell^i(X)),$$

where $H_\ell^i(X) = H^i_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_\ell)$, and the twisted LLV representation (see (3.3))

$$R\colon \mathrm{GSpin}(H_\ell^2(X)) \to \prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H_\ell^i(X)).$$

4.1. **Galois representations for known deformation types.** If $\overline{L} \subset \mathbb{C}$, the Artin comparison theorem provides a natural identification

$$\prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H_\ell^i(X)) \simeq \prod_i \mathrm{GL}_{\mathbb{Q}}(H_B^i(X_{\mathbb{C}})) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_\ell. \tag{4.1}$$

The following theorem is known as the *Mumford–Tate conjecture* of hyper-Kähler varieties, which is confirmed by André ([And96]) in degree 2 when $b_2(X) \geq 4$. For the full cohomology, the validity of this conjecture for the four known deformation types is established in a series of works [Flo22b, Theorem 1.1] and [FFZ21, Theorem 1.18].

Theorem 4.1. *We assume that L is finitely generated over $\mathbb{Q}$. We fix an embedding $\overline{L} \hookrightarrow \mathbb{C}$. Assume that X is in one of four known deformation types ($\star$). Let $G_\ell^\circ(X)$ be the identity component of the Zariski closure of the image of ϕ_X. Then we have*

$$G_\ell^\circ(X) = \mathrm{MT}(X_{\mathbb{C}}) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_\ell$$

under the identification (4.1).

Without the assumption that L is finitely generated over $\mathbb{Q}$, we can still prove

Corollary 4.2. *Assume that X is in one of four known deformation types ($\star$). There exists an open subgroup $H \subset G_L$ such that $\phi_X(H)$ is contained in the image of the twisted LLV representation $R\colon \mathrm{GSpin}(H_\ell^2(X)) \to \prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H_\ell^i(X))$.*

Proof. By replacing L by a subfield $L' \subset L$ which is finitely generated over $\mathbb{Q}$ such that X is defined over L', we may assume that we are in the situation of Theorem 4.1. Then the assertion follows from Theorem 4.1 and Theorem 3.21 (see also Remark 3.22). □

An important consequence is

Corollary 4.3. *Let X be a hyper-Kähler variety over K which is in one of four known deformation types ($\star$) and $\ell \neq p$. Then the ℓ-adic monodromy operator*

$$N_\ell = (N_{\ell,i})_i \in \prod_i \mathrm{End}_{\mathbb{Q}_\ell}(H_\ell^i(X))$$

is contained in the reduced LLV Lie algebra $\overline{\mathfrak{g}}_\ell(X)$.

Proof. It follows from Corollary 4.2 that $N_\ell \in \overline{\mathfrak{g}}_\ell(X) \oplus \mathbb{Q}_\ell h^{\deg} \subset \prod_i \mathrm{End}_{\mathbb{Q}_\ell}(H^i_\ell(X))$ (cf. (3.4)). Since N_ℓ is nilpotent, its trace is zero, and thus we have $N_\ell \in \overline{\mathfrak{g}}_\ell(X)$. □

Remark 4.4. Corollary 4.2 (together with the Kuga–Satake construction) also implies that the p-adic monodromy operator $N_p = (N_{p,i})_i \in \prod_i \mathrm{End}_{K_0^{\mathrm{ur}}}(H^i_{\mathrm{pst}}(X))$ is contained in $\overline{\mathfrak{g}}_{\mathrm{pst}}(X)$. The details will be given in Section 6. In fact, we will prove this by applying Sen's theory instead of Theorem 4.1, which enables us to prove the same result without assuming that X is of one of the four known deformation types ($\star$); see Theorem 6.3.

4.2. **Kuga–Satake constructions.** We recall the Kuga–Satake construction for hyper-Kähler varieties. We assume that $b_2(X) \geq 4$. Let $\mathscr{L}$ be an ample line bundle on X.

After replacing L by its finite extension, there exists an abelian variety $\mathrm{KS}(X, \mathscr{L})$ over L of dimension $2^{b_2(X)-2}$ with a $\mathbb{Z}/2\mathbb{Z}$-grading

$$\mathrm{KS}(X, \mathscr{L}) = \mathrm{KS}^+(X, \mathscr{L}) \times \mathrm{KS}^-(X, \mathscr{L})$$

satisfying the following properties. Let $P^2_\ell(X)(1) \subset H^2_\ell(X)(1)$ be the primitive part with respect to $\mathscr{L}$. Let $\mathrm{Cl} := \mathrm{Cl}(P^2_\ell(X)(1))$ be the (abstract) Clifford algebra over $\mathbb{Q}_\ell$. Then $H^1_\ell(\mathrm{KS}(X, \mathscr{L}))$ admits a right action of Cl compatible with $\mathbb{Z}/2\mathbb{Z}$-gradings and the G_L-action on $H^1_\ell(\mathrm{KS}(X, \mathscr{L}))$, where we equip Cl with the trivial G_L-action. Moreover, there exist G_L-equivariant isomorphisms

$$\mathrm{Cl}(P^2_\ell(X)(1)) \simeq \mathrm{End}_{\mathrm{Cl}}(H^1_\ell(\mathrm{KS}(X, \mathscr{L}))), \tag{4.2}$$

$$\mathrm{Cl}^+(P^2_\ell(X)(1)) \simeq \mathrm{End}_{\mathrm{Cl}^+}(H^1_\ell(\mathrm{KS}^+(X, \mathscr{L}))). \tag{4.3}$$

Here the left hand sides are equipped with the G_L-action induced from that on $P^2_\ell(X)(1)$. There exists an isomorphism

$$H^1_\ell(\mathrm{KS}(X, \mathscr{L})) \simeq \mathrm{Cl} \tag{4.4}$$

compatible with $\mathbb{Z}/2\mathbb{Z}$-gradings and right actions of Cl, such that via this isomorphism, the isomorphisms (4.2) and (4.3) are given by left multiplication. We call $\mathrm{KS}(X, \mathscr{L})$ a *Kuga-Satake abelian variety.*

Remark 4.5. The existence of such an abelian variety was originally observed by Deligne [Del72] and André [And96]. One can also prove this result by using the period map over $\mathbb{Q}$ given in [Bin21], as in the case of $K3$ surfaces [Mad15]. Here, as in [Cha13; Mad15], we use the full Clifford algebra rather than the even part.

Lemma 4.6. *(1) There exists a unique homomorphism $G_L \to \mathrm{GSpin}(P^2_\ell(X)(1))$ which induces the usual actions of G_L on $H^1_\ell(\mathrm{KS}(X, \mathscr{L}))$ via (4.2) and on $P^2_\ell(X)(1)$.*

(2) There exist an isomorphism

$$H^1_\ell(\mathrm{KS}(X, \mathscr{L})^{\times 2}) \simeq \mathrm{Cl}(H^2_\ell(X)(1)) \tag{4.5}$$

of $\mathbb{Q}_\ell$-vector spaces and a homomorphism

$$\phi_{\mathrm{KS}} \colon G_L \to \mathrm{GSpin}(H^2_\ell(X)(1))$$

which induces the usual actions of G_L on $H^1_\ell(\mathrm{KS}(X, \mathscr{L})^{\times 2})$ via (4.5), and on $H^2_\ell(X)(1)$.

Proof. (1) follows immediately. For (2), using the orthogonal decomposition $H^2_\ell(X)(1) = P^2_\ell(X)(1) \oplus \mathbb{Q}_\ell \cdot c_1(\mathscr{L})$, we obtain

$$\mathrm{Cl}(H^2_\ell(X)(1)) = \mathrm{Cl}(P^2_\ell(X)(1)) \oplus \mathrm{Cl}(P^2_\ell(X)(1)) \cdot c_1(\mathscr{L}).$$

In particular we have an inclusion $\mathrm{GSpin}(P^2_\ell(X)(1)) \hookrightarrow \mathrm{GSpin}(H^2_\ell(X)(1))$. Moreover, the isomorphism (4.4) induces (4.5) such that the composition

$$G_L \to \mathrm{GSpin}(P^2_\ell(X)(1)) \hookrightarrow \mathrm{GSpin}(H^2_\ell(X)(1))$$

induces the usual action of G_L on $H^1_\ell(\mathrm{KS}(X,\mathscr{L})^{\times 2})$. (See also [And96, Variant 4.1.3].) □

We consider the homomorphism ϕ_{KS} constructed in Lemma 4.6. Here, as in Remark 3.8, we identify $\mathrm{GSpin}(H^2_\ell(X)(1))$ with $\mathrm{GSpin}(H^2_\ell(X))$ by choosing an isomorphism $\mathbb{Q}_\ell(1) \simeq \mathbb{Q}_\ell$ (but we will distinguish the actions of G_L on $H^2_\ell(X)(1)$ and $H^2_\ell(X)$).

Proposition 4.7. *Assume that the homomorphism $\phi_X \colon G_L \to \prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H^i_\ell(X))$ factors through $R(\mathrm{GSpin}(H^2_\ell(X))) \subset \prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H^i_\ell(X))$. Then, the composition*

$$G_L \xrightarrow{\phi_{\mathrm{KS}}} \mathrm{GSpin}(H^2_\ell(X)) \xrightarrow{R} \prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H^i_\ell(X))$$

agrees with the homomorphism ϕ_X after restricting to an open subgroup of G_L.

Proof. By [FFZ21, Lemma 6.5], the projection $\prod_i \mathrm{GL}_{\mathbb{Q}_\ell}(H^i_\ell(X)) \to \mathrm{GL}_{\mathbb{Q}_\ell}(H^2_\ell(X))$ induces an isogeny with finite kernel from $R(\mathrm{GSpin}(H^2_\ell(X)))$ onto its image. Therefore, it suffices to show that the action of G_L on $H^2_\ell(X)$ induced by the composition

$$G_L \xrightarrow{\phi_{\mathrm{KS}}} \mathrm{GSpin}(H^2_\ell(X)) \xrightarrow{R_2} \mathrm{GL}_{\mathbb{Q}_\ell}(H^2_\ell(X)) \tag{4.6}$$

is equal to the usual Galois action, where R_2 is the composition of R with the projection. We remark that R_2 is not the same as the usual homomorphism

$$\mathrm{GSpin}(H^2_\ell(X)) \to \mathrm{SO}(H^2_\ell(X)) \subset \mathrm{GL}_{\mathbb{Q}_\ell}(H^2_\ell(X)),$$

but rather its twist by the spinor norm $\mathrm{GSpin}(H^2_\ell(X)) \to \mathbb{G}_m$. By construction of $\mathrm{KS}(X,\mathscr{L})$, the composition

$$G_L \xrightarrow{\phi_{\mathrm{KS}}} \mathrm{GSpin}(H^2_\ell(X)) \to \mathbb{G}_m$$

agrees with the inverse of the cyclotomic character. Thus, the action of G_L induced by (4.6) is the twist of the usual action on $H^2_\ell(X)(1)$ by the inverse of the cyclotomic character, that is $H^2_\ell(X)$. □

Corollary 4.8. *We set $W := H^1_\ell(\mathrm{KS}(X,\mathscr{L})^{\times 2})$, viewed as $\mathrm{GSpin}(H^2_\ell(X))$-module by the identification (4.5). Let*

$$\widetilde{\iota} \colon H^\bullet_\ell(X) \hookrightarrow \bigoplus_{j\in J} W^{\otimes m_j} \otimes W^{\vee,\otimes m'_j}$$

be a $\mathrm{GSpin}(H^2_\ell(X))$-equivariant embedding [1] *for a finite set J and a family of integers $\{m_j, m'_j\}_{j\in J}$. Assume that ϕ_X factors through $R(\mathrm{GSpin}(H^2_\ell(X)))$. Then there exists an open subgroup $H \subset G_L$ such that $\widetilde{\iota}$ is H-equivariant, and admits an H-equivariant splitting.*

[1] Since $\mathrm{GSpin}(H^2_\ell(X))$ is reductive and its action on W is faithful, such an embedding always exists.

Proof. This follows from Proposition 4.7. For the second statement, we have used that every finite-dimensional representation of $\mathrm{GSpin}(H^2_\ell(X))$ is semisimple. $\square$

In the rest of this section, we assume that $L = K$ and $\ell = p$. We shall recall a lemma concerning the compatibility between the p-adic monodromy operators and the Kuga–Satake construction. This result will be used in the proof of Theorem 6.3.

For $W := H^1_p(\mathrm{KS}(X, \mathscr{L})^{\times 2}) \simeq \mathrm{Cl}(H^2_p(X)(1))$, the natural embedding

$$H^2_p(X)(1) \hookrightarrow \mathrm{End}_{\mathbb{Q}_p}(W) \tag{4.7}$$

defined by left multiplication by elements in $H^2_p(X)(1)$ is $\mathrm{GSpin}(H^2_p(X)(1))$-equivariant, and hence G_K-equivariant. We identify the Lie algebra of $\mathrm{GSpin}(H^2_p(X)(1))$ with the orthogonal Lie algebra $\mathfrak{so}(H^2_p(X)(1)) \oplus \mathbb{Q}_p h^{\deg}$ as in (3.4), and let

$$\rho^{\mathrm{KS}} \colon \mathfrak{so}(H^2_p(X)(1)) \oplus \mathbb{Q}_p h^{\deg} \hookrightarrow \mathrm{End}_{\mathbb{Q}_p}(W)$$

be the induced Lie algebra representation, where $h^{\deg}$ acts as the identity on W. Since ρ^{KS} is G_K-equivariant, we then have the induced homomorphism

$$\rho^{\mathrm{KS}} \colon \mathfrak{so}(H^2_{\mathrm{pst}}(X)(1)) \oplus K_0^{\mathrm{ur}} h^{\deg} \hookrightarrow \mathrm{End}_{K_0^{\mathrm{ur}}}(D_{\mathrm{pst}}(W)),$$

which we denote by the same symbol.

Lemma 4.9. *Let $N_{\mathrm{KS}} \in \mathrm{End}_{K_0^{\mathrm{ur}}}(D_{\mathrm{pst}}(W))$ be the monodromy operator on $D_{\mathrm{pst}}(W)$. Then $N_{\mathrm{KS}} = \rho^{\mathrm{KS}}(N_{p,2})$ where $N_{p,2} \in \mathfrak{so}(H^2_{\mathrm{pst}}(X)(1))$ is the monodromy operator on $H^2_{\mathrm{pst}}(X)(1)$.*

Proof. Let $\{s_\alpha\}_\alpha \subset W^\otimes$ be a set of tensors defining $\mathrm{GSpin}(H^2_p(X)(1)) \subset \mathrm{GL}_{\mathbb{Q}_p}(W)$. Here $W^\otimes$ is the direct sum of all the $\mathbb{Q}_p$-vector spaces which can be constructed from W by taking duals and tensor products. Then $\mathfrak{so}(H^2_p(X)(1)) \oplus \mathbb{Q}_p h^{\deg} \subset \mathrm{End}_{\mathbb{Q}_p}(W)$ agrees with the subspace consisting of all endomorphisms that annihilate $\{s_\alpha\}_\alpha$.

Since each s_α is G_K-invariant, it induces a tensor $s_{\alpha,\mathrm{pst}} \in D_{\mathrm{pst}}(W)^\otimes$ which is annihilated by the monodromy operator. Since $\mathfrak{so}(H^2_{\mathrm{pst}}(X)(1)) \oplus K_0^{\mathrm{ur}} h^{\deg} \subset \mathrm{End}_{K_0^{\mathrm{ur}}}(D_{\mathrm{pst}}(W))$ agrees with the subspace consisting of all endomorphisms that annihilate $\{s_{\alpha,\mathrm{pst}}\}_\alpha$, we have $N_{\mathrm{KS}} \in \mathfrak{so}(H^2_{\mathrm{pst}}(X)(1)) \oplus K_0^{\mathrm{ur}} h^{\deg}$. Since N_{KS} is nilpotent, it follows that $N_{\mathrm{KS}} \in \mathfrak{so}(H^2_{\mathrm{pst}}(X)(1))$. The homomorphism

$$H^2_{\mathrm{pst}}(X)(1) \to \mathrm{End}_{K_0^{\mathrm{ur}}}(D_{\mathrm{pst}}(W))$$

induced by (4.7) is compatible with monodromy operators, which implies that $N_{\mathrm{KS}} = N_{p,2}$ in $\mathfrak{so}(H^2_{\mathrm{pst}}(X)(1))$. $\square$

5. Reduction types

In this section, we will establish an arithmetic analogue of [SS20], which is essential in the proof of the main theorems. In particular, we will show that there is a geometric description for the reduction types of the second cohomology of a hyper-Kähler variety over K.

5.1. **Reduction types of Kuga–Satake abelian varieties.** Let X be a hyper-Kähler variety over K and $\mathscr{L}$ an ample line bundle on X. We assume that $b_2(X) \geq 4$. By extending K if necessary, we assume that the Kuga–Satake abelian variety $\mathrm{KS}(X, \mathscr{L})$ and the additional structures on it as in Section 4.2 are defined over K. In particular, we have the $\mathbb{Z}/2\mathbb{Z}$-grading $\mathrm{KS}(X, \mathscr{L}) = \mathrm{KS}^+(X, \mathscr{L}) \times \mathrm{KS}^-(X, \mathscr{L})$.

We set

$$A \coloneqq \mathrm{KS}^+(X, \mathscr{L}) \quad (\text{resp. } A \coloneqq \mathrm{KS}(X, \mathscr{L}))$$

if $b_2(X)$ is even (resp. odd). Let ℓ be a prime. If $\ell \neq p$, then let $P^2(X)(1)$ be the primitive part of $H^2_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_\ell)(1)$ with respect to $\mathscr{L}$ and $H^1(A) \coloneqq H^1_{\text{ét}}(A_{\overline{K}}, \overline{\mathbb{Q}}_\ell)$. If $\ell = p$, then let $P^2(X)(1)$ be the primitive part of $D_{\mathrm{pst}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)(1)) \otimes_{K_0^{\mathrm{ur}}} \overline{K}$ with respect to $\mathscr{L}$ and $H^1(A) \coloneqq D_{\mathrm{pst}}(H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)) \otimes_{K_0^{\mathrm{ur}}} \overline{K}$. Here we work over the algebraically closed fields $F = \overline{\mathbb{Q}}_\ell$ or $F = \overline{K}$, respectively.

Let $M_\bullet(X)$ be the monodromy filtration on $P^2(X)(1)$ associated with the monodromy operator N_2 on $P^2(X)(1)$ as in [Del80, Proposition 1.6.1]. Similarly let $M_\bullet(A)$ be the monodromy filtration on $H^1(A)$. For an integer $i \in \mathbb{Z}$, let

$$r_i(X) \coloneqq \dim_F \mathrm{gr}_i^M(P^2(X)(1)) = \dim_F M_i(X)/M_{i-1}(X),$$
$$r_i(A) \coloneqq \dim_F \mathrm{gr}_i^M(H^1(A)) = \dim_F M_i(A)/M_{i-1}(A).$$

Remark 5.1. By Lemma 2.5 and Lemma 2.7, $r_i(X) = 0$ if $|i| > 2$ and $r_i(A) = 0$ if $|i| > 1$. By definition of monodromy filtrations, $r_i(X) = r_{-i}(X)$ and $r_i(A) = r_{-i}(A)$.

Theorem 5.2. *In the above setting, assume that the abelian variety A over K has semi-abelian reduction.*

(1) If $\nu(N_2) = 0$, then we have

$$r_2(X) = r_{-2}(X) = r_1(X) = r_{-1}(X) = 0 \quad \text{and} \quad r_0(X) = b_2(X) - 1.$$

Also, we have $r_1(A) = r_{-1}(A) = 0$ and $r_0(A) = b_1(A)$, and A admits good reduction.

(2) If $\nu(N_2) = 1$, then $b_2(X) \geq 5$, and we have

$$r_2(X) = r_{-2}(X) = 0, \quad r_1(X) = r_{-1}(X) = 2, \quad \text{and} \quad r_0(X) = b_2(X) - 5.$$

Moreover we have $\dim_F(\mathrm{Im}\, N_2) = 2$. Also, we have $r_0(A) = b_1(A)/2$, and $r_1(A) = r_{-1}(A) = b_1(A)/4$, and the torus rank of a special fiber of a Néron model of A is a half of $\dim A$.

(3) If $\nu(N_2) = 2$, then we have

$$r_2(X) = r_{-2}(X) = 1, \quad r_1(X) = r_{-1}(X) = 0, \quad \text{and} \quad r_0(X) = b_2(X) - 3.$$

Moreover we have $\dim_F(\mathrm{Im}\, N_2) = 2$ and $\dim_F(\mathrm{Im}\, N_2^2) = 1$. Also, we have $r_1(A) = r_{-1}(A) = b_1(A)/2$ and $r_0(A) = 0$, and A admits totally toric reduction.

Proof. To simplify the notation, we write $b \coloneqq b_2(X)$. Let B denote the even Clifford algebra $\mathrm{Cl}^+(P^2(X)(1))$ if b is even and denote the full Clifford algebra $\mathrm{Cl}(P^2(X)(1))$ if b is odd. Then there exist a right action of B on $H^1(A)$ which is compatible with the monodromy operator, and an isomorphism

$$\mathrm{Cl}^+(P^2(X)(1)) \simeq \mathrm{End}_B(H^1(A)) \quad (\text{resp. } \mathrm{Cl}(P^2(X)(1)) \simeq \mathrm{End}_B(H^1(A)))$$

which is compatible with monodromy operators. Here the monodromy operator on the left (resp. right) hand side is induced from that on $P^2(X)(1)$ (resp. $H^1(A)$).

Since the even (resp. full) Clifford algebra B is split as a central simple algebra over F, there exists a simple B-submodule $H \subset H^1(A)$ which is preserved by the monodromy operator, and we have an isomorphism $H^1(A) \simeq H^{\oplus m}$ which is compatible with B-actions and monodromy operators. It follows that we have an isomorphism

$$\mathrm{Cl}^+(P^2(X)(1)) \simeq \mathrm{End}_F(H) \quad (\text{resp. } \mathrm{Cl}(P^2(X)(1)) \simeq \mathrm{End}_F(H)) \tag{5.1}$$

which is compatible with monodromy operators. We note that $m = \dim_F(H) = 2^{(b-2)/2}$ (resp. $m = \dim_F(H) = 2^{(b-1)/2}$).

Let us compute the monodromy filtrations on both sides of (5.1). Let $du\colon \mathfrak{sl}_2 \to \mathfrak{so}(P^2(X)(1))$ be a representation with $du(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}) = N_2$ given by the Jacobson–Morozov theorem. We consider the semisimple element $\varphi := (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$ and let $V_j \subset P^2(X)(1)$ be the subspace consisting of elements v such that $\varphi(v) = jv$. Then we have $M_i(X) = \bigoplus_{j \le i} V_j$; see [Del80, (1.6.8)]. In particular we have $\mathrm{gr}^M_i P^2(X)(1) \simeq V_i$ and $r_i(X) = \dim_F V_i$. Let

$$\{-2 \le \beta_1 \le \cdots \le \beta_{b-1} \le 2\}$$

be the (ordered) multiset of eigenvalues of φ acting on $P^2(X)(1)$ so that for any integer $-2 \le i \le 2$, the number of occurrences of i in this multiset is equal to $r_i(X)$. Let $\mathfrak{sl}_2 \to \mathrm{End}_F(\mathrm{Cl}^+(P^2(X)(1)))$ (resp. $\mathfrak{sl}_2 \to \mathrm{End}_F(\mathrm{Cl}(P^2(X)(1))))$ be the homomorphism induced from $du\colon \mathfrak{sl}_2 \to \mathfrak{so}(P^2(X)(1))$. We see that the multiset of eigenvalues of φ acting on $\mathrm{Cl}^+(P^2(X)(1))$ (resp. $\mathrm{Cl}(P^2(X)(1)))$ is given by

$$\begin{gathered}\{0\} \cup \{\beta_{j_1} + \cdots + \beta_{j_{2k}}\}_{k>0,\, 1 \le j_1 < j_2 < \cdots < j_{2k} \le b-1} \\ (\text{resp. } \{0\} \cup \{\beta_{j_1} + \cdots + \beta_{j_k}\}_{k>0,\, 1 \le j_1 < j_2 < \cdots < j_k \le b-1})\end{gathered} \tag{5.2}$$

via the identification

$$\mathrm{Cl}^+(P^2(X)(1)) = \bigoplus_{k \ge 0} \bigwedge^{2k} P^2(X)(1) \quad (\text{resp. } \mathrm{Cl}(P^2(X)(1) = \bigoplus_{k \ge 0} \bigwedge^{k} P^2(X)(1)).$$

By the same reasoning as above, we see that the dimension of the i-th successive quotient

$$\mathrm{gr}^M_i(\mathrm{Cl}^+(P^2(X)(1))) \quad (\text{resp. } \mathrm{gr}^M_i(\mathrm{Cl}(P^2(X)(1))))$$

of the monodromy filtration of $\mathrm{Cl}^+(P^2(X)(1))$ (resp. $\mathrm{Cl}(P^2(X)(1)))$ is equal to the number of occurrences of i in the multiset (5.2).

Using the same description of monodromy filtrations in terms of the Jacobson–Morozov theorem, one can show (see [Del80, Proposition 1.6.9]) that

$$\mathrm{gr}^M_i(\mathrm{End}_F(H)) \simeq \bigoplus_{i'+i''=i} \mathrm{gr}^M_{i'}(H) \otimes \mathrm{gr}^M_{i''}(H^\vee) \simeq \bigoplus_{i'+i''=i} \mathrm{gr}^M_{i'}(H) \otimes \mathrm{gr}^M_{-i''}(H)^\vee.$$

If follows that

$$\dim_F \mathrm{gr}^M_i(\mathrm{End}_F(H)) = \frac{1}{m^2} \sum_{i'+i''=i} r_{i'}(A) r_{i''}(A).$$

In particular, we have $\mathrm{gr}^M_i(\mathrm{End}_F(H)) = 0$ if $|i| > 2$.

By comparing these results via isomorphisms

$$\mathrm{gr}^M_i(\mathrm{Cl}^+(P^2(X)(1))) \simeq \mathrm{gr}^M_i(\mathrm{End}_F(H)) \quad (\text{resp. } \mathrm{gr}^M_i(\mathrm{Cl}(P^2(X)(1))) \simeq \mathrm{gr}^M_i(\mathrm{End}_F(H)))$$

induced from (5.1), we see that one of the following conditions holds:

(a) $r_2(X) = r_{-2}(X) = r_1(X) = r_{-1}(X) = 0$ and $r_0(X) = b - 1$.

(b) $r_2(X) = r_{-2}(X) = 0$, $r_1(X) = r_{-1}(X) = 2$, and $r_0(X) = b - 5$.

(c) $r_2(X) = r_{-2}(X) = 1$, $r_1(X) = r_{-1}(X) = 0$, and $r_0(X) = b - 3$.

(d) $r_2(X) = r_{-2}(X) = 0$, $r_1(X) = r_{-1}(X) = 1$, and $r_0(X) = b - 3$.

In the case (a), we have $\nu(N_2) = 0$. Since every eigenvalue in (5.2) is 0, we have that $r_1(A) = r_{-1}(A) = 0$. This means that A admits good reduction by [ST68, Theorem 1] and [CI99, Part II, Theorem 4.7].

Next, we treat the case (b). Note that (b) does not occur when $b = 4$. In the following, we may assume that $b \geq 5$. In this case, we have $\nu(N_2) = 1$ and $\dim_F(\operatorname{Im} N_2) = 2$. Since we have

$$\dim_F \operatorname{gr}_2^M(\operatorname{End}_F(H)) = r_1(A)^2/m^2$$

and

$$\dim_F \operatorname{gr}_2^M(\operatorname{Cl}^+(P^2(X)(1))) = 2^{b-6} \quad (\text{resp. } \dim_F \operatorname{gr}_2^M(\operatorname{Cl}(P^2(X)(1))) = 2^{b-5}),$$

it follows that $r_1(A) = 2^{(b-4)}$ (resp. $r_1(A) = 2^{(b-3)}$), and thus $r_1(A) = b_1(A)/4$. By [SGA7-I, Exposé IX] (especially [SGA7-I, Exposé IX, Proposition 3.5]) and [CI99, Part II, Proposition 4.5], the torus rank of the special fiber of a Néron model of A is half of $\dim A$.

In the case (c), we see that $\nu(N_2) = 2$, $\dim_F(\operatorname{Im} N_2) = 2$, and $\dim_F(\operatorname{Im} N_2^2) = 1$. Moreover, since eigenvalues in (5.2) (counting without multiplicity) are $\{-2, 0, 2\}$, we have

$$\dim_F \operatorname{gr}_1^M(\operatorname{End}_F(H)) = 0 \quad \text{and} \quad \dim_F \operatorname{gr}_2^M(\operatorname{End}_F(H)) \neq 0.$$

It follows that $r_0(A) = 0$. Thus A admits totally toric reduction (i.e., the torus rank of the special fiber of the Néron model of A is $\dim A$) again by [SGA7-I, Exposé IX] and [CI99, Part II, Proposition 4.5].

In the case (d), we have that $\operatorname{gr}_2^M(\operatorname{End}_F(H)) = 0$ since eigenvalues in (5.2) are contained $\{-1, 0, 1\}$. This implies that $r_1(A) = r_{-1}(A) = 0$. But then we have $\operatorname{gr}_1^M(\operatorname{End}_F(H)) = 0$, which contradicts that 1 occurs at least once in (5.2) (where we use that $r_0(X) = b - 3 > 0$). Therefore, this case does not occur. Since all cases are covered in (a)-(c), it finishes the proof. □

Corollary 5.3. *For a hyper-Kähler variety X over K with $b_2(X) \geq 4$ and any $\ell \neq p$, we have $\nu(N_{\ell,2}) = \nu(N_{p,2})$, where $N_{\ell,2}$ (resp. $N_{p,2}$) is the monodromy operator on $H^2_\ell(X)$ (resp. $H^2_{\mathrm{pst}}(X)$).*

Proof. By extending K if necessary, we may assume that the assumption of Theorem 5.2 is satisfied. Then the result follows from Theorem 5.2 since the conditions on the reduction the Néron model are independent of ℓ and p. □

5.2. **Reduction types of hyper-Kähler varieties.** Let X be a hyper-Kähler variety over K. Let $N_{\ell,2}$ (resp. $N_{p,2}$) be the monodromy operator on $H^2_\ell(X)$ (resp. $H^2_{\mathrm{pst}}(X)$).

Definition 5.4. For a prime number ℓ (including $\ell = p$), we say that X has

- Type I reduction if $\nu(N_{\ell,2}) = 0$,

- Type II reduction if $\nu(N_{\ell,2}) = 1$, or
- Type III reduction if $\nu(N_{\ell,2}) = 2$.

By Corollary 5.3 and Lemma 5.5 below, this definition does not depend on ℓ.

Lemma 5.5. *If $b_2(X) = 3$, we have $N_{\ell,2} = 0$ for any ℓ (including $\ell = p$).*

Proof. For simplicity, we treat the $\ell \neq p$ case; the p-adic case follows by the same argument. Let $\mathscr{L}$ be an ample line bundle on X. Consider the orthogonal decomposition $H^2_\ell(X)(1) = \mathbb{Q}_\ell \cdot c_1(\mathscr{L}) \oplus V$ where $V := P^2_\ell(X)(1)$. It suffices to show that $N_{\ell,2}|_V = 0$. Since $\mathfrak{so}(V)_{\overline{\mathbb{Q}}_\ell} \simeq \mathfrak{gl}_1$ and $N_{\ell,2}|_V \in \mathfrak{so}(V)$ is nilpotent, it follows that $N_{\ell,2}|_V = 0$ as desired. □

The following fact allows us to describe the monodromy operators on a hyper-Kähler variety in a normalized basis.

Proposition 5.6. *Let X be a hyper-Kähler variety over K and ℓ a prime number (including $\ell = p$). Let $r := \lfloor \frac{b_2(X)}{2} \rfloor$. We put*

$$H^2(X) := \begin{cases} H^2_\ell(X) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell & \text{if } \ell \neq p, \\ H^2_{\mathrm{pst}}(X) \otimes_{K_0^{\mathrm{ur}}} \overline{K} & \text{if } \ell = p. \end{cases}$$

The following statements hold.

(1) If X has Type II reduction, then $b_2(X) \geq 5$, and there exists a basis

$$\{e_1, \ldots, e_r, e'_1, \ldots, e'_r\} \quad (\text{resp. } \{e_1, \ldots, e_r, e'_1, \ldots, e'_r, e_{r+1}\})$$

of $H^2(X)$ such that the matrix of q is given by

$$\begin{pmatrix} 0 & \mathrm{id}_{r\times r} \\ \mathrm{id}_{r\times r} & 0 \end{pmatrix} \quad (\text{resp. } \begin{pmatrix} 0 & \mathrm{id}_{r\times r} & 0 \\ \mathrm{id}_{r\times r} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}), \tag{5.3}$$

and

$$N_{\ell,2}(\sum_{i=1}^{r \ (\text{resp. } r+1)} a_i e_i + \sum_{i=1}^{r} a'_i e'_i) = -a_2 e'_1 + a_1 e'_2$$

if $b_2(X)$ is even (resp. odd).

(2) If X has Type III reduction, then $b_2(X) \geq 4$, and there exists a basis

$$\{e_1, \ldots, e_r, e'_1, \ldots, e'_r\} \quad (\text{resp. } \{e_1, \ldots, e_r, e'_1, \ldots, e'_r, e_{r+1}\})$$

of $H^2(X)$ such that the matrix of q is given by (5.3) and

$$N_{\ell,2}(\sum_{i=1}^{r \ (\text{resp. } r+1)} a_i e_i + \sum_{i=1}^{r} a'_i e'_i) = a_1 e_2 - (a_2 + a'_2) e'_1 + a_1 e'_2,$$

if $b_2(X)$ is even (resp. odd).

Proof. By Lemma 5.5, we have $b_2(X) \geq 4$ in both cases. If X has Type II reduction, then $b_2(X) \geq 5$ by Theorem 5.2. For the existence of the required basis, the same proof as in [Gre+22, Lemma 5.10] works by using Theorem 5.2. □

6. Sen theory and Looijenga–Lunts–Verbitsky Lie algebras

For a p-adic Galois representation, the Sen operator plays a role analogous to that of the Weil operator in Hodge theory. In Section 6.1, we will review some facts on Sen's theory and prove that the Sen operator belongs to the LLV Lie algebra for hyper-Kähler varieties. Using this result, we prove in Section 6.2 that the p-adic monodromy operators are contained in the LLV Lie algebra.

6.1. **Sen operator.** Let C be the completion of $\overline{K}$. Let V be a p-adic G_K-representation over $\mathbb{Q}_p$. We assume that V is Hodge–Tate (for simplicity), so that we have a natural G_K-equivariant decomposition

$$V_C = \bigoplus_{i\in\mathbb{Z}} D^i_{\mathrm{HT}}(V) \otimes_K C(-i),$$

where we set $D^i_{\mathrm{HT}}(V) := (V \otimes_{\mathbb{Q}_p} C(i))^{G_K}$ and $C(i) = C \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(i)$ denotes the Tate twist. The *Sen operator* $\Theta\colon V_C \to V_C$ is the C-linear homomorphism such that $\Theta(x) = -ix$ for all $x \in D^i_{\mathrm{HT}}(V) \otimes_K C(-i)$.

Theorem 6.1 (Sen)**.** *Let $\phi\colon G_K \to \mathrm{GL}_{\mathbb{Q}_p}(V)$ be a Hodge–Tate representation. Assume that the residue field k is algebraically closed.*

(1) The Lie algebra $\mathfrak{g}$ of the p-adic Lie group $\phi(G_K)$ is the minimal subalgebra of $\mathrm{End}_{\mathbb{Q}_p}(V)$ such that $\mathfrak{g}_C$ contains Θ.

(2) There exists an open subgroup of G_K that acts on $V \cap \ker\Theta$ trivially.

Proof. See [Sen73, Theorem 1] for (1). The assertion (2) follows by applying the first one to the Hodge–Tate representation $V \cap \ker\Theta$; see [Sen73, Corollary 1]. □

For a smooth proper variety X over K, there is a G_K-equivariant decomposition

$$H^i_p(X)_C \simeq \bigoplus_{s+t=i} H^t(X, \Omega^s_X) \otimes_K C(-s).$$

In this paper, we choose a decomposition induced by the comparison isomorphism given in [Tsu02, Theorem A1]. We call it the *Hodge–Tate decomposition.* Via this decomposition, the $(-s)$-eigenspace of the Sen operator Θ on $H^i_p(X)_C$ is $H^t(X, \Omega^s_X) \otimes_K C(-s)$.

Recall the twisted LLV representation $R\colon \mathrm{GSpin}(H^2_p(X)) \to \prod_i \mathrm{GL}_{\mathbb{Q}_p}(H^i_p(X))$ from (3.3) and the induced Lie subalgebra

$$\mathfrak{g}_p(X)^{\mathrm{tw}}_0 := \overline{\mathfrak{g}}_p(X) \oplus \mathbb{Q}_p h^{\deg} \subset \prod_i \mathrm{End}_{\mathbb{Q}_p}(H^i_p(X))$$

from Remark 3.20. Similar to the relation between the Weil operator and the LLV Lie algebra (Theorem 3.21), we observe the following relation between the Sen operator and the LLV Lie algebra. In fact, we deduce this result from Theorem 3.21.

Theorem 6.2. *Let X be a hyper-Kähler variety over K.*

(1) The Sen operator $\Theta \in \prod_i \mathrm{End}_C(H^i_p(X)_C)$ of $H^\bullet_p(X)$ is contained in $\mathfrak{g}_p(X)^{\mathrm{tw}}_0 \otimes_{\mathbb{Q}_p} C$.

(2) There exists an open subgroup $I \subset I_K$ whose image under $G_K \to \prod_i \mathrm{GL}_{\mathbb{Q}_p}(H^i_p(X))$ is contained in $R(\mathrm{GSpin}(H^2_p(X)))$.

Proof. (1) Let $K' \subset K$ be a subfield which is finitely generated over $\mathbb{Q}$ such that X has a model X' over K'. We regard K' as a subfield of $\mathbb{C}$ and let $Y := X'_{\mathbb{C}}$. We further choose a field L which sits in the following commutative diagram:

$$\begin{array}{ccccc} K' & \longrightarrow & & & \mathbb{C} \\ \downarrow & & & & \downarrow \\ K & \longrightarrow & C & \longrightarrow & L \end{array}$$

Let W be the Weil operator on $H_B^\bullet(Y)_{\mathbb{C}}$. We claim that there exists an isomorphism of graded algebras over L

$$\gamma\colon H_p^\bullet(X)_C \otimes_C L \simeq H_B^\bullet(Y)_{\mathbb{C}} \otimes_{\mathbb{C}} L$$

such that under this isomorphism, we have

$$\Theta = \frac{1}{2\sqrt{-1}} W - \frac{1}{2} h^{\deg}. \tag{6.1}$$

This claim, together with Theorem 3.21 and Lemma 3.1, implies that $\Theta \in \mathfrak{g}_p(X)_0^{\mathrm{tw}} \otimes_{\mathbb{Q}_p} C$.

We shall prove the claim. We fix an isomorphism $\mathbb{Q}_p(1) \simeq \mathbb{Q}_p$ of $\mathbb{Q}_p$-vector spaces. Then, by [Tsu02, Theorem A1, (A1.2)], the Hodge–Tate decomposition induces isomorphisms of graded algebras over L:

$$\bigoplus_i H_p^i(X)_C \otimes_C L \simeq \bigoplus_i \bigoplus_{s+t=i} H^t(X_L, \Omega^s_{X_L}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-s) \simeq \bigoplus_i \bigoplus_{s+t=i} H^t(X_L, \Omega^s_{X_L}),$$

where the algebra structure on the target is induced by the cup product on the Hodge cohomology. On the other hand, by the Hodge decomposition, we have the following isomorphisms of graded algebras over L:

$$\bigoplus_i \bigoplus_{s+t=i} H^t(X_L, \Omega^s_{X_L}) \simeq \bigoplus_i \bigoplus_{s+t=i} H^t(Y, \Omega^s_Y) \otimes_{\mathbb{C}} L \simeq \bigoplus_i H_B^i(Y)_{\mathbb{C}} \otimes_{\mathbb{C}} L.$$

By composing these isomorphisms, we obtain $\gamma\colon H_p^\bullet(X)_C \otimes_C L \simeq H_B^\bullet(Y)_{\mathbb{C}} \otimes_{\mathbb{C}} L$. With this construction, it is easy to check that the equality (6.1) holds.

(2) This can be deduced from (1) and Theorem 6.1 by the same argument as in [FFZ21, Lemma 6.7]. □

6.2. **p-adic monodromy of hyper-Kähler varieties.** We are ready to show that the monodromy operator $N_p = (N_{p,i})_i \in \prod_i \mathrm{End}_{K_0^{\mathrm{ur}}}(H^i_{\mathrm{pst}}(X))$ is contained in the reduced LLV Lie algebra.

Theorem 6.3. *Let X be a hyper-Kähler variety over K.*

(1) If $b_2(X) \geq 4$, then the monodromy operator $N_p \in \prod_i \mathrm{End}_{K_0^{\mathrm{ur}}}(H^i_{\mathrm{pst}}(X))$ of $H^\bullet_{\mathrm{pst}}(X)$ is contained in $\bar{\mathfrak{g}}_{\mathrm{pst}}(X)$.

(2) If $b_2(X) = 3$, then the monodromy operator $N_{p,2i}$ of $H^{2i}_{\mathrm{pst}}(X)$ is zero for any integer i.

Proof. Without loss of generality, we may assume that the residue field k is algebraically closed (cf. [Fon94b, §5.1.5]). Assume that $b_2(X) \geq 4$. We choose an ample line bundle $\mathscr{L}$ on X. Then, after replacing K by its finite extension, we can consider the Kuga–Satake abelian variety $\mathrm{KS}(X, \mathscr{L})$ over K and use the computations given in Section 4.2. We keep the notations there.

As in Corollary 4.8, for $W := H^1_p(\mathrm{KS}(X, \mathscr{L})^{\times 2})$, there exists a $\mathrm{GSpin}(H^2_p(X))$-equivariant embedding

$$\widetilde{\iota}\colon H^\bullet_p(X) \hookrightarrow \bigoplus_{j\in J} W^{\otimes m_j} \otimes W^{\vee,\otimes m'_j}$$

for a finite set J and some integers $\{m_j, m'_j\}_{j\in J}$. By Theorem 6.2 and Corollary 4.8, after replacing K by its finite extension, the embedding $\widetilde{\iota}$ becomes G_K-equivariant. Then we can apply D_{pst} to obtain an embedding

$$\widetilde{\iota}_{\mathrm{pst}}\colon H^\bullet_{\mathrm{pst}}(X) \hookrightarrow \bigoplus_{j\in J} D_{\mathrm{pst}}(W)^{\otimes m_j} \otimes D_{\mathrm{pst}}(W)^{\vee,\otimes m'_j}. \tag{6.2}$$

By construction, this is compatible with the action of

$$\overline{\mathfrak{g}}_{\mathrm{pst}}(X) \simeq \mathfrak{so}(H^2_{\mathrm{pst}}(X)) \simeq \mathfrak{so}(H^2_{\mathrm{pst}}(X)(1)).$$

Let N_{KS} be the monodromy operator of $D_{\mathrm{pst}}(W)$. By Lemma 4.9, we can view $N_{\mathrm{KS}} \in \overline{\mathfrak{g}}_{\mathrm{pst}}(X)$ (which agrees with the monodromy operator $N_{p,2}$ of $H^2_{\mathrm{pst}}(X)$). Since $\widetilde{\iota}_{\mathrm{pst}}$ is also compatible with monodromy operators, we see that $N_{\mathrm{KS}} \in \overline{\mathfrak{g}}_{\mathrm{pst}}(X)$ acts on $H^\bullet_{\mathrm{pst}}(X)$ as the monodromy operator N_p. In particular we have $N_p \in \overline{\mathfrak{g}}_{\mathrm{pst}}(X)$, which proves (1).

One can argue for the even part as follows; this also applies to the case where $b_2(X) = 3$. Consider the integration

$$\widetilde{\rho}\colon \mathrm{Spin}(H^2_p(X)) \to \prod_i \mathrm{GL}_{\mathbb{Q}_p}(H^i_p(X))$$

of $\rho\colon \overline{\mathfrak{g}}_p(X) \to \prod_i \mathrm{End}_{\mathbb{Q}_p}(H^i_p(X))$. The restriction of this representation to the even part $\bigoplus_i H^{2i}_p(X)$ factors through $\mathrm{SO}(H^2_p(X))$. Therefore one can use the faithful $\mathrm{SO}(H^2_p(X))$-representation $H^2_p(X)$ instead of W to carry out the same argument as above to conclude that the monodromy operator $N^+_p \in \prod_i \mathrm{End}_{K_0^{\mathrm{ur}}}(H^{2i}_{\mathrm{pst}}(X))$ of the even part is contained in (the image of) $\overline{\mathfrak{g}}_{\mathrm{pst}}(X)$. In particular $N^+_p = 0$ if $b_2(X) = 3$ since in this case we have $N_{p,2} = 0$ by Lemma 5.5. This proves (2). □

We conclude this section with an application of our results. For a potentially semistable G_K-representation V, we say that $D_{\mathrm{pst}}(V)$ satisfies *Griffiths transversality* if $N(F^\bullet) \subset F^{\bullet-1}$ for its monodromy operator N and Hodge filtration $F^\bullet$ (see [Pau04, Definition 1.3.1] for example). A typical example of $D_{\mathrm{pst}}(V)$ which satisfies Griffiths transversality is $D_{\mathrm{pst}}(H^1_{\mathrm{ét}}(A_{\overline{K}}, \mathbb{Q}_p))$ for an abelian variety A over K (in this case the transversality follows since $F^0 = D_{\mathrm{pst}}(H^1_{\mathrm{ét}}(A_{\overline{K}}, \mathbb{Q}_p))$ and $F^2 = 0$). This, together with the results obtained above, allows us to prove the following.

Corollary 6.4. *Let X be a hyper-Kähler variety over K with $b_2(X) \geq 4$. Then $H^i_{\mathrm{pst}}(X) = D_{\mathrm{pst}}(H^i_{\mathrm{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ satisfies Griffiths transversality for all i.*

Proof. We may assume k is algebraically closed. Replacing K by its finite extension, we can take an embedding

$$\widetilde{\iota}_{pst}\colon \bigoplus_i H^i_{\mathrm{pst}}(X) \hookrightarrow \bigoplus_{j\in J} H^1_{\mathrm{pst}}(\mathrm{KS}(X, \mathscr{L})^{\times 2})^{\otimes m_j} \otimes H^1_{\mathrm{pst}}(\mathrm{KS}(X, \mathscr{L})^{\times 2})^{\vee,\otimes m'_j}$$

as in (6.2). We recall that this filtered embedding is strict; see [Fon94b, Théorème 5.3.5]. Therefore, the assertion follows from the Griffiths transversality of $H^1_{\mathrm{pst}}(\mathrm{KS}(X, \mathscr{L})^{\times 2})$. □

7. Arithmetic analogue of Nagai's conjecture

In this section, we will prove our main results on the ℓ-adic and p-adic analogues of Nagai's conjecture (Conjecture 1.3 and Conjecture 1.4).

7.1. **A criterion of Green–Kim–Laza–Robles.** In [Gre+22, Theorem 5.2], it is shown that if X is a complex hyper-Kähler variety of dimension $2n$ with $b_2(X) \geq 5$ such that the highest weight of any irreducible factor $V_{\mu,\mathbb{C}}$ in the LLV decomposition (Definition 3.12) appearing in the even part $H_B^+(X)_\mathbb{C} = \bigoplus_i H_B^{2i}(X)_\mathbb{C}$ satisfies the inequality

$$\mu_0 + \mu_1 + \mu_2 \leq n,$$

then the Nagai conjecture holds for Type II degenerations of X. In the following, we establish an analogue of their result in the arithmetic setting.

Let X be a hyper-Kähler variety over K and ℓ a prime number (including $\ell = p$). Following Set-up 3.7, we write

$$H^\bullet(X) \coloneqq H_\ell^\bullet(X) \quad \text{and} \quad H^\bullet(X)_F \coloneqq H_\ell^\bullet(X) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \quad \text{if} \quad \ell \neq p$$

and

$$H^\bullet(X) \coloneqq H_{\mathrm{pst}}^\bullet(X) \quad \text{and} \quad H^\bullet(X)_F \coloneqq H_{\mathrm{pst}}^\bullet(X) \otimes_{K_0^{\mathrm{ur}}} \overline{K} \quad \text{if} \quad \ell = p.$$

Refer to Definition 3.12 for the LLV decomposition of X, i.e., the decomposition into irreducible $\mathfrak{g}(X)_F$-modules:

$$H^\bullet(X)_F = \bigoplus_{\mu \in \Lambda^+} V_{\mu,F}^{\oplus m_\mu}.$$

(Here, we have fixed a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}(X)_F$ and a positive system of roots as in Section 3.2. These choices do not affect the conclusions below by Remark 2.8.)

Let $N_{\ell,2} \in \overline{\mathfrak{g}}(X) \simeq \mathfrak{so}(H^2(X))$ be the monodromy operator on $H^2(X)$. Let

$$N'_{\ell,i} \colon H^i(X)_F \to H^i(X)_F$$

be the image of $N_{\ell,2}$ under the projection $\rho_i \colon \overline{\mathfrak{g}}(X)_F \to \mathrm{End}_F(H^i(X))$. See Section 3.2 for the notation used here.

Theorem 7.1. *Let X be a hyper-Kähler variety of dimension $2n$ over K and ℓ a prime number. We assume that X has Type II reduction. Then the following conditions are equivalent.*

(1) We have $\nu(N'_{\ell,2i}) = i$ for all $0 \leq i \leq n$.

(2) For all irreducible factors $V_{\mu,F} \neq 0$ in the LLV decomposition which are contained in the even part $H^+(X)_F = \bigoplus_i H^{2i}(X)_F$, the following inequality holds:

$$\mu_0 + \mu_1 + \mu_2 \leq n.$$

With Proposition 5.6, the same argument as given in [Gre+22] implies the theorem. Here, we will provide a slightly different proof using the branching rule of successive restrictions of representations of orthogonal Lie algebras (cf. [FH91, Section 25.3] and [Gre+22, Appendix B.2]).

Remark 7.2. If X has Type II reduction, then $b_2(X) \geq 5$ by Proposition 5.6. In the rest of this subsection, we will always assume that $b_2(X) \geq 5$.

Recall that $h \in \mathfrak{g}(X)$ is the shifted degree map which acts on $H^i(X)_F$ multiplication by $i - 2n$. In the following lemma, we will use the notation of Remark 3.13.

Lemma 7.3. *We assume that $b_2(X) \geq 5$. Let $r = \lfloor \frac{b_2(X)}{2} \rfloor$. Let $V_{\mu,F}$ be the irreducible $\mathfrak{g}(X)_F$-module corresponding to a dominant integral weight $\mu = (\mu_0, \mu_1, \ldots, \mu_r) \in \Lambda^+$.*

(1) Let $V_{\mu,F}(-2\mu_0)$ be the $(-2\mu_0)$-eigenspace of $h \in \mathfrak{g}(X)$. If $b_2(X)$ is odd, then the $\overline{\mathfrak{g}}(X)_F$-module $V_{\mu,F}(-2\mu_0)$ contains the irreducible $\overline{\mathfrak{g}}(X)_F$-module $\overline{V}_{(\mu_1,\mu_2,\ldots,\mu_r),F}$ with highest weight $(\mu_1, \mu_2, \ldots, \mu_r) \in \overline{\Lambda}^+$. If $b_2(X)$ is even, then $V_{\mu,F}(-2\mu_0)$ contains at least one of $\overline{V}_{(\mu_1,\mu_2,\ldots,\mu_r),F}$ or $\overline{V}_{(\mu_1,\mu_2,\ldots,-\mu_r),F}$.

(2) Assume that $V_{\mu,F}$ is contained in $H^\bullet(X)_F$. For an irreducible $\overline{\mathfrak{g}}(X)_F$-submodule $\overline{V}_{\lambda,F} \subset V_{\mu,F}$ (with $\lambda = (\lambda_1, \ldots, \lambda_r) \in \overline{\Lambda}^+$) which is contained in $H^i(X)_F$ for some $0 \leq i \leq 2n$, we have the following inequality:

$$\mu_0 + \mu_1 + \mu_2 \geq \lambda_1 + \lambda_2 + n - i/2.$$

Proof. This is well-known and follows easily from the branching rule. We provide a direct proof for the convenience of the reader. We may assume that Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}(X)_F$ and $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}(X)_F$ are given as follows. (Note that $\mu_0, \mu_1, \mu_2, \lambda_1, \lambda_2$ are non-negative since $b_2(X) \geq 5$, and will not change if we change Cartan subalgebras and positive systems of roots by Remark 2.8.) Recall $\psi_F \colon \mathfrak{g}(X)_F \simeq \mathfrak{so}(\widetilde{H}^2(X)_F)$ from Theorem 3.10. Here $\widetilde{H}^2(X)_F = H^2(X)_F \oplus Fv \oplus Fw$ is the Mukai completion. If $b_2(X)$ is odd (resp. even), we can choose a basis

$$\{v_0, \ldots, v_r, w_0, \ldots, w_r, v_{r+1}\} \quad (\text{resp. } \{v_0, \ldots, v_r, w_0, \ldots, w_r\})$$

of $\widetilde{H}^2(X)_F$ such that $v_0 = v$, $w_0 = w$ and other vectors v_i, w_i are contained in $H^2(X)_F$, and the associated matrix of the quadratic form is given as in (5.3). By the standard process of constructing Cartan subalgebras in terms of this basis (see [FH91, Lecture 18] or [Mil17, Section 21 (j)] for example), we can choose a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}(X)_F$ such that

- the elements $v_1, \ldots, v_r, w_1, \ldots, w_r$ are weight vectors of the non-trivial $\overline{\mathfrak{h}}$-weights $\epsilon'_1, \ldots, \epsilon'_r, -\epsilon'_1, \ldots, -\epsilon'_r$ of $H^2(X)_F$, respectively, and
- $\mathfrak{h} := Fh \oplus \overline{\mathfrak{h}}$ is a Cartan subalgebra of $\mathfrak{g}(X)_F$ and we can write the non-trivial $\mathfrak{h}$-weights $\{\pm\epsilon_m\}_{0 \leq m \leq r}$ of $\widetilde{H}^2(X)_F$ such that under the identification $\mathfrak{h}^* = (Fh)^* \oplus \overline{\mathfrak{h}}^*$, we have $\pm\epsilon_0 \in (Fh)^*$ with $\epsilon_0(h) = -2$ and $\epsilon_m = \epsilon'_m \in \overline{\mathfrak{h}}^*$ for all $1 \leq m \leq r$.

We then fix positive systems of roots as in Section 2.4.

The highest weight vector $v_\mu \in V_{\mu,F}$ is contained in the $(-2\mu_0)$-eigenspace $V_{\mu,F}(-2\mu_0)$ of $h \in \mathfrak{g}(X)$ and the $\overline{\mathfrak{g}}(X)_F$-submodule of $V_{\mu,F}(-2\mu_0)$ generated by v_μ is an irreducible $\overline{\mathfrak{g}}(X)_F$-module with highest weight $(\mu_1, \ldots, \mu_r) \in \overline{\Lambda}^+$. This proves (1).

Now we prove (2). The shifted degree operator h acts on $\overline{V}_{\lambda,F}$ as multiplication by $i-2n$, since $\overline{V}_{\lambda,F} \subset H^i(X)_F$. Let $\overline{v}_\lambda \in \overline{V}_{\lambda,F}$ be the highest weight vector. Then $\overline{v}_\lambda$, viewed as an element of $V_{\mu,F}$, is a weight vector for the $\mathfrak{h}$-weight

$$(n - i/2)\epsilon_0 + \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_r\epsilon_r.$$

Since $\mu = \mu_0\epsilon_0 + \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_r\epsilon_r$ is the highest weight, we obtain the inequality $\mu_0 + \mu_1 + \mu_2 \geq \lambda_1 + \lambda_2 + n - i/2$. $\square$

Lemma 7.4. *We assume that X has Type II reduction. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \overline{\Lambda}^+$ be a dominant integral weight with $\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq r$ and let $\rho\colon \overline{\mathfrak{g}}(X)_F \to \mathrm{End}_F(\overline{V}_{\lambda,F})$ be the corresponding irreducible $\overline{\mathfrak{g}}(X)_F$-module with highest weight λ. Then we have $\nu(\rho(N_{\ell,2})) = \lambda_1 + \lambda_2$.*

Proof. As X has Type II reduction, by using the normalized basis given in Proposition 5.6 (1), we can prove this lemma by the same argument as in [Gre+22, Lemma 5.8]. □

Proof of Theorem 7.1. (1) $\Rightarrow$ (2): We argue by contradiction and assume that there is some irreducible $\mathfrak{g}(X)_F$-submodule $V_{\mu,F} \subset H^+(X)_F$ such that $\mu_0 + \mu_1 + \mu_2 > n$. Since the $(-2\mu_0)$-eigenspace $V_{\mu,F}(-2\mu_0)$ of h is contained in $H^{2n-2\mu_0}(X)_F$, it follows from Lemma 7.3 that $H^{2n-2\mu_0}(X)_F$ contains the irreducible $\overline{\mathfrak{g}}(X)_F$-module $\overline{V}_{\lambda,F}$ where $\lambda = (\mu_1, \dots, \mu_r)$ if $b_2(X)$ is odd, and $\lambda = (\mu_1, \dots, \mu_r)$ or $\lambda = (\mu_1, \dots, -\mu_r)$ if $b_2(X)$ is even. We note that $\mu_i \in \mathbb{Z}$ by Proposition 3.18. It then follows from Lemma 7.4 that

$$n - \mu_0 = \nu(N'_{\ell,2n-2\mu_0}) \geq \nu(N'_{\ell,2n-2\mu_0}|_{\overline{V}_{\lambda,F}}) = \mu_1 + \mu_2 > n - \mu_0,$$

which is a contradiction.

(2) $\Rightarrow$ (1): We fix an integer $0 \leq i \leq n$. By Corollary 3.15, we have $\nu(N'_{\ell,2i}) \geq i$. In order to show that $\nu(N'_{\ell,2i}) = i$, it suffices to show the inequality $\nu(N'_{\ell,2i}|_{\overline{V}_{\lambda,F}}) \leq i$ for every irreducible $\overline{\mathfrak{g}}(X)_F$-module component $\overline{V}_{\lambda,F} \subset H^{2i}(X)_F$. Let us choose an irreducible $\mathfrak{g}(X)_F$-module component $V_{\mu,F} \subset H^+(X)_F$ which contains $\overline{V}_{\lambda,F}$. Then, by our assumption and Lemma 7.3, we have

$$n \geq \mu_0 + \mu_1 + \mu_2 \geq \lambda_1 + \lambda_2 + n - i.$$

This, together with Lemma 7.4 (and Proposition 3.18), implies $\nu(N'_{\ell,2i}|_{\overline{V}_{\lambda,F}}) = \lambda_1 + \lambda_2 \leq i$ as required. □

7.2. **Arithmetic analogue of Nagai's conjecture.** Now we can prove our main results. Let X be a $2n$-dimensional hyper-Kähler variety over K. Let $N_{\ell,i}$ be the ℓ-adic monodromy operator on $H^i_\ell(X) = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ if $\ell \neq p$ and let $N_{p,i}$ be the p-adic monodromy operator on $H^i_{\mathrm{pst}}(X) = D_{\mathrm{pst}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ if $\ell = p$.

Proof of Theorem 1.5. The proof will be divided into three cases based on the reduction type. We start with the Type III case.

Theorem 7.5 (Type III)**.** *Assume that X has Type III reduction. Then we have $\nu(N_{\ell,2i}) = 2i$ for all $0 \leq i \leq n$ and for all ℓ.*

Proof. This follows from Lemma 2.5, Lemma 2.7, and Corollary 3.16. □

For the Type I case, we provide a slightly expanded version.

Theorem 7.6 (Type I)**.** *Let $1 \leq i \leq 4n-1$ be an integer such that $b_i(X) \neq 0$. If $b_2(X) = 3$, we further assume that i is even.*

(1) Then $N_{p,2} = 0$ if and only if $N_{p,i} = 0$.

(2) Let $\ell \neq p$ be a prime number. Assume that X is in one of four known deformation types ($\star$). Then $N_{\ell,2} = 0$ if and only if $N_{\ell,i} = 0$.

Proof. We have $N_{p,i} = \rho_i(N_{p,2})$ by Theorem 6.3. The assertion (1) follows from this equality, together with the injectivity of ρ_i (see Corollary 3.19). The assertion (2) follows from the same argument using Corollary 4.3. □

Finally, we treat the Type II case.

Theorem 7.7 (Type II)**.** *Assume that X has Type II reduction. We suppose that X is in one of the four known deformation types ($\star$). Then we have $\nu(N_{\ell,2i}) = i$ for all $0 \le i \le n$ and for all prime numbers ℓ.*

Proof. By Corollary 4.3 and Theorem 6.3, the monodromy operator $N_{\ell,2i}$ agrees with $N'_{\ell,2i} = \rho_{2i}(N_{\ell,2})$. Thus, the result now follows from Theorem 3.17 and Theorem 7.1. □

Now, Theorem 1.5 can be concluded from Theorem 7.5, Theorem 7.6, and Theorem 7.7. Furthermore, in the $\ell = p$ case, even if X is not known to be in one of the four deformation types ($\star$), we still have the following consequence.

Theorem 7.8. *Assume that X has Type II reduction. Then the equality $\nu(N_{p,2i}) = i$ holds for all $0 \le i \le n$ if and only if the inequality $\mu_0 + \mu_1 + \mu_2 \le n$ holds for every irreducible factor $V_{(\mu_0,\mu_1,\dots,\mu_r),\overline{K}} \neq 0$ in the LLV decomposition of $H^+_{\mathrm{pst}}(X)_{\overline{K}}$.*

Proof. By Theorem 6.3, the p-adic monodromy operator $N_{p,2i}$ agrees with $N'_{p,2i} = \rho_{2i}(N_{p,2})$. So this is a consequence of Theorem 7.1. □

7.3. **Nilpotency indices in odd degrees.** In this part, we want to make further remarks on the nilpotency indices of the monodromy operators on cohomology in odd degrees and their relation to those on the second cohomology. In the four known deformation types ($\star$), $\mathrm{Kum}_n (n \ge 2)$ is the sole example that has non-trivial odd cohomology groups (see Remark 2.1).

In [Sol20, Proposition 3.15] and [HM22, Proposition 3.12], it is shown that, for a Type III projective one-parameter degeneration $\mathcal{X}/\Delta$ of a hyper-Kähler manifold $\mathcal{X}_0$ of dimension $2n$ with $b_3(\mathcal{X}_0) \neq 0$, we have $\nu(N_{2i+1}) = 2i - 1$ for the monodromy operator N_{2i+1} on $H^{2i+1}_B(\mathcal{X}_0)$ for any $1 \le i \le n-1$.

Let X be a hyper-Kähler variety over K of dimension $2n$. Similarly, we obtain the following description of monodromy nilpotency indices for $H^{2i+1}_{\mathrm{pst}}(X)$:

Theorem 7.9. *Suppose that $b_2(X) \ge 4$.*

(1) We have $\nu(N_{p,2i+1}) \le 2i - 1$ for all $1 \le i \le n-1$.

(2) If X has Type III reduction and $b_3(X) \neq 0$, then $\nu(N_{p,2i+1}) = 2i-1$ for all $1 \le i \le n-1$.

Proof. Since $H^{2i+1}(X, \mathcal{O}_X) = H^0(X, \Omega^{2i+1}_X) = 0$ (cf. [Bea83, Proposition 3]), (1) follows from Lemma 2.7. Note that we have the normalized basis for $H^2_{\mathrm{pst}}(X) \otimes_{K_0^{\mathrm{ur}}} \overline{K}$ as in Proposition 5.6. Then the claim (2) follows from the same computation as in [Sol20, Proposition 3.15]. □

Corollary 7.10. *Suppose that $b_2(X) \geq 4$ and $b_3(X) \neq 0$. We have $\nu(N_{p,3}) = 0$ if X is of type I, and $\nu(N_{p,3}) = 1$ if X is of type II or III.*

Proof. This follows by combining Theorem 7.9 and Theorem 7.6. □

We can prove the ℓ-adic analogue of Corollary 7.10 for generalized Kummer types.

Proposition 7.11. *Assume that X is of generalized Kummer type and $\ell \neq p$. If X is of type I, then $\nu(N_{\ell,3}) = 0$. If X is of type II or type III, then $\nu(N_{\ell,3}) = 1$.*

Proof. After replacing K by a finite extension, we may assume that we are in the situation of Proposition 4.7. So there exists a nilpotent operator $N \in \overline{\mathfrak{g}}_\ell(X)$ which acts on both $H^3_\ell(X)$ and $W := H^1_\ell(\mathrm{KS}(X, \mathscr{L})^{\times 2})$ as monodromy operators. It follows from [Sol20, Lemma 3.11] that $H^3_\ell(X)_{\overline{\mathbb{Q}}_\ell}$ is a direct sum of the spin representation $\overline{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2}),\overline{\mathbb{Q}}_\ell}$ as a $\overline{\mathfrak{g}}_\ell(X)$-module. (In fact, we know that $H^3_\ell(X)_{\overline{\mathbb{Q}}_\ell} = \overline{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2}),\overline{\mathbb{Q}}_\ell}$ since $b_3(X) = 8$.) Thus the desired description follows from Theorem 5.2 and the fact that $W_{\overline{\mathbb{Q}}_\ell}$ is a direct sum of $\overline{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2}),\overline{\mathbb{Q}}_\ell}$ as an $\mathfrak{so}(H^2_\ell(X))_{\overline{\mathbb{Q}}_\ell}$-module. □

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