

SEMISIMPLICITY, PURITY AND MUMFORD–TATE CONJECTURE FOR HYPER-KÄHLER VARIETIES

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ABSTRACT. We prove the Mumford–Tate conjecture in every degree for hyper-Kähler varieties over fields finitely generated over $\mathbb{Q}$. The proof establishes semisimplicity of ℓ -adic cohomology by eliminating the unipotent radical of the algebraic monodromy group of total cohomology. We also prove the weight–monodromy conjecture for hyper-Kähler varieties over p -adic fields. For hyper-Kähler varieties over number fields with $b_2 \geq 4$, we establish, after suitable finite extensions, strong compatibility of the associated Weil–Deligne representations valued in Mumford–Tate groups and its integral refinement away from finitely many primes.

1. INTRODUCTION

1.1. The Mumford–Tate conjecture and semisimplicity. The Mumford–Tate conjecture relates the ℓ -adic Galois representations attached to a smooth projective variety in characteristic zero to the Hodge structures on its Betti cohomology. More precisely, let X be a smooth projective variety over a field F finitely generated over $\mathbb{Q}$. Let $\overline{F}$ be an algebraic closure of F and fix an embedding $\overline{F} \hookrightarrow \mathbb{C}$. For a prime ℓ , we write $G_{\ell,i}(X)$ for the algebraic monodromy group, that is, the Zariski closure of the image of the representation

$$G_F = \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(H_{\mathrm{\acute{e}t}}^i(X_{\overline{F}}, \mathbb{Q}_{\ell})),$$

and $\mathrm{MT}_i(X_{\mathbb{C}})$ for the Mumford–Tate group of $H_B^i(X_{\mathbb{C}}, \mathbb{Q})$. The Mumford–Tate conjecture predicts that, under the étale–Betti comparison,

$$G_{\ell,i}(X)^{\circ} = \mathrm{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}}$$

for every i and ℓ . This conjecture was originally formulated for abelian varieties and remains open in general, even in that case; see [Moo17] and the references therein for an overview.

In this paper, we prove the Mumford–Tate conjecture in every degree for hyper-Kähler varieties. Here a *hyper-Kähler variety* over a field K of characteristic zero means a smooth projective geometrically simply connected variety X with $H^0(X, \Omega_{X/K}^2) = K\sigma$ for a nowhere-degenerate 2-form σ . Hyper-Kähler varieties are higher-dimensional analogues of K3 surfaces and occur among the factors in the Beauville–Bogomolov decomposition [Bog74; Bea83]. For K3 surfaces, the Mumford–Tate conjecture was proved by Tankeev [Tan91]. André [And96] proved it in degree two for hyper-Kähler varieties with $b_2(X) \geq 4$. In all degrees, the conjecture was known for the four standard deformation types, $K3^{[n]}$, generalized Kummer, OG6, and OG10, by work of Floccari, Soldatenkov, and Floccari–Fu–Zhang [Flo22; Sol22; FFZ21].

Tang and the second author proved the Mumford–Tate conjecture for arbitrary hyper-Kähler varieties after semisimplifying the Galois representations [TZ26]. The remaining problem is therefore to establish semisimplicity, or equivalently to prove that the unipotent radical of the algebraic monodromy group is trivial, which is predicted by the *semisimplicity conjecture* of Grothendieck and Serre [Gro69; Ser94].

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We prove a general restriction on the possible unipotent radicals of algebraic monodromy groups arising from geometry: the connected algebraic monodromy group of a smooth proper variety over F has no nontrivial unipotent algebraic quotient ([Theorem 2.3](#)). For a hyper-Kähler variety X over F , write $G_\ell(X)$ and $\mathrm{MT}(X_{\mathbb{C}})$ for the algebraic monodromy and Mumford–Tate groups of total cohomology, respectively, and $R_{u,\ell}$ for the unipotent radical of the identity component $G_\ell(X)^\circ$. The structural results of [\[TZ26\]](#) identify $\mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell}$ with a Levi subgroup centralizing $R_{u,\ell}$, giving

$$G_\ell(X)^\circ = \mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_\ell} \times R_{u,\ell}.$$

The unipotent radical is therefore a quotient of $G_\ell(X)^\circ$ and must be trivial by the preceding result. This proves semisimplicity of the ℓ -adic cohomology, thereby completing the proof of the Mumford–Tate conjecture for hyper-Kähler varieties. More precisely, we prove the following theorem.

Theorem A ([Theorem 2.5](#)). *Let X be a hyper-Kähler variety over a field F finitely generated over $\mathbb{Q}$. Then $H_{\mathrm{ét}}^i(X_{\overline{F}}, \mathbb{Q}_\ell)$ is a semisimple G_F -representation for every i and every prime ℓ , and the Mumford–Tate conjecture holds for X in every degree.*

The weight–monodromy conjecture in degree i asserts that the associated Weil–Deligne representation is pure of weight i (see [Section 3.3](#)). Using [Theorem A](#), we show that, after a finite extension of K , the tensor square $H_{\mathrm{ét}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)^{\otimes 2}$ is a direct summand of a finite sum of tensor products of $H_{\mathrm{ét}}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ and its dual. The known degree-two case (see [Lemma 3.8](#)) then implies that the Weil–Deligne representation associated with this tensor square is pure of weight $2i$. Since purity can be detected on tensor squares, the conjecture follows in every degree.

Theorem B ([Theorem 4.4](#)). *Let K be a finite extension of $\mathbb{Q}_p$, and let X be a hyper-Kähler variety over K . The weight–monodromy conjecture holds for $H_{\mathrm{ét}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ for every i and every prime ℓ , including $\ell = p$.*

1.2. Strong compatibility. Our next aim is to compare the local Weil–Deligne representations valued in Mumford–Tate groups as ℓ varies. Let X be a hyper-Kähler variety over a number field F with $b_2(X) \geq 4$. Fix an embedding $\overline{F} \hookrightarrow \mathbb{C}$, and put $\mathbf{G} = \mathrm{MT}(X_{\mathbb{C}})$. After replacing F by a finite extension, the Mumford–Tate conjecture implies that the Galois actions on total cohomology factor through representations

$$\phi_\ell^{\mathrm{MT}}: G_F \longrightarrow \mathbf{G}(\mathbb{Q}_\ell)$$

for every ℓ . For a finite place $v \mid p$ of F , the restriction of ϕ_ℓ^{MT} to G_{F_v} gives rise to a $\mathbf{G}$ -valued Weil–Deligne representation $\rho_{\ell,v}^{\mathrm{MT}}$ of F_v over $\overline{\mathbb{Q}_\ell}$ for every prime ℓ , including $\ell = p$. We refer to [Section 3](#) for details on the notion of $\mathbf{G}$ -valued Weil–Deligne representations. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ for each ℓ . We prove the following strong compatibility result.

Theorem C ([Theorem 5.8](#)). *After replacing F by a further finite extension, the family $\{\rho_{\ell,v}^{\mathrm{MT}}\}_\ell$ is rationally strongly compatible for every finite place v of F . More precisely, there exists a $\mathbf{G}$ -valued Weil–Deligne representation ρ_v of F_v over $\overline{\mathbb{Q}}$ whose $\mathbf{G}(\overline{\mathbb{Q}})$ -conjugacy class is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and whose base change to $\overline{\mathbb{Q}_\ell}$ is equivalent to $\rho_{\ell,v}^{\mathrm{MT}}$ for every ℓ , including $\ell = p$.*

There has recently been substantial progress on the corresponding statement for abelian varieties. Noot obtained partial results at places of semistable reduction [\[Noo13\]](#). Kisin–Zhou proved that Frobenius conjugacy classes in the Mumford–Tate group are independent of ℓ at places of good reduction with odd residue characteristic [\[KZ25a\]](#). They subsequently established strong compatibility for the Weil–Deligne representations valued in Mumford–Tate groups at places of semistable reduction, including $\ell = p$ [\[KZ25b, Theorem 1.2\]](#).

Using a Kuga–Satake abelian variety and Looijenga–Lunts–Verbitsky (LLV) theory, we deduce our theorem from their result. We refer to [Section 5](#) for the proof and a more precise description of the required finite extension of F .

Strong compatibility refines the classical independence of ℓ of Frobenius characteristic polynomials at places of good reduction [[Del74](#); [Ser94](#)]. The theorem implies that, after replacing F by a finite extension, the characteristic polynomials of Frobenius on the associated Weil–Deligne representations in every degree are independent of ℓ at every finite place, including places of bad reduction; see [Corollary 5.10](#) for a purely local result.

We further prove an integral refinement away from finitely many primes ℓ . Keeping the notation above, we summarize it as follows; the precise formulation is given in [Section 5.3](#). Let $\overline{\mathbb{Z}} \subset \overline{\mathbb{Q}}$ and $\overline{\mathbb{Z}}_\ell \subset \overline{\mathbb{Q}}_\ell$ be the integral closures of $\mathbb{Z}$ and $\mathbb{Z}_\ell$, respectively.

Theorem D ([Theorem 5.17](#)). *We keep the notation of [Theorem C](#) and fix a finite place $v \mid p$ of F . There exist a positive integer D divisible by p and a reductive model $\mathcal{G}$ of $\mathbf{G}$ over $\mathbb{Z}[1/D]$ such that ρ_v extends to a $\mathcal{G}$ -valued Weil–Deligne representation ρ_v^{int} of F_v over $\overline{\mathbb{Z}}[1/D]$ and, for every prime $\ell \nmid D$, its base change $(\rho_v^{\text{int}})_{\overline{\mathbb{Z}}_\ell}$ to $\overline{\mathbb{Z}}_\ell$ is equivalent to the $\mathcal{G}$ -valued Weil–Deligne representation $\rho_{\ell,v}^{\text{MT,int}}$ of F_v over $\overline{\mathbb{Z}}_\ell$ associated with the Galois action on $H_{\text{ét}}^\bullet(X_{\overline{F}_v}, \mathbb{Z}_\ell)$.*

We also prove an analogous integral refinement for abelian varieties with semistable reduction at v . The key input for the integral refinement is torsion-freeness of cokernels of all powers of the monodromy operator on $H_{\text{ét}}^\bullet(X_{\overline{F}_v}, \mathbb{Z}_\ell)$ for all but finitely many $\ell \neq p$. This property was studied in [[Ito21](#)] in connection with a torsion analogue of the weight–monodromy conjecture. As in the proof of the weight–monodromy conjecture, we reduce to the degree-two case to establish this property for hyper-Kähler varieties in every degree ([Proposition 5.14](#)).

Outline. [Section 2](#) shows that the connected algebraic monodromy group of a smooth proper variety over a field finitely generated over $\mathbb{Q}$ has no nontrivial unipotent algebraic quotient ([Theorem 2.3](#)) and deduces the Mumford–Tate conjecture and semisimplicity ([Theorem 2.5](#)) for hyper-Kähler varieties from this result. In [Section 3](#), we collect what we need about $\mathbf{G}$ -valued Weil–Deligne representations. We also introduce the notion of purity for $\mathbf{G}$ -valued Weil–Deligne representations with integral coefficients, which plays an important role in the proof of [Theorem D](#). In [Section 4](#), we prove the weight–monodromy conjecture ([Theorem 4.4](#)). Finally, in [Section 5](#), we prove rational strong compatibility ([Theorem 5.8](#)) and its integral refinement ([Theorem 5.17](#)).

Notation and conventions. For a field F , we write $\overline{F}$ for a separable closure and $G_F = \text{Gal}(\overline{F}/F)$ for the absolute Galois group. We denote base change by a subscript. For example, for a homomorphism of rings $R \rightarrow S$, we write $X_S := X \times_{\text{Spec } R} \text{Spec } S$ for an R -scheme X and $M_S := M \otimes_R S$ for an R -module M . We use the same convention for group schemes, representations, etc.

2. MUMFORD–TATE CONJECTURE FOR HYPER-KÄHLER VARIETIES

2.1. Unipotent quotients of algebraic monodromy groups. Let F be a number field. For a finite place v of F , let F_v be the completion of F at v . For $v \mid \ell$, let $\mathbb{C}_\ell$ be the completion of $\overline{F}_v$. Let $\chi_\ell: G_{F_v} \rightarrow \mathbb{Z}_\ell^\times$ be the cyclotomic character, describing the action on ℓ -power roots of unity, and write $\mathbb{C}_\ell(r)$ for $\mathbb{C}_\ell$ with action $g \cdot z = \chi_\ell(g)^r g(z)$. A finite-dimensional continuous G_{F_v} -representation W over $\mathbb{Q}_\ell$ is *Hodge–Tate* if there is a G_{F_v} -equivariant $\mathbb{C}_\ell$ -linear isomorphism

$$\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} W \simeq \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_\ell(-j)^{\oplus m_j},$$

with the diagonal action on the left, where the m_j are nonnegative integers, almost all zero. This is an ℓ -adic analogue of the Hodge decomposition. The integers j with $m_j > 0$, counted with multiplicity m_j , are the *Hodge–Tate weights* of W . In this convention, the cyclotomic

representation $\mathbb{Q}_\ell(1)$ has Hodge–Tate weight -1 . The multiplicity of each Hodge–Tate weight is additive in short exact sequences of Hodge–Tate representations. A finite-dimensional G_F -representation over $\mathbb{Q}_\ell$ is Hodge–Tate at v if its restriction to G_{F_v} has this property.

Let $c: G_F \rightarrow \mathbb{Q}_\ell$ be a continuous additive character of G_F . Denote by V_c the two-dimensional representation $g \mapsto \begin{pmatrix} 1 & c(g) \\ 0 & 1 \end{pmatrix}$. We write I_v for the inertia subgroup of G_F at v .

Lemma 2.1. *If V_c is Hodge–Tate at every place $v \mid \ell$, then $c = 0$.*

Proof. Let $v \mid \ell$. The representation $V_c|_{G_{F_v}}$ is an extension of the trivial representation $\mathbb{Q}_\ell$ by itself. Since it is Hodge–Tate by hypothesis, additivity of the multiplicities of Hodge–Tate weights in short exact sequences shows that all its weights are zero. Applying Sen’s theorem [Sen73, Corollary 1] over the completion of the maximal unramified extension of F_v shows that I_v has finite image on V_c . Thus $c(I_v)$ is finite and hence zero, since the additive group of $\mathbb{Q}_\ell$ is torsion-free.

Now let $v \nmid \ell$, and let p be the residue characteristic of F_v . Since c is additive, its restriction to G_{F_v} factors through the abelianization. Local class field theory therefore identifies $c(I_v)$ with the image of a continuous homomorphism $\mathcal{O}_{F_v}^\times \rightarrow \mathbb{Q}_\ell$. The group $\mathcal{O}_{F_v}^\times$ has an open pro- p subgroup, whereas every compact subgroup of $\mathbb{Q}_\ell$ is pro- ℓ . Since $p \neq \ell$, this open subgroup has trivial image, so $c(I_v)$ is again finite and hence zero.

We have shown that c is unramified at every finite place of F . Since the target of c is abelian, global class field theory makes c factor through the narrow class group of F , which is finite. As $\mathbb{Q}_\ell$ has no nonzero finite subgroup, we conclude that $c = 0$. $\square$

Proposition 2.2. *Let $\rho: G_F \rightarrow \mathrm{GL}(V)$ be a continuous finite-dimensional representation over $\mathbb{Q}_\ell$. Suppose that V is Hodge–Tate at every place $v \mid \ell$. If G is the algebraic monodromy group of V , then G° has no nontrivial unipotent algebraic quotient.*

Proof. By replacing F by a finite extension, we may assume that $G = G^\circ$. Let $c: G \rightarrow \mathbb{G}_a$ be an algebraic homomorphism and consider the two-dimensional representation V_c of G as above. By [Mil17, Theorem 4.14], V_c is a subquotient of a finite direct sum of tensor products of V and its dual $V^\vee$, and hence V_c is Hodge–Tate at every place $v \mid \ell$. It then follows from Lemma 2.1 that c is trivial. A nontrivial connected unipotent quotient of G would admit $\mathbb{G}_a$ as a quotient [Mil17, Propositions 14.21 and 14.22], contradicting what we have just proved. $\square$

Theorem 2.3. *Let F be a finitely generated field over $\mathbb{Q}$ and let X be a smooth proper variety over F . Let $G_\ell(X)$ be the algebraic monodromy group of the representation*

$$\rho_\ell: G_F \longrightarrow \mathrm{GL}(H_{\text{ét}}^\bullet(X_{\overline{F}}, \mathbb{Q}_\ell)).$$

Then $G_\ell(X)^\circ$ has no nontrivial unipotent algebraic quotient.

Proof. By the standard specialization argument using Hilbert irreducibility [Ser97, Section 10.6] (see also [Cad13, Section 5]), it suffices to treat the case where F is a number field. In this case, $H_{\text{ét}}^\bullet(X_{\overline{F}}, \mathbb{Q}_\ell)$ is Hodge–Tate at every place above ℓ by [Fal88; Tsu99]. The assertion therefore follows from Proposition 2.2. $\square$

2.2. The Mumford–Tate conjecture and semisimplicity. Let X be a hyper-Kähler variety over a field F finitely generated over $\mathbb{Q}$, and fix an embedding $\overline{F} \hookrightarrow \mathbb{C}$.

Write $\mathrm{MT}(X_{\mathbb{C}})$ for the Mumford–Tate group of total Betti cohomology and $\mathrm{MT}_i(X_{\mathbb{C}})$ for that of $H_B^i(X_{\mathbb{C}}, \mathbb{Q})$. For any algebraic group acting on total Betti cohomology and preserving the

grading, write p_i for its degree- i projection; we use the same notation on étale cohomology via the étale–Betti comparison. In particular, $p_i(\mathrm{MT}(X_{\mathbb{C}})) = \mathrm{MT}_i(X_{\mathbb{C}})$.

For each prime ℓ , write $G_{\ell}(X)$ for the algebraic monodromy group of $H_{\mathrm{\acute{e}t}}^{\bullet}(X_{\overline{F}}, \mathbb{Q}_{\ell})$ and $R_{u,\ell}$ for the unipotent radical of $G_{\ell}(X)^{\circ}$. Let $G_{\ell,i}(X) = p_i(G_{\ell}(X))$ be the algebraic monodromy group in degree i . The Mumford–Tate conjecture $(\mathrm{MTC})_i$ asserts that

$$G_{\ell,i}(X)^{\circ} = \mathrm{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}}$$

for every prime ℓ . We recall the main result of [TZ26] in the following form.

Theorem 2.4 ([TZ26]). *Under the comparison isomorphism*

$$H_{\mathrm{\acute{e}t}}^{\bullet}(X_{\overline{F}}, \mathbb{Q}_{\ell}) \simeq H_B^{\bullet}(X_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell},$$

we have $\mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}} \subset G_{\ell}(X)^{\circ}$. Moreover, $R_{u,\ell}$ commutes with $\mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}}$, and multiplication induces a canonical decomposition

$$G_{\ell}(X)^{\circ} = \mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}} \times R_{u,\ell}. \quad (2.1)$$

Proof. See [TZ26, Lemma 3.3.2 and Proposition 5.2.2]. $\square$

The direct-product decomposition makes $R_{u,\ell}$ a quotient of the connected monodromy group. The preceding subsection therefore gives the following theorem.

Theorem 2.5. *For every prime ℓ , we have*

$$G_{\ell}(X)^{\circ} = \mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}} \quad (2.2)$$

under the étale–Betti comparison. Moreover, for each $0 \leq i \leq 2 \dim X$, $H_{\mathrm{\acute{e}t}}^i(X_{\overline{F}}, \mathbb{Q}_{\ell})$ is a semisimple G_F -representation and the Mumford–Tate conjecture $(\mathrm{MTC})_i$ holds for X .

Proof. Projection onto the second factor in (2.1) makes $R_{u,\ell}$ a unipotent algebraic quotient of $G_{\ell}(X)^{\circ}$. Thus $R_{u,\ell} = 1$ by Theorem 2.3, which proves (2.2). For every i , the degree- i projection satisfies $p_i(G_{\ell}(X)^{\circ}) = G_{\ell,i}(X)^{\circ}$. Applying p_i to (2.2) therefore gives

$$G_{\ell,i}(X)^{\circ} = p_i(\mathrm{MT}(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}}) = \mathrm{MT}_i(X_{\mathbb{C}})_{\mathbb{Q}_{\ell}},$$

which proves $(\mathrm{MTC})_i$ in every degree. Reductivity of Mumford–Tate groups then implies semisimplicity by [Mil17, Corollary 22.43]. $\square$

3. $\mathbf{G}$ -VALUED WEIL–DELIGNE REPRESENTATIONS

3.1. Review of $\mathbf{G}$ -valued Weil–Deligne representations. Let K be a complete discretely valued field of characteristic zero with residue field $\mathbb{F}_q$, where q is a power of a prime number p . Let I_K be the inertia subgroup of G_K . Let W_K be the Weil group of K . By definition, we have a short exact sequence

$$1 \longrightarrow I_K \longrightarrow W_K \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 1,$$

where α sends a lift Frob_q of the geometric Frobenius element to -1 .

Let R be a $\mathbb{Z}[1/q]$ -algebra and let $\mathbf{G}$ be a reductive group scheme over $\mathrm{Spec} R$. Let $\mathrm{Lie}(\mathbf{G})$ be the Lie algebra over R of $\mathbf{G}$. Let Ω be an R -algebra. In this paper, by a $\mathbf{G}$ -valued Weil–Deligne representation (of K) over Ω , we mean a pair $\rho = (r, N)$ consisting of a continuous homomorphism $r: W_K \rightarrow \mathbf{G}(\Omega)$ (where $\mathbf{G}(\Omega)$ is endowed with the discrete topology) and an element $N \in \mathrm{Lie}(\mathbf{G})_{\Omega}$, which we call the monodromy operator, such that $\mathrm{Ad}(r(w))(N) = q^{\alpha(w)} N$ for every $w \in W_K$. If ρ takes values in $\mathbf{G} = \mathrm{GL}_n$ for some nonnegative integer n , we simply call ρ a Weil–Deligne representation. Two $\mathbf{G}$ -valued Weil–Deligne representations $\rho_1 = (r_1, N_1)$ and $\rho_2 = (r_2, N_2)$ over Ω are said to be equivalent if they are conjugate by an element of $\mathbf{G}(\Omega)$, that is, if there exists

$g \in \mathbf{G}(\Omega)$ such that $r_2(w) = \text{Ad}(g)(r_1(w))$ for every $w \in W_K$ and $N_2 = \text{Ad}(g)(N_1)$. In this case, we write $\rho_1 \sim_{\mathbf{G}(\Omega)} \rho_2$.

For a homomorphism $\eta: \mathbf{G} \rightarrow \mathbf{G}'$ of reductive group schemes over R , we denote by $\eta \circ \rho$ the induced $\mathbf{G}'$ -valued Weil–Deligne representation $(\eta \circ r, d\eta(N))$, where $d\eta: \text{Lie}(\mathbf{G}) \rightarrow \text{Lie}(\mathbf{G}')$ is the homomorphism induced by η on Lie algebras. For a homomorphism $\iota: \Omega \rightarrow \Omega'$ of R -algebras, we denote by $\rho \otimes_{\Omega} \Omega'$ or $\rho_{\Omega'}$ the base change of ρ along ι .

If Ω is a field of characteristic zero, we say that ρ is *Frobenius semisimple* if $r(\text{Frob}_q) \in \mathbf{G}(\Omega)$ is semisimple for some (and hence any) geometric Frobenius lift Frob_q . For a finite extension L/K , the restriction $\rho|_L := (r|_{W_L}, N)$ is Frobenius semisimple if and only if ρ is Frobenius semisimple.

3.2. $\mathbf{G}$ -valued Weil–Deligne representations associated with Galois representations.

Let ℓ be a prime number and let $\mathbf{G}$ be a reductive group over $\mathbb{Q}_{\ell}$. A $\mathbf{G}$ -valued Weil–Deligne representation can be attached to a $\mathbf{G}$ -valued representation of G_K (which is potentially semistable when $\ell = p$). Here we recall the construction.

Example 3.1. Assume that $\ell \neq p$. Let $\phi: G_K \rightarrow \mathbf{G}(\mathbb{Q}_{\ell})$ be a continuous homomorphism. We fix an isomorphism $\mathbb{Z}_{\ell}(1) \simeq \mathbb{Z}_{\ell}$ (as $\mathbb{Z}_{\ell}$ -modules). Let $t_{\ell}: I_K \rightarrow \mathbb{Z}_{\ell}(1) \simeq \mathbb{Z}_{\ell}$ be the usual maximal pro- ℓ quotient. By Grothendieck’s monodromy theorem, there exists a unique nilpotent element $N \in \text{Lie}(\mathbf{G})$ such that $\phi(\sigma) = \exp(t_{\ell}(\sigma)N)$ for all elements σ of some open subgroup of I_K . We fix a geometric Frobenius lift $\text{Frob}_q \in W_K$. We define $r: W_K \rightarrow \mathbf{G}(\mathbb{Q}_{\ell})$ by $\text{Frob}_q^n \sigma \mapsto \phi(\text{Frob}_q^n \sigma) \exp(-t_{\ell}(\sigma)N)$. After passing to $\overline{\mathbb{Q}_{\ell}}$, we obtain the $\mathbf{G}$ -valued Weil–Deligne representation $\rho(\phi) := (r, N)$ of K over $\overline{\mathbb{Q}_{\ell}}$, whose equivalence class is independent of the choices of $\mathbb{Z}_{\ell}(1) \simeq \mathbb{Z}_{\ell}$ and Frob_q (see [Del73, Lemma 8.4.3 and Variant 8.11]).

Example 3.2. Assume that $\ell = p$. Let $\phi: G_K \rightarrow \mathbf{G}(\mathbb{Q}_{\ell})$ be a continuous homomorphism. We assume that ϕ is potentially semistable, that is, for every finite-dimensional representation $V \in \text{Rep}_{\mathbb{Q}_p}(\mathbf{G})$, the induced G_K -action on V is potentially semistable¹. Let K_0 be the maximal unramified subfield of K and $K_0^{\text{ur}} \subset \overline{K}$ the maximal unramified extension of K_0 . Every $V \in \text{Rep}_{\mathbb{Q}_p}(\mathbf{G})$ gives rise to the (φ, N) -module

$$D_{\text{pst}}(V) := \varinjlim_L (V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{G_L}$$

over K_0^{ur} , where $L \subset \overline{K}$ runs over all finite Galois extensions of K . We choose an embedding $\tau: K_0^{\text{ur}} \hookrightarrow \overline{\mathbb{Q}_p}$ (over $\mathbb{Q}_p$). The construction $V \mapsto D_{\text{pst}}(V) \otimes_{K_0^{\text{ur}}} \overline{\mathbb{Q}_p}$ induces an exact tensor functor $\omega': \text{Rep}_{\mathbb{Q}_p}(\mathbf{G}) \rightarrow \text{Vect}_{\overline{\mathbb{Q}_p}}$, where $\text{Vect}_{\overline{\mathbb{Q}_p}}$ is the category of finite-dimensional $\overline{\mathbb{Q}_p}$ -vector spaces. Let $\omega: \text{Rep}_{\mathbb{Q}_p}(\mathbf{G}) \rightarrow \text{Vect}_{\overline{\mathbb{Q}_p}}$ be the canonical fiber functor $V \mapsto V_{\overline{\mathbb{Q}_p}}$. By Tannakian formalism, there exists an isomorphism $\omega \simeq \omega'$ of tensor functors, and this induces an isomorphism $\mathbf{G}_{\overline{\mathbb{Q}_p}} \simeq \underline{\text{Aut}}^{\otimes}(\omega')$. For each $V \in \text{Rep}_{\mathbb{Q}_p}(\mathbf{G})$, the Weil group W_K acts on $D_{\text{pst}}(V)$ by

$$\sigma \otimes \sigma \varphi^{-\alpha(\sigma)[\mathbb{F}_q: \mathbb{F}_p]}, \quad \forall \sigma \in W_K.$$

This action and the monodromy operators on the (φ, N) -modules are compatible with tensor operations in V . Hence, via the chosen tensor isomorphism $\omega \simeq \omega'$, they define a continuous homomorphism $r: W_K \rightarrow \mathbf{G}(\overline{\mathbb{Q}_p})$ together with an element $N \in \text{Lie}(\mathbf{G})_{\overline{\mathbb{Q}_p}}$ such that $\rho(\phi) := (r, N)$ is a $\mathbf{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Q}_p}$. The equivalence class of $\rho(\phi)$ is independent of the choices of τ and $\omega \simeq \omega'$.

Remark 3.3. The constructions of $\mathbf{G}$ -valued Weil–Deligne representations $\phi \mapsto \rho(\phi)$ in Examples 3.1 and 3.2 are both functorial in $\mathbf{G}$, up to equivalence.

¹It suffices to check this condition for one faithful representation; see [IM20, Lemma 2.8] for example.

For use in [Section 5.3](#), we record an integral version of [Example 3.1](#).

Example 3.4. Assume that $\ell \neq p$. Let

$$\mathcal{G} \hookrightarrow \prod_i \mathrm{GL}(V_i)$$

be a reductive closed subgroup scheme over $\mathbb{Z}_\ell$, where $\prod_i \mathrm{GL}(V_i)$ is a finite product and each V_i is a finite free $\mathbb{Z}_\ell$ -module of rank n_i . Assume that $\ell > \max_i n_i$. Let $\phi: G_K \rightarrow \mathcal{G}(\mathbb{Z}_\ell)$ be a continuous homomorphism, and let $\overline{\mathbb{Z}_\ell}$ denote the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}_\ell}$. We choose an isomorphism $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ and a geometric Frobenius lift $\mathrm{Frob}_q \in W_K$. Let (r, N) be the $\mathcal{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Q}_\ell}$ associated with the generic fiber of ϕ using our fixed choices $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ and Frob_q . We shall show that $r: W_K \rightarrow \mathcal{G}(\overline{\mathbb{Q}_\ell})$ takes values in $\mathcal{G}(\overline{\mathbb{Z}_\ell})$ and N belongs to $\mathrm{Lie}(\mathcal{G})_{\overline{\mathbb{Z}_\ell}}$.

Choose $\sigma \in I_K$ with $t_\ell(\sigma) = 1$, and write g_i and N_i for the actions of $\phi(\sigma)$ and N on $V_i \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$. The eigenvalues of g_i are roots of unity, and its unipotent part is $\exp(N_i)$. If an eigenvalue has order divisible by ℓ , then the inequality $n_i < \ell$ forces the characteristic polynomial of g_i to be irreducible over $\mathbb{Q}_\ell$. Thus g_i is semisimple and $N_i = 0$. If the semisimple part of g_i has order e_i prime to ℓ , then $g_i^{e_i} = \exp(e_i N_i)$ is unipotent. This, together with $(g_i^{e_i} - 1)^{n_i} = 0$ and $n_i < \ell$, implies that

$$N_i = \frac{1}{e_i} \log(g_i^{e_i}) = \frac{1}{e_i} \sum_{j=1}^{n_i-1} \frac{(-1)^{j+1}}{j} (g_i^{e_i} - 1)^j \in \mathrm{End}_{\overline{\mathbb{Z}_\ell}}(V_i \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}_\ell}).$$

Similarly, $\exp(tN_i)$ is defined over $\overline{\mathbb{Z}_\ell}$ for every $t \in \mathbb{Z}_\ell$. Since $\mathcal{G}$ is closed in $\prod_i \mathrm{GL}(V_i)$, it follows that

$$N \in \mathrm{Lie}(\mathcal{G})_{\overline{\mathbb{Z}_\ell}}, \quad r(W_K) \subset \mathcal{G}(\overline{\mathbb{Z}_\ell}),$$

as desired. The resulting $\mathcal{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Z}_\ell}$ is also denoted by $\rho(\phi) = (r, N)$. In contrast to the rational case, its equivalence class may depend on the choices of $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ and Frob_q . The following proposition shows that it is independent of these choices under an additional hypothesis.

Proposition 3.5. *Keep the notation and assumptions of [Example 3.4](#). Assume that $\ell \nmid (q-1)$ and that there exists a geometric Frobenius lift $\mathrm{Frob}_q \in W_K$ such that, for every i , any two distinct eigenvalues a, b of $\phi(\mathrm{Frob}_q)$ on $V_i \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$ satisfy $a - b \in \overline{\mathbb{Z}_\ell}^\times$. Then the equivalence class of $\rho(\phi)$ over $\overline{\mathbb{Z}_\ell}$ is independent of the choices of $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ and the geometric Frobenius lift.*

Proof. We follow the arguments of [\[Del73, Lemma 8.4.3 and Variant 8.11\]](#). If $N = 0$, then $r = \phi|_{W_K}$, and the assertion is immediate. We henceforth assume that $N \neq 0$.

Since $q-1 \in \mathbb{Z}_\ell^\times$, the same argument as in [\[Del73, Lemma 8.4.3\]](#) shows that, with the identification $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ fixed, changing the geometric Frobenius lift yields an equivalent $\mathcal{G}$ -valued Weil–Deligne representation over $\overline{\mathbb{Z}_\ell}$.

Changing the identification $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ leaves r unchanged and replaces N by aN for some $a \in \mathbb{Z}_\ell^\times$. Let $x = \phi(\mathrm{Frob}_q)$ and let x_s be its semisimple part. By the assumption, the eigenspace decomposition of x_s on each $V_i \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$ is defined over $\overline{\mathbb{Z}_\ell}$. It follows that x_s belongs to $\mathcal{G}(\overline{\mathbb{Z}_\ell})$. Since $r(I_K)$ is finite, there is an integer $m > 0$ such that x_s^m commutes with $r(I_K)$. Let D be the schematic closure of $\langle x_s^m \rangle$ in $\mathcal{G}_{\overline{\mathbb{Z}_\ell}}$. Then D is diagonalizable and commutes with $r(W_K)$. The relation $\mathrm{Ad}(x_s^m)(N) = q^{-m}N$ shows that D acts on N via a character $\psi: D \rightarrow \mathbb{G}_{m, \overline{\mathbb{Z}_\ell}}$. Since $\psi(x_s^m) = q^{-m}$, the order of ψ is infinite. We claim that $\psi: D(\overline{\mathbb{Z}_\ell}) \rightarrow \overline{\mathbb{Z}_\ell}^\times$ is surjective. Indeed, under the identification $D(\overline{\mathbb{Z}_\ell}) = \mathrm{Hom}(X^*(D), \overline{\mathbb{Z}_\ell}^\times)$, this map is the restriction

$$\mathrm{Hom}(X^*(D), \overline{\mathbb{Z}_\ell}^\times) \longrightarrow \mathrm{Hom}(\mathbb{Z}\psi, \overline{\mathbb{Z}_\ell}^\times) \simeq \overline{\mathbb{Z}_\ell}^\times.$$

Since $\overline{\mathbb{Z}}_\ell^\times$ is divisible, every homomorphism on $\mathbb{Z}\psi \subset X^*(D)$ extends to $X^*(D)$. An element $h \in D(\overline{\mathbb{Z}}_\ell)$ with $\psi(h) = a$ therefore conjugates (r, N) to (r, aN) . $\square$

For later use, we record the following lemma.

Lemma 3.6. *Keep the notation of Example 3.4. Let $\rho = (r, N)$ be a $\mathcal{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Z}}_\ell$ whose generic fiber is Frobenius semisimple. Assume that there exists a geometric Frobenius lift $\text{Frob}_q \in W_K$ such that, for every i , any two distinct eigenvalues a, b of $r(\text{Frob}_q)$ on $V_i \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$ satisfy $a - b \in \overline{\mathbb{Z}}_\ell^\times$. Then the scheme-theoretic centralizer $Z_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(r(\text{Frob}_q))$ is smooth over $\overline{\mathbb{Z}}_\ell$. Moreover, $Z_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(r(\text{Frob}_q))$ contains a maximal torus of $\mathcal{G}_{\overline{\mathbb{Z}}_\ell}$.*

Proof. Put $x := r(\text{Frob}_q)$. As in the proof of Proposition 3.5, the assumptions imply that the eigenspace decomposition of x on each $V_i \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$ is defined over $\overline{\mathbb{Z}}_\ell$. Let D be the schematic closure of $\langle x \rangle$ in $\mathcal{G}_{\overline{\mathbb{Z}}_\ell}$. Then D is diagonalizable and

$$Z_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(x) = Z_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(D).$$

Since $Z_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(D)$ is smooth by [Con14, Lemma 2.2.4], the first assertion follows. The second assertion follows from [Con14, Remark 3.1.5 and Theorem 3.2.6]. $\square$

3.3. Purity. Let $\rho = (r, N)$ be a Weil–Deligne representation of K over $\overline{\mathbb{Q}}_\ell$ and let V be its underlying $\overline{\mathbb{Q}}_\ell$ -vector space. We note that the monodromy operator N is nilpotent. Thus it induces the monodromy filtration $\{M_\bullet V\}$ on V ; see [Del80, Proposition (1.6.1)]. Let i be an integer. We say that ρ is *pure of weight i* if the eigenvalues of $r(\text{Frob}_q)$ on $\text{gr}_j^M V := M_j V / M_{j-1} V$ are Weil q^{i+j} -numbers for any geometric Frobenius lift $\text{Frob}_q \in W_K$.

Conjecture 3.7 (Weight–monodromy conjecture). *Let X be a smooth proper scheme over K . Then, for every prime ℓ and every integer i , the Weil–Deligne representation associated with $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is pure of weight i .*

The conjecture is a local analogue of the Weil conjectures: in the good-reduction case, the monodromy operator vanishes, and the assertion reduces to purity for the special fiber. Several cases are known for $\ell \neq p$; see, for example, [Ito21, Theorem 3.2] and the references therein. (We note that the equal-characteristic analogue is already known.) The conjecture is also known for $\ell = p$ in many of these cases; see [BKV25] and the references therein. It remains open in general in dimension at least three. For our argument, we need only the following low-degree case.

Lemma 3.8. *Let X be a smooth proper scheme over K . The weight–monodromy conjecture (Conjecture 3.7) holds for $i \leq 2$.*

Proof. For $\ell \neq p$, see [Sai03, Lemma 3.9]. For $\ell = p$, see [Shi22, Theorem 10.5]. $\square$

Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$. We next extend the notion of purity to $\mathbf{G}$ -valued Weil–Deligne representations. Let $\mu: \mathbb{G}_m \rightarrow \mathbf{G}$ be a cocharacter over $\mathbb{Q}$ whose image is contained in the center $Z(\mathbf{G})$ of $\mathbf{G}$. Let ℓ be a prime number, and let $\rho = (r, N)$ be a $\mathbf{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Q}}_\ell$. We say that ρ is *pure* (with respect to μ) if for every finite-dimensional $\mathbf{G}$ -representation V over $\overline{\mathbb{Q}}_\ell$ such that $\mu(z)$ acts on V as z^i for some integer i , the induced Weil–Deligne representation on V is pure of weight i .

Let $\rho = (r, N)$ be a $\mathbf{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Q}}_\ell$ and assume that ρ is pure. By Tannakian formalism, there exists a unique cocharacter $w: \mathbb{G}_m \rightarrow \mathbf{G}_{\overline{\mathbb{Q}}_\ell}$ with the following property: Let V be a finite-dimensional $\mathbf{G}$ -representation over $\overline{\mathbb{Q}}_\ell$. Since μ is central, we have a decomposition $V = \bigoplus_i V_i$ where V_i is the subspace on which $\mu(z)$ acts as z^i . Let $V_i = \bigoplus_j V_{i,j}$ be

a decomposition into W_K -representations where the eigenvalues of Frob_q acting on $V_{i,j}$ are Weil q^{i+j} -numbers for any geometric Frobenius lift $\text{Frob}_q \in W_K$. Then $w(z)$ acts on $V_{i,j}$ as z^j .

Definition 3.9. The cocharacter $w: \mathbb{G}_m \rightarrow \mathbf{G}_{\overline{\mathbb{Q}}_\ell}$ characterized by the above property is called the *Frobenius-weight cocharacter* of r .

Lemma 3.10. Let $\rho = (r, N)$ be a $\mathbf{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Q}}_\ell$ and assume that ρ is pure. Let $w: \mathbb{G}_m \rightarrow \mathbf{G}_{\overline{\mathbb{Q}}_\ell}$ be the Frobenius-weight cocharacter of r and put $\lambda := w^{-1}$. Then λ is associated to N in the sense of [Jan04, 5.3, Definition].

Proof. We choose a faithful representation V of $\mathbf{G}_{\overline{\mathbb{Q}}_\ell}$. By [Jan04, 5.12, Claim], it suffices to show that λ is associated to N in $\text{GL}(V)$. We write $V = \bigoplus_{j \in \mathbb{Z}} V(\lambda, j)$ for the weight decomposition with respect to λ . We have $N(V(\lambda, j)) \subset V(\lambda, j+2)$ and $N^j: V(\lambda, -j) \xrightarrow{\sim} V(\lambda, j)$ for $j \geq 0$ by purity. By the Jacobson–Morozov theorem, these isomorphisms give an $\mathfrak{sl}_2$ -triple $(N, d\lambda(1), F)$ in $\text{End}_{\overline{\mathbb{Q}}_\ell}(V)$. Hence λ is associated to N in $\text{GL}(V)$ by [Jan04, 5.5, Proposition]. See also [Del80, Proposition (1.6.9)]. $\square$

The following proposition generalizes [TY07, Lemma 1.4(4)] to general $\mathbf{G}$. This also follows from [Gan+24, Lemma 3.5]. Here, we give a proof in terms of associated cocharacters.

Proposition 3.11. Let $\rho_i = (r_i, N_i)$, for $i = 1, 2$, be Frobenius semisimple $\mathbf{G}$ -valued Weil–Deligne representations of K over $\overline{\mathbb{Q}}_\ell$ which are pure with respect to μ . If r_1 and r_2 are $\mathbf{G}(\overline{\mathbb{Q}}_\ell)$ -conjugate, then

$$\rho_1 \sim_{\mathbf{G}(\overline{\mathbb{Q}}_\ell)} \rho_2.$$

Proof. We may assume that $r := r_1 = r_2$, and we let w be the Frobenius-weight cocharacter determined by r . Let $\mathfrak{g} := \text{Lie}(\mathbf{G})_{\overline{\mathbb{Q}}_\ell}$ and let $\mathfrak{g} = \bigoplus_{\tau} \mathfrak{g}_{\tau}$ be the isotypic decomposition of $\mathfrak{g}$ as a W_K -representation, where τ runs over the isomorphism classes of irreducible W_K -representations over $\overline{\mathbb{Q}}_\ell$ and $\mathfrak{g}_{\tau}$ denotes the τ -isotypic component. Put $\lambda := w^{-1}$. Since r commutes with λ , each component $\mathfrak{g}_{\tau}$ is stable under the action of λ . Hence we have the weight decomposition

$$\mathfrak{g}_{\tau} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\tau}(\lambda, n),$$

where $\lambda(z)$ acts on $\mathfrak{g}_{\tau}(\lambda, n)$ as z^n .

Let $P := P_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(\lambda)$ be the parabolic subgroup associated with λ . We note that $\text{Lie}(P) = \bigoplus_{\tau, n \geq 0} \mathfrak{g}_{\tau}(\lambda, n)$. By Lemma 3.10, the cocharacter λ is associated to both N_1 and N_2 . By [Jan04, 5.9, Proposition] (see also [Cot26, Lemma 5.7]), the map $\bigoplus_{\tau, n \geq 0} \mathfrak{g}_{\tau}(\lambda, n) \rightarrow \bigoplus_{\tau, n \geq 2} \mathfrak{g}_{\tau}(\lambda, n)$ defined by $Y \mapsto [Y, N_1]$ is surjective. Since each component $\mathfrak{g}_{\tau}(\lambda, n)$ is mapped to $\mathfrak{g}_{q^{\alpha} \cdot \tau}(\lambda, n+2)$, where $q^{\alpha} \cdot \tau$ is the W_K -representation defined by $g \mapsto q^{\alpha(g)} \cdot \tau(g)$, we see that the induced map $\bigoplus_{n \geq 0} \mathfrak{g}_1(\lambda, n) \rightarrow \bigoplus_{n \geq 2} \mathfrak{g}_{q^{\alpha}}(\lambda, n)$ is surjective, where 1 denotes the trivial W_K -representation.

Let $Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r)$ be the scheme-theoretic centralizer of $r(W_K)$. In our setting, we have

$$Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r) \subset Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(\lambda) \subset P,$$

and thus the equalities $Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r) \cap P = Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r)$ and $\bigoplus_{n \geq 0} \mathfrak{g}_1(\lambda, n) = \text{Lie}(Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r))$ hold. We also note that $Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r)$ is smooth since the characteristic is zero. It follows that the orbit map $Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r) \rightarrow \bigoplus_{n \geq 2} \mathfrak{g}_{q^{\alpha}}(\lambda, n)$ defined by $g \mapsto \text{Ad}(g)(N_1)$ has a dense open image. Similarly, the orbit map $Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r) \rightarrow \bigoplus_{n \geq 2} \mathfrak{g}_{q^{\alpha}}(\lambda, n)$ defined by $g \mapsto \text{Ad}(g)(N_2)$ also has a dense open image. Since these two images are dense open subsets of the irreducible affine space $\bigoplus_{n \geq 2} \mathfrak{g}_{q^{\alpha}}(\lambda, n)$, they intersect. In particular, N_1 and N_2 are conjugate by an element of $Z_{\mathbf{G}_{\overline{\mathbb{Q}}_\ell}}(r)(\overline{\mathbb{Q}}_\ell)$, as desired. $\square$

3.4. Purity with integral coefficients. Assume that $\mathbf{G}$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}[1/D_0]$ for some positive integer D_0 divisible by p , and μ extends to a central cocharacter of this model. Let $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$. Let $\ell \nmid D_0$ be a prime number.

Definition 3.12. Let $\rho = (r, N)$ be a $\mathcal{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Z}}_\ell$. We say that ρ is *pure* (with respect to μ) if the following conditions hold:

- (1) The generic fiber $\rho_{\overline{\mathbb{Q}}_\ell}$ is pure with respect to μ .
- (2) The Frobenius-weight cocharacter $w: \mathbb{G}_m \rightarrow \mathbf{G}_{\overline{\mathbb{Q}}_\ell}$ of r extends over $\overline{\mathbb{Z}}_\ell$.
- (3) The scheme-theoretic centralizer $H := Z_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(r)$ of $r(W_K)$ is smooth over $\overline{\mathbb{Z}}_\ell$.
- (4) Put $\lambda := w^{-1}$ and

$$\mathcal{V} := \{X \in \bigoplus_{n \geq 2} \mathfrak{g}_{\overline{\mathbb{Z}}_\ell}(\lambda, n) \mid \mathrm{Ad}(r(\sigma))(X) = q^{\alpha(\sigma)} X \text{ for every } \sigma \in W_K\},$$

where $\mathfrak{g}_{\overline{\mathbb{Z}}_\ell} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\overline{\mathbb{Z}}_\ell}(\lambda, n)$ is the weight decomposition with respect to λ . Then the map $\mathrm{Lie}(H) \rightarrow \mathcal{V}$ defined by $Y \mapsto [Y, N]$ is surjective.

This definition is motivated by the principle that purity determines the monodromy operator, up to conjugacy, from the underlying representation of W_K . These properties hold automatically on the generic fiber if it is pure and Frobenius semisimple, and they are used in the proof of [Proposition 3.11](#). The following proposition establishes the corresponding uniqueness statement with integral coefficients.

Proposition 3.13. Let $\rho_i = (r_i, N_i)$ for $i = 1, 2$ be $\mathcal{G}$ -valued Weil–Deligne representations of K over $\overline{\mathbb{Z}}_\ell$ which are pure with respect to μ (in the sense of [Definition 3.12](#)) and whose generic fibers are Frobenius semisimple. If r_1 and r_2 are $\mathcal{G}(\overline{\mathbb{Z}}_\ell)$ -conjugate, then

$$\rho_1 \sim_{\mathcal{G}(\overline{\mathbb{Z}}_\ell)} \rho_2.$$

Proof. Our argument is inspired by the proof of [[Cot26](#), Theorem 5.11]. We may assume that $r := r_1 = r_2$. Let $w: \mathbb{G}_m \rightarrow \mathcal{G}_{\overline{\mathbb{Z}}_\ell}$ be the Frobenius-weight cocharacter of r and put $\lambda := w^{-1}$. Let H and $\mathcal{V}$ be as in [Definition 3.12](#). The adjoint action of H preserves $\mathcal{V}$ and $N_1, N_2 \in \mathcal{V}$. We also write $\mathcal{V}$ for the corresponding affine space. For $i = 1, 2$, consider the orbit map $f_i: H \rightarrow \mathcal{V}$, $h \mapsto \mathrm{Ad}(h)(N_i)$. Its differential at the identity is surjective by the assumption. Since H and $\mathcal{V}$ are smooth, this implies that f_i is smooth. In particular, the reduction $(f_i)_{\overline{\mathbb{F}}_\ell}$ has a dense open image for $i = 1, 2$. These two images intersect, and there is an element $\bar{h} \in H(\overline{\mathbb{F}}_\ell)$ such that $\mathrm{Ad}(\bar{h})(N_1 \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell) = N_2 \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$. The fiber $f_1^{-1}(N_2)$ is smooth over $\overline{\mathbb{Z}}_\ell$ and has the $\overline{\mathbb{F}}_\ell$ -point $\bar{h}$. Since $\overline{\mathbb{Z}}_\ell$ is henselian, this point lifts to an element $h \in H(\overline{\mathbb{Z}}_\ell)$ satisfying $\mathrm{Ad}(h)(N_1) = N_2$. As h centralizes r , it conjugates ρ_1 to ρ_2 . $\square$

The following lemma gives a sufficient criterion for integral purity in terms of torsion-freeness of the cokernel of the adjoint action of N , which will be used in [Section 5.3](#).

Lemma 3.14. Let $\rho = (r, N)$ be a $\mathcal{G}$ -valued Weil–Deligne representation of K over $\overline{\mathbb{Z}}_\ell$ whose generic fiber is Frobenius semisimple. Assume that ρ satisfies conditions (1)–(3) of [Definition 3.12](#). Let $\mathfrak{g}_{\overline{\mathbb{Q}}_\ell} = \bigoplus_{\tau} \mathfrak{g}_{\overline{\mathbb{Q}}_\ell, \tau}$ be the isotypic decomposition for the W_K -action, and put $\mathfrak{g}_{\overline{\mathbb{Z}}_\ell, \tau} := \mathfrak{g}_{\overline{\mathbb{Z}}_\ell} \cap \mathfrak{g}_{\overline{\mathbb{Q}}_\ell, \tau}$. Assume that

$$\mathfrak{g}_{\overline{\mathbb{Z}}_\ell} = \bigoplus_{\tau} \mathfrak{g}_{\overline{\mathbb{Z}}_\ell, \tau}$$

and that the cokernel of

$$d: \mathfrak{g}_{\overline{\mathbb{Z}}_\ell} \longrightarrow \mathfrak{g}_{\overline{\mathbb{Z}}_\ell}, \quad Y \mapsto [Y, N]$$

is torsion-free. Then ρ is pure with respect to μ .

Proof. It remains to verify condition (4). Let $\lambda := w^{-1}$ and let $\mathfrak{g}_{\overline{\mathbb{Z}}_\ell, \tau} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\overline{\mathbb{Z}}_\ell, \tau}(\lambda, n)$ be the weight decomposition with respect to λ . As in the proof of [Proposition 3.11](#),

$$d: \bigoplus_{n \geq 0} \mathfrak{g}_{\overline{\mathbb{Z}}_\ell, 1}(\lambda, n) \longrightarrow \bigoplus_{n \geq 2} \mathfrak{g}_{\overline{\mathbb{Z}}_\ell, q^\alpha}(\lambda, n)$$

is surjective after tensoring with $\overline{\mathbb{Q}}_\ell$. Its cokernel is a torsion-free $\overline{\mathbb{Z}}_\ell$ -module by the assumptions, and hence it is zero. As λ is central in H , the source is $\text{Lie}(H)$, and the target is $\mathcal{V}$. This shows that ρ satisfies condition (4). $\square$

4. WEIGHT–MONODROMY CONJECTURE FOR HYPER-KÄHLER VARIETIES

Throughout this section, let K be a finite extension of $\mathbb{Q}_p$ and let X be a hyper-Kähler variety over K . We fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. Let $\mathbf{G} := \text{MT}(X_{\mathbb{C}})$ and $\mathbf{G}_2 := \text{MT}_2(X_{\mathbb{C}})$.

In this section, we prove the weight–monodromy conjecture for X as an application of [Theorem 2.5](#). After replacing K by a finite extension, the Mumford–Tate conjecture enables us to construct the associated $\mathbf{G}$ -valued Weil–Deligne representations. Using these representations and the properties of the projection $\mathbf{G} \rightarrow \mathbf{G}_2$, we deduce the weight–monodromy conjecture in every degree from the degree-two case, which is already known. We also prove Frobenius semisimplicity by a similar reduction to the degree-two case.

4.1. Weil–Deligne representations valued in Mumford–Tate groups. We choose a model of X over a finitely generated subfield $F \subset K$. Let $\overline{F} \subset \overline{K}$ be the algebraic closure of F . Let F^c/F be a finite extension in $\overline{F}$ such that for every ℓ , the algebraic monodromy group associated with the representation $G_{F^c} \rightarrow \text{GL}(H_{\text{ét}}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell))$ is connected. Such an extension exists by a theorem of Serre [\[Ser00\]](#); see also [\[LP92, Proposition \(6.14\)\]](#). Let $K^c := KF^c$. Then [Theorem 2.5](#) implies that the action of G_{K^c} on $H_{\text{ét}}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell)$ factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for every ℓ (including $\ell = p$) under the étale–Betti comparison. We denote this representation by

$$\phi_\ell^{\text{MT}}: G_{K^c} \longrightarrow \mathbf{G}(\mathbb{Q}_\ell)$$

and the associated $\mathbf{G}$ -valued Weil–Deligne representation of K^c over $\overline{\mathbb{Q}}_\ell$ by ρ_ℓ^{MT} . Let $\rho_{\ell, i}$ be the Weil–Deligne representation of K over $\overline{\mathbb{Q}}_\ell$ associated with $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)$. For each i , let $p_i: \mathbf{G} \rightarrow \text{GL}(H_B^i(X_{\mathbb{C}}, \mathbb{Q}))$ be the natural representation. Then $p_i \circ \rho_\ell^{\text{MT}}$ is equivalent to the restriction $\rho_{\ell, i}|_{K^c}$.

Let $s \in \mathbf{G}(\mathbb{Q})$ be the *parity operator*, obtained by evaluating the weight cocharacter at -1 ; it acts on $H_B^i(X_{\mathbb{C}}, \mathbb{Q})$ as $(-1)^i$. We use the following consequence of LLV theory: the degree-two projection

$$p_2: \mathbf{G} \longrightarrow \mathbf{G}_2$$

is a central isogeny with kernel $\langle s \rangle$; see [\[TZ26, Lemma 3.2.5\]](#). This kernel is trivial when the odd cohomology vanishes and has order two otherwise.

Using these facts, we can prove the following result.

Proposition 4.1. *For every prime ℓ and every i , the Weil–Deligne representation $\rho_{\ell, i}$ of K over $\overline{\mathbb{Q}}_\ell$ associated with $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is Frobenius semisimple. The $\mathbf{G}$ -valued Weil–Deligne representation ρ_ℓ^{MT} is also Frobenius semisimple.*

Proof. Since Frobenius semisimplicity is invariant under finite extensions of K , we may assume that $K = K^c$. We first show that $\rho_{\ell, 2}$ is Frobenius semisimple for every ℓ .

If $b_2(X) \geq 4$, the Kuga–Satake construction [\[And96\]](#) (see also [\[Ito+25, Lemma 4.6\]](#) and [Section 5.2](#)) reduces the assertion to the corresponding result for abelian varieties, which follows from [\[SGA7-I, Exposé IX\]](#) when $\ell \neq p$ and from [\[CI99\]](#) when $\ell = p$.

If $b_2(X) = 3$, then $H^{1,1}(X_{\mathbb{C}})$ is spanned by a polarization whose orthogonal complement in $H_B^2(X_{\mathbb{C}}, \mathbb{Q})$ has rank two. The corresponding special orthogonal group is a torus, which in turn implies that $\mathbf{G}_2$ is a torus. This implies the assertion.

Now write $\rho_{\ell}^{\text{MT}} = (r_{\ell}^{\text{MT}}, N_{\ell}^{\text{MT}})$, and let x be the value of r_{ℓ}^{MT} at a geometric Frobenius lift. Write $x = x_s x_u$ for its Jordan decomposition. Since $p_2(x)$ is semisimple, we have $p_2(x_u) = 1$. The kernel of p_2 contains no nontrivial unipotent elements, so $x_u = 1$. Therefore ρ_{ℓ}^{MT} is Frobenius semisimple. This implies that $\rho_{\ell,i}$ is also Frobenius semisimple for every i . $\square$

4.2. Weight–monodromy conjecture for hyper-Kähler varieties. We now prove the weight–monodromy conjecture for hyper-Kähler varieties. The degree-two case is already known by [Lemma 3.8](#). To pass to higher degrees, we use the following construction.

Lemma 4.2. *Let $V_i = H_B^i(X_{\mathbb{C}}, \mathbb{Q})$ for each integer $0 \leq i \leq 2 \dim X$.*

- (1) *The action of $\mathbf{G}$ on $V_i^{\otimes 2} := V_i \otimes V_i$ factors through $\mathbf{G}_2$.*
- (2) *As a $\mathbf{G}_2$ -representation, $V_i^{\otimes 2}$ is a direct summand of a finite direct sum of representations $V_2^{\otimes m} \otimes V_2^{\vee, \otimes n}$ for some $m, n \geq 0$ with $m - n = i$.*

Proof. (1) The kernel of p_2 is generated by s , which acts on V_i as $(-1)^i$ and hence trivially on $V_i^{\otimes 2}$. The assertion follows since p_2 is surjective.

(2) The group $\mathbf{G}_2$ is reductive, and V_2 is faithful as a $\mathbf{G}_2$ -representation. Thus $V_i^{\otimes 2}$ is a direct summand of a finite direct sum of $V_2^{\otimes m} \otimes V_2^{\vee, \otimes n}$ for some $m, n \geq 0$ by [\[Mil17, Theorems 4.14 and 22.42\]](#). The scalar subgroup $\mathbf{G}_m \subset \mathbf{G}_2$ acts on $V_i^{\otimes 2}$ as z^i and on $V_2^{\otimes m} \otimes V_2^{\vee, \otimes n}$ as z^{m-n} . Thus we may assume that $m - n = i$, as required. $\square$

The following lemma detects purity from a tensor square.

Lemma 4.3. *Let $\rho = (r, N)$ be a Weil–Deligne representation of K over $\overline{\mathbb{Q}}_{\ell}$. If $\rho \otimes \rho$ is pure of weight $2i$ for some integer i , then ρ is pure of weight i .*

Proof. Let V be the underlying vector space of ρ and let $M_{\bullet}V$ be the monodromy filtration of N . The monodromy operator on $V \otimes V$ is $N \otimes 1 + 1 \otimes N$. Its monodromy filtration is the convolution of the two filtrations, so

$$\text{gr}_m^M(V \otimes V) \simeq \bigoplus_{a+b=m} \text{gr}_a^M V \otimes \text{gr}_b^M V.$$

This follows, for instance, from the Jacobson–Morozov theorem; see [\[Del80, Proposition \(1.6.9\)\]](#).

Let a be an eigenvalue of geometric Frobenius on $\text{gr}_j^M V$. Then a^2 is an eigenvalue on the summand $\text{gr}_j^M V \otimes \text{gr}_j^M V$ of $\text{gr}_{2j}^M(V \otimes V)$. Purity of $V \otimes V$ says that a^2 is a Weil $q^{2(i+j)}$ -number. It follows that a is a Weil q^{i+j} -number, as desired. $\square$

We can now prove the weight–monodromy conjecture ([Conjecture 3.7](#)) for hyper-Kähler varieties.

Theorem 4.4. *Let X be a hyper-Kähler variety over K . For every prime ℓ and every integer $0 \leq i \leq 2 \dim X$, the Weil–Deligne representation $\rho_{\ell,i}$ is pure of weight i .*

Proof. Since purity is invariant under finite extensions, we may assume that $K = K^c$. Using [Lemma 4.2](#) and the étale–Betti comparison, we see that $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_{\ell})^{\otimes 2}$ is a direct summand of a finite direct sum of $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_{\ell})^{\otimes m} \otimes H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_{\ell})^{\vee, \otimes n}$ with $m - n = i$ as a $\mathbf{G}_2$ -representation over $\mathbb{Q}_{\ell}$. Since the action of G_K on total cohomology factors through $\phi_{\ell}^{\text{MT}}: G_K \rightarrow \mathbf{G}(\mathbb{Q}_{\ell})$, it follows

that this splitting is also G_K -equivariant. Thus $\rho_{\ell,i}^{\otimes 2}$ is a direct summand of a finite direct sum of Weil–Deligne representations $\rho_{\ell,2}^{\otimes m} \otimes \rho_{\ell,2}^{\vee, \otimes n}$ with $m - n = i$. By Lemma 3.8, $\rho_{\ell,2}$ is pure of weight 2, so each $\rho_{\ell,2}^{\otimes m} \otimes \rho_{\ell,2}^{\vee, \otimes n}$ is pure of weight $2(m - n) = 2i$. Hence $\rho_{\ell,i}^{\otimes 2}$ is pure of weight $2i$. The assertion now follows from Lemma 4.3. $\square$

5. STRONGLY COMPATIBLE SYSTEMS WITH RATIONAL AND INTEGRAL COEFFICIENTS

In this section, we establish strong compatibility for the Weil–Deligne representations valued in Mumford–Tate groups attached to hyper-Kähler varieties over number fields and its integral refinement.

5.1. Compatibility of Weil–Deligne representations. Let F be a number field, and fix an embedding $\overline{F} \hookrightarrow \mathbb{C}$. Let X be an abelian variety or a hyper-Kähler variety over F . Put

$$V_B = \begin{cases} H_B^1(X_{\mathbb{C}}, \mathbb{Q}), & \text{if } X \text{ is an abelian variety,} \\ H_B^\bullet(X_{\mathbb{C}}, \mathbb{Q}), & \text{if } X \text{ is a hyper-Kähler variety,} \end{cases}$$

and, for every prime ℓ , put

$$V_\ell = \begin{cases} H_{\text{ét}}^1(X_{\overline{F}}, \mathbb{Q}_\ell), & \text{if } X \text{ is an abelian variety,} \\ H_{\text{ét}}^\bullet(X_{\overline{F}}, \mathbb{Q}_\ell), & \text{if } X \text{ is a hyper-Kähler variety.} \end{cases}$$

Let $\mathbf{G} = \text{MT}(X_{\mathbb{C}})$ denote the Mumford–Tate group of V_B . Write

$$\mu: \mathbb{G}_m \longrightarrow Z(\mathbf{G})$$

for the weight cocharacter, normalized so that $\mu(z)$ acts as z on V_B if X is an abelian variety and as z^i on $H_B^i(X_{\mathbb{C}}, \mathbb{Q})$ for each i if X is a hyper-Kähler variety. By replacing F by a finite extension, we assume that the algebraic monodromy group of V_ℓ is connected for every prime ℓ . By Deligne’s theorem on absolute Hodge cycles for abelian varieties [Del82] and by Theorem 2.5 for hyper-Kähler varieties, the Galois representation on V_ℓ factors through

$$\phi_\ell^{\text{MT}}: G_F \longrightarrow \mathbf{G}(\mathbb{Q}_\ell),$$

where we identify V_ℓ with $V_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ via the étale–Betti comparison.

Let $v \mid p$ be a finite place of F . Applying the construction of Section 3.2 to $\phi_\ell^{\text{MT}}|_{G_{F_v}}$ gives a $\mathbf{G}$ -valued Weil–Deligne representation

$$\rho_{\ell,v}^{\text{MT}} = (r_{\ell,v}^{\text{MT}}, N_{\ell,v})$$

of F_v over an algebraic closure $\overline{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$ for every prime ℓ , including $\ell = p$.

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and fix an embedding $\iota_\ell: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ for every ℓ .

Definition 5.1. The family $\{\rho_{\ell,v}^{\text{MT}}\}_\ell$ is *rationally strongly compatible* if there exists a $\mathbf{G}$ -valued Weil–Deligne representation ρ_v of F_v over $\overline{\mathbb{Q}}$ satisfying the following conditions:

- (1) ${}^\sigma \rho_v \sim_{\mathbf{G}(\overline{\mathbb{Q}})} \rho_v$ for every $\sigma \in G_{\mathbb{Q}}$, where ${}^\sigma \rho_v$ denotes the base change of ρ_v along $\sigma: \overline{\mathbb{Q}} \xrightarrow{\sim} \overline{\mathbb{Q}}$.
- (2) $\rho_v \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}_\ell} \sim_{\mathbf{G}(\overline{\mathbb{Q}_\ell})} \rho_{\ell,v}^{\text{MT}}$ for every ℓ , including $\ell = p$.

Remark 5.2. Extension of scalars from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ induces a bijection between the $G_{\mathbb{Q}}$ -invariant equivalence classes of $\mathbf{G}$ -valued Weil–Deligne representations of F_v over $\overline{\mathbb{Q}}$ and the $\text{Aut}(\mathbb{C}/\mathbb{Q})$ -invariant equivalence classes of $\mathbf{G}$ -valued Weil–Deligne representations of F_v over $\mathbb{C}$. The latter are called “defined over $\mathbb{Q}$ ” in [KZ25b, Definition 4.1.7]. Indeed, after fixing a finite quotient through which the inertia subgroup acts, $\mathbf{G}$ -valued Weil–Deligne representations are parametrized by an affine scheme of finite type over $\mathbb{Q}$. An $\text{Aut}(\mathbb{C}/\mathbb{Q})$ -invariant $\mathbf{G}$ -conjugacy orbit is locally closed and descends to $\overline{\mathbb{Q}}$, so it has a $\overline{\mathbb{Q}}$ -point. Moreover, two $\mathbf{G}$ -valued Weil–Deligne representations

over $\overline{\mathbb{Q}}$ that are equivalent over $\mathbb{C}$ are already equivalent over $\overline{\mathbb{Q}}$ since their transporter is a scheme of finite type over $\overline{\mathbb{Q}}$, and hence has a $\overline{\mathbb{Q}}$ -point whenever it has a $\mathbb{C}$ -point. This proves the desired bijection.

Thus [Definition 5.1](#) is equivalent to the formulation of strong compatibility used in [\[KZ25b\]](#).

Theorem 5.3 ([\[KZ25b, Theorem 1.2\]](#)). *In the above setting, suppose that X is an abelian variety with semistable reduction at v . Then $\{\rho_{\ell,v}^{\text{MT}}\}_{\ell}$ is rationally strongly compatible.*

Proof. This is [\[KZ25b, Theorem 1.2\]](#), reformulated using [Remark 5.2](#). $\square$

5.2. Strong compatibility for hyper-Kähler varieties. Now we assume that X is a hyper-Kähler variety over F with $b_2(X) \geq 4$. We shall deduce from [Theorem 5.3](#) its analogue for X after replacing F by a finite extension. For this, we use the Kuga–Satake construction [\[Del72; And96\]](#).

The $\mathbb{Q}$ -vector space $H_B^2(X_{\mathbb{C}}, \mathbb{Q})(1)$ carries a natural non-degenerate $\mathbb{Q}$ -quadratic form q , which is called the Beauville–Bogomolov–Fujiki (BBF) form. We denote this quadratic space over $\mathbb{Q}$ by $V_B^2(1)$ and let $\text{Cl}(V_B^2(1))$ be the associated Clifford algebra over $\mathbb{Q}$. Let $\text{GSpin} := \text{GSpin}(V_B^2(1))$ be the associated general spin group and let $\text{SO} := \text{SO}(V_B^2(1))$. We write

$$\pi: \text{GSpin} \longrightarrow \text{SO}, \quad \nu: \text{GSpin} \longrightarrow \mathbb{G}_m$$

for the natural projection and the spinor norm, respectively.

In this paper, we employ the following definition.

Definition 5.4. Let $E \subset \overline{F}$ be a finite extension of F . We say that X admits a Kuga–Satake abelian variety over E of dimension $2^{b_2(X)-1}$ if there exists an abelian variety A over E with an isomorphism of $\mathbb{Q}$ -vector spaces

$$\theta: H_B^1(A_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} \text{Cl}(V_B^2(1))$$

satisfying the following conditions:

(1) Let

$$\text{GSpin} \hookrightarrow \text{GL}(H_B^1(A_{\mathbb{C}}, \mathbb{Q}))$$

be the injection obtained via θ from left multiplication on $\text{Cl}(V_B^2(1))$. The Hodge cocharacter $\mu_A: \mathbb{G}_{m,\mathbb{C}} \rightarrow \text{GL}(H_B^1(A_{\mathbb{C}}, \mathbb{C}))$ factors through $\text{GSpin}_{\mathbb{C}}$, and the composition $\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu_A} \text{GSpin}_{\mathbb{C}} \xrightarrow{\pi} \text{SO}_{\mathbb{C}}$ is equal to the Hodge cocharacter associated with $V_B^2(1)$.

(2) For every prime ℓ , the Galois representation

$$\phi_{\ell}^{\text{KS}}: G_E \longrightarrow \text{GL}(H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{Q}_{\ell}))$$

factors through $\text{GSpin}(\mathbb{Q}_{\ell})$ (via the étale–Betti comparison), and via $\pi: \text{GSpin}_{\mathbb{Q}_{\ell}} \rightarrow \text{SO}_{\mathbb{Q}_{\ell}}$, this induces the usual G_E -action on $V_{\ell}^2(1) := H_{\text{ét}}^2(X_{\overline{F}}, \mathbb{Q}_{\ell})(1)$. Moreover, the composition

$$G_E \xrightarrow{\phi_{\ell}^{\text{KS}}} \text{GSpin}(\mathbb{Q}_{\ell}) \xrightarrow{\nu} \mathbb{Q}_{\ell}^{\times}$$

is the inverse of the cyclotomic character χ_{ℓ} .

We call such an abelian variety A , equipped with θ , a Kuga–Satake abelian variety of X over E .

As recalled in [\[Ito+25, Section 4.2\]](#), the Kuga–Satake construction [\[Del72; And96\]](#) provides a Kuga–Satake abelian variety of X over some finite extension E/F in the sense above. In fact, a result of André [\[And96, Theorem 8.4.3\]](#) allows us to control the extension E/F as follows.

Lemma 5.5. *Let $\mathcal{L}$ be an ample line bundle on X and put*

$$\Lambda_{\mathbb{Z}} := H_{\text{ét}}^2(X_{\overline{F}}, \widehat{\mathbb{Z}}(1)), \quad P_{\mathbb{Z}} := c_1(\mathcal{L})^\perp \subset \Lambda_{\mathbb{Z}},$$

where the orthogonal complement is taken with respect to the BBF form. Write $P_{\mathbb{A}_f} := P_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{A}_f$ for finite adèles $\mathbb{A}_f$ of $\mathbb{Q}$. For $m = 3$ or $m = 4$, let K_m^{ad} be the image of

$$K_m := \{g \in \text{GSpin}(P_{\mathbb{A}_f}) \cap \text{Cl}(P_{\mathbb{Z}})^\times \mid g - 1 \in m \text{Cl}(P_{\mathbb{Z}})\}$$

under the natural projection to $\text{SO}(P_{\mathbb{Z}})$. Let E/F be a finite extension such that the image of $G_E \rightarrow \text{O}(P_{\mathbb{Z}})$ is contained in K_m^{ad} . Then X admits a Kuga–Satake abelian variety A over E in the sense of [Definition 5.4](#). Moreover, A can be chosen to have semistable reduction at every finite place of E of residue characteristic prime to m .

Proof. By [\[And96, Theorem 8.4.3\]](#), the assumption ensures that the Kuga–Satake construction associated with the even Clifford algebra of the primitive cohomology can be carried out over E , yielding a Kuga–Satake package in the sense of [\[And96, Definition 4.5.1\]](#) such that all m -torsion points of the underlying abelian variety B/E are rational over E . (Strictly speaking, we use a different quadratic form from the one in [\[And96\]](#), but the same argument applies.) We set $A := B^4$. Using [\[And96, Variant 4.1.3 and Corollary 6.4.4\]](#), together with [\[Ito+25, Lemma 4.6\]](#), one can check that there is an isomorphism $\theta: H_B^1(A_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} \text{Cl}(V_B^2(1))$ such that A , equipped with θ , is a Kuga–Satake abelian variety over E in the sense of [Definition 5.4](#).

Since all m -torsion points of A are E -rational, it follows from Raynaud’s criterion [\[SGA7-I, Exposé IX, Proposition 4.7\]](#) that A has semistable reduction at every finite place of E of residue characteristic prime to m . $\square$

Assume now that X admits a Kuga–Satake abelian variety A over E of dimension $2^{b_2(X)-1}$. To relate the higher cohomology of X to the cohomology of A , we use the twisted Looijenga–Lunts–Verbitsky (LLV) representation

$$R: \text{GSpin} \longrightarrow \prod_i \text{GL}(H_B^i(X_{\mathbb{C}}, \mathbb{Q})).$$

Here the restriction of R to $\text{Spin}(V_B^2(1))$ integrates the action of the degree-zero semisimple part of the LLV Lie algebra, and $z \in \mathbb{G}_m \subset \text{GSpin}$ acts in degree i as z^i . We refer to [\[Ito+25, Section 3.2\]](#) and the references therein for the construction. It follows from a theorem of Verbitsky [\[Ver90, Theorem 1.4\]](#) that $\mathbf{G} = \text{MT}(X_{\mathbb{C}})$ is contained in the image $R(\text{GSpin})$; see also [\[Ito+25, Theorem 3.21 and Remark 3.22\]](#).

Lemma 5.6. *The composition $\mathbb{G}_{m, \mathbb{C}} \xrightarrow{\mu_A} \text{GSpin}_{\mathbb{C}} \xrightarrow{R} \prod_i \text{GL}(H_B^i(X_{\mathbb{C}}, \mathbb{C}))$ is the Hodge cocharacter μ_X of $H_B^\bullet(X_{\mathbb{C}}, \mathbb{Q})$. In particular, we have the following commutative diagram:*

$$\begin{array}{ccc} \text{MT}(A_{\mathbb{C}}) & \longrightarrow & \mathbf{G} \\ \downarrow & & \downarrow \\ \text{GSpin} & \xrightarrow{R} & R(\text{GSpin}) \hookrightarrow \prod_i \text{GL}(H_B^i(X_{\mathbb{C}}, \mathbb{Q})). \end{array}$$

The upper horizontal map is the restriction of R and is an isogeny.

We denote the restriction $\text{MT}(A_{\mathbb{C}}) \rightarrow \mathbf{G}$ by the same notation R .

Proof. Both μ_X and $R \circ \mu_A$ take values in $R(\text{GSpin})_{\mathbb{C}}$. Since the projection

$$p_2: \prod_i \text{GL}(H_B^i(X_{\mathbb{C}}, \mathbb{Q})) \longrightarrow \text{GL}(H_B^2(X_{\mathbb{C}}, \mathbb{Q}))$$

induces a central isogeny from $R(\mathrm{GSpin})$ onto its image, it is enough to show that their degree-two projections are the same. Since $p_2 \circ R = \nu \cdot \pi$, the required equality follows immediately. By the defining property of Mumford–Tate groups, the equality $R \circ \mu_A = \mu_X$ implies $R(\mathrm{MT}(A_{\mathbb{C}})) = \mathbf{G}$. Since $\nu \cdot \pi$ has finite kernel, so does R . Thus the restriction $R: \mathrm{MT}(A_{\mathbb{C}}) \rightarrow \mathbf{G}$ is an isogeny. $\square$

Recall that we assume that the algebraic monodromy group of the action of G_F on $H_{\text{ét}}^{\bullet}(X_{\overline{F}}, \mathbb{Q}_{\ell})$ is connected for every prime ℓ .

Proposition 5.7. *Keep the notation and assumptions above. Let $s \in Z(\mathbf{G})(\mathbb{Q})$ be the element which acts as $(-1)^i$ on $H_B^i(X_{\mathbb{C}}, \mathbb{Q})$.*

- (1) *For every ℓ , the homomorphism $\phi_{\ell}^{\mathrm{KS}}: G_E \rightarrow \mathrm{GSpin}(\mathbb{Q}_{\ell})$ factors through $\mathrm{MT}(A_{\mathbb{C}})(\mathbb{Q}_{\ell})$.*
- (2) *There is a unique continuous character $\epsilon: G_E \rightarrow \langle s \rangle = \{1, s\}$ such that the following diagram commutes for every ℓ :*

$$\begin{array}{ccc} G_E & \xrightarrow{\phi_{\ell}^{\mathrm{KS}}} & \mathrm{MT}(A_{\mathbb{C}})(\mathbb{Q}_{\ell}) \\ & \searrow \epsilon \phi_{\ell}^{\mathrm{MT}} & \downarrow R \\ & & \mathbf{G}(\mathbb{Q}_{\ell}). \end{array}$$

Proof. (1) We choose an ample line bundle on X , and write η for its class in $V_B^2(1)$. Let $H \subset \mathrm{GSpin}$ be the stabilizer of η with respect to its usual action on $V_B^2(1)$ via π . By the properties of the Kuga–Satake abelian variety, we have $\mathrm{MT}(A_{\mathbb{C}}) \subset H$ and $\phi_{\ell}^{\mathrm{KS}}(G_E) \subset H(\mathbb{Q}_{\ell})$. We define $f: H \rightarrow \mathrm{GL}(H_B^2(X_{\mathbb{C}}, \mathbb{Q}))$ by $f = (p_2 \circ R)|_H = (\nu \cdot \pi)|_H$. Then the kernel of f is $\{\pm 1\} \subset H$. Let $\mathrm{MT}_2(X_{\mathbb{C}}) \subset \mathrm{GL}(H_B^2(X_{\mathbb{C}}, \mathbb{Q}))$ be the Mumford–Tate group of degree 2. Since $\mathrm{MT}(A_{\mathbb{C}})$ contains $\{\pm 1\}$ and $f(\mathrm{MT}(A_{\mathbb{C}})) = \mathrm{MT}_2(X_{\mathbb{C}})$, it follows that $f^{-1}(\mathrm{MT}_2(X_{\mathbb{C}})) = \mathrm{MT}(A_{\mathbb{C}})$. By our assumption, the Galois representation $\phi_{\ell}^{\mathrm{KS}}$ takes values in $f^{-1}(\mathrm{MT}_2(X_{\mathbb{C}}))$, which proves (1).

(2) For each ℓ , we define $\epsilon_{\ell}: G_E \rightarrow \langle s \rangle$ by $\epsilon_{\ell}(\sigma) = \phi_{\ell}^{\mathrm{MT}}(\sigma) \cdot (R \circ \phi_{\ell}^{\mathrm{KS}})(\sigma)^{-1}$. We have to show that ϵ_{ℓ} is independent of ℓ . For this, it suffices to show that the subgroup $\ker(\epsilon_{\ell}) \subset G_E$ is independent of ℓ .

Let $J_{\ell} \subset \mathrm{MT}(A_{\mathbb{C}})_{\mathbb{Q}_{\ell}} \times \mathbf{G}_{\mathbb{Q}_{\ell}}$ be the Zariski closure of the image of

$$\tau_{\ell} := (\phi_{\ell}^{\mathrm{KS}}, \phi_{\ell}^{\mathrm{MT}}): G_E \longrightarrow \mathrm{MT}(A_{\mathbb{C}})(\mathbb{Q}_{\ell}) \times \mathbf{G}(\mathbb{Q}_{\ell}).$$

We claim that $\tau_{\ell}^{-1}(J_{\ell}^{\circ}(\mathbb{Q}_{\ell})) = \ker(\epsilon_{\ell})$. This implies that $\ker(\epsilon_{\ell})$ is independent of ℓ by a theorem of Serre [Ser00] (see also [LP92, Proposition (6.14)]). To show the claim, let $\Gamma_R \subset \mathrm{MT}(A_{\mathbb{C}}) \times \mathbf{G}$ be the graph of R . Since $\mathbf{G} \rightarrow \mathrm{MT}_2(X_{\mathbb{C}})$ is an isogeny, the degree-two comparison shows that $J_{\ell}^{\circ} \subset (\Gamma_R)_{\mathbb{Q}_{\ell}}$; see also the proof of [Ito+25, Proposition 4.7]. Since $R: \mathrm{MT}(A_{\mathbb{C}}) \rightarrow \mathbf{G}$ is an isogeny, Theorem 2.5 implies that J_{ℓ}° and $(\Gamma_R)_{\mathbb{Q}_{\ell}}$ have the same dimension. Hence $J_{\ell}^{\circ} = (\Gamma_R)_{\mathbb{Q}_{\ell}}$. Consequently, we have $\tau_{\ell}^{-1}(J_{\ell}^{\circ}(\mathbb{Q}_{\ell})) = \ker(\epsilon_{\ell})$. This completes the proof of (2). $\square$

Theorem 5.8. *Let X be a hyper-Kähler variety over a number field F with $b_2(X) \geq 4$. Assume that the algebraic monodromy group of $H_{\text{ét}}^{\bullet}(X_{\overline{F}}, \mathbb{Q}_{\ell})$ is connected for every ℓ , and that X admits a Kuga–Satake abelian variety A over F with semistable reduction at a finite place $v \mid p$ of F . Let $\mathbf{G} := \mathrm{MT}(X_{\mathbb{C}})$ and consider the associated $\mathbf{G}$ -valued Weil–Deligne representation $\rho_{\ell,v}^{\mathrm{MT}}$ of F_v over $\overline{\mathbb{Q}_{\ell}}$. Then $\{\rho_{\ell,v}^{\mathrm{MT}}\}_{\ell}$ is rationally strongly compatible in the sense of Definition 5.1.*

Proof. By Lemma 5.6 and Proposition 5.7, there are an isogeny $R: \mathrm{MT}(A_{\mathbb{C}}) \rightarrow \mathbf{G}$ over $\mathbb{Q}$ and a continuous character $\epsilon: G_F \rightarrow \langle s \rangle$ such that $R \circ \phi_{\ell}^{\mathrm{KS}} = \epsilon \phi_{\ell}^{\mathrm{MT}}$ for every ℓ . By Theorem 5.3, there

exists an $\mathrm{MT}(A_{\mathbb{C}})$ -valued Weil–Deligne representation $\rho_v^A = (r_v^A, N_v^A)$ of F_v over $\overline{\mathbb{Q}}$ realizing the resulting rationally strongly compatible system. Write ϵ_v for the restriction of ϵ to W_{F_v} and set

$$\rho_v := (\epsilon_v \cdot (R \circ r_v^A), dR(N_v^A)).$$

Since R is defined over $\mathbb{Q}$ and ϵ_v takes values in $Z(\mathbf{G})(\mathbb{Q})$, the $\mathbf{G}(\overline{\mathbb{Q}})$ -conjugacy class of ρ_v is $G_{\mathbb{Q}}$ -invariant. The constructions given in [Examples 3.1](#) and [3.2](#) are compatible with tensor products and send the finite-order character $\epsilon|_{G_{F_v}}$ to $(\epsilon_v, 0)$. Together with functoriality under R , we see that

$$(\rho_v)_{\overline{\mathbb{Q}}_\ell} \sim_{\mathbf{G}(\overline{\mathbb{Q}}_\ell)} \rho_{\ell,v}^{\mathrm{MT}}$$

for every ℓ , including $\ell = p$. This proves the assertion. $\square$

Remark 5.9. The assertion of [Theorem 5.8](#) also holds over a finite extension $K/\mathbb{Q}_p$ equipped with an embedding $K \hookrightarrow \mathbb{C}$. More precisely, let X be a hyper-Kähler variety over K with $b_2(X) \geq 4$. Assume that, for every ℓ , the Galois representation on $H_{\mathrm{\acute{e}t}}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell)$ takes values in $\mathrm{MT}(X_{\mathbb{C}})(\mathbb{Q}_\ell)$, and that X admits a Kuga–Satake abelian variety over K with semistable reduction. Then the associated $\mathrm{MT}(X_{\mathbb{C}})$ -valued Weil–Deligne representations form a rationally strongly compatible system. (Here the notions of a Kuga–Satake abelian variety and rational strong compatibility are understood by extending the definitions above to this local setting in the obvious way.) Indeed, the comparison with Kuga–Satake abelian varieties also holds in this setting by descent to a finitely generated subfield of K and then the same proof applies using the local version of [Theorem 5.3](#); see [\[KZ25b, Remark 5.3.13\]](#).

Corollary 5.10. *Let K be a finite extension of $\mathbb{Q}_p$, and let X be a hyper-Kähler variety over K . For every ℓ and i , let $\rho_{\ell,i} = (r_{\ell,i}, N_{\ell,i})$ be the Weil–Deligne representation of K over $\overline{\mathbb{Q}}_\ell$ associated with $H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)$. There exists a finite extension L/K such that, for every i and every geometric Frobenius lift $\mathrm{Frob}_L \in W_L$, the characteristic polynomial $\det(T - r_{\ell,i}(\mathrm{Frob}_L))$ belongs to $\mathbb{Z}[T]$ and is independent of ℓ , including $\ell = p$. If $b_2(X) = 3$, one may take $L = K$.*

Proof. Fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. Suppose first that $b_2(X) = 3$. Then $\mathrm{MT}(X_{\mathbb{C}})$ is a torus. Thus, by [Theorem 2.5](#), we see that $H_{\mathrm{\acute{e}t}}^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is potentially unramified if $\ell \neq p$ and potentially crystalline if $\ell = p$ for all i . By the weight–monodromy theorem ([Theorem 4.4](#)), all the Frobenius eigenvalues in degree i are therefore of weight i . By [\[Och99, Theorems B and D\]](#), the alternating traces of all positive powers of Frob_K are rational and independent of ℓ , including $\ell = p$. Thus the alternating product of characteristic polynomials is rational and independent of ℓ . Since different cohomological degrees have distinct weights, each factor has the same properties as well.

Suppose now that $b_2(X) \geq 4$. After a finite extension L/K , the Galois representations are valued in the Mumford–Tate group, and X_L admits a Kuga–Satake abelian variety over L with semistable reduction. The assertion then follows from [Remark 5.9](#).

Finally, the Frobenius eigenvalues for $\ell \neq p$ are algebraic integers by [\[Och99, Proposition 2.1\]](#). Hence the common characteristic polynomials belong to $\mathbb{Z}[T]$ for all ℓ (including $\ell = p$). $\square$

The following example shows that the character ϵ in [Proposition 5.7](#) need not be trivial in general.

Example 5.11. Let F be a number field, let $v \mid 3$, and let C be an elliptic curve over F such that $\mathrm{End}_F(C)_{\mathbb{Q}} = \mathrm{End}_{\overline{F}}(C_{\overline{F}})_{\mathbb{Q}} \simeq \mathbb{Q}(\sqrt{-1})$ and all 3-torsion points of C are rational over F . Suppose that there is a quadratic character $\epsilon: G_F \rightarrow \{\pm 1\}$, ramified at v , such that the quadratic twist C^ϵ of C by ϵ has good reduction at v . Then for every $\ell \neq 3$, the inertia subgroup I_v acts on $H_{\mathrm{\acute{e}t}}^1(C_{\overline{F}}, \mathbb{Q}_\ell)$ through the character ϵ .

Let $S := C \times C$ and let $X = K_2(S)$ be the associated generalized Kummer fourfold over F . We have G_F -equivariant decompositions

$$\begin{aligned} H_{\text{ét}}^2(X_{\overline{F}}, \mathbb{Q}_\ell) &\simeq H_{\text{ét}}^2(S_{\overline{F}}, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell(-1), \\ H_{\text{ét}}^3(X_{\overline{F}}, \mathbb{Q}_\ell) &\simeq H_{\text{ét}}^3(S_{\overline{F}}, \mathbb{Q}_\ell) \oplus H_{\text{ét}}^1(S_{\overline{F}}, \mathbb{Q}_\ell)(-1), \\ H_{\text{ét}}^4(X_{\overline{F}}, \mathbb{Q}_\ell) &\simeq \text{Sym}^2 H_{\text{ét}}^2(X_{\overline{F}}, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell(-2)^{\oplus 80} \end{aligned}$$

and similarly for Betti cohomology groups. For the third identification, see [FH23, Theorem 1.1]. These decompositions are obtained from algebraic correspondences defined over F and are compatible with the étale–Betti comparison.

Put

$$T := \text{MT}(C_{\mathbb{C}}) \simeq \text{Res}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}} \mathbb{G}_m.$$

Since all endomorphisms of $C_{\overline{F}}$ are defined over F , the Galois representation on $H_{\text{ét}}^1(C_{\overline{F}}, \mathbb{Q}_\ell)$ takes values in $T(\mathbb{Q}_\ell)$ for every ℓ . The above decompositions give a representation

$$\Phi: T \longrightarrow \prod_j \text{GL}(H_B^j(X_{\mathbb{C}}, \mathbb{Q}))$$

such that $\Phi(T) = \text{MT}(X_{\mathbb{C}})$ and, for every ℓ , the composition

$$G_F \longrightarrow T(\mathbb{Q}_\ell) \xrightarrow{\Phi} \prod_i \text{GL}(H_{\text{ét}}^i(X_{\overline{F}}, \mathbb{Q}_\ell))$$

is the usual Galois representation on the total cohomology of X . In particular, the algebraic monodromy group of X is connected (for every ℓ).

Finally, let $U \subset V_B^2(1)$ be the orthogonal complement of $\text{Pic}(X)_{\mathbb{Q}} = \text{Pic}(X_{\overline{F}})_{\mathbb{Q}}$. The CM structure gives

$$T \simeq \text{GSpin}(U) \longrightarrow \text{GSpin}(V_B^2(1)).$$

Since C^ϵ and C are isomorphic over $\mathbb{C}$, they have the same T -representation on H_B^1 . Thus there exists a T -equivariant isomorphism

$$\theta: H_B^1((C^\epsilon)_{\mathbb{C}}, \mathbb{Q})^{\oplus 64} \xrightarrow{\sim} \text{Cl}(V_B^2(1)).$$

We set $A := (C^\epsilon)^{64}$. By construction, one can check that A , equipped with the isomorphism θ , is a Kuga–Satake abelian variety over F in the sense of Definition 5.4. Since A has good reduction at v , the inertia subgroup I_v acts trivially on $H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{Q}_\ell)$ for $\ell \neq 3$. On the other hand, it acts on $H_{\text{ét}}^3(X_{\overline{F}}, \mathbb{Q}_\ell)$ through the nontrivial character $\epsilon|_{I_v}$. Thus the character in Proposition 5.7 is nontrivial.

5.3. An integral refinement of strong compatibility. Keep the notation and hypotheses of Section 5.1. In particular, X denotes an abelian variety over F , or a hyper-Kähler variety over F . Assume in addition that $b_2(X) \geq 4$ if X is a hyper-Kähler variety. Put

$$\Lambda_B = \begin{cases} H_B^1(X_{\mathbb{C}}, \mathbb{Z}), & \text{if } X \text{ is an abelian variety,} \\ H_B^\bullet(X_{\mathbb{C}}, \mathbb{Z}), & \text{if } X \text{ is a hyper-Kähler variety.} \end{cases}$$

Let $\mathbf{G} := \text{MT}(X_{\mathbb{C}})$. Let $v \mid p$ be a finite place of F and let q be the order of the residue field of F_v . If X is an abelian variety, assume that it has semistable reduction at v . If X is a hyper-Kähler variety, assume that it admits a Kuga–Satake abelian variety over F with semistable reduction at v .

By Theorems 5.3 and 5.8, we may choose a Frobenius semisimple $\mathbf{G}$ -valued Weil–Deligne representation $\rho_v = (r_v, N_v)$ over $\mathbb{Q}$ such that

$$(\rho_v)_{\overline{\mathbb{Q}}_\ell} \sim_{\mathbf{G}(\overline{\mathbb{Q}}_\ell)} \rho_{\ell, v}^{\text{MT}}$$

for every ℓ . We note that $r_v(I_{F_v})$ is trivial in the abelian case by the semistable reduction assumption. In the hyper-Kähler case, it is contained in $Z(\mathbf{G})(\overline{\mathbb{Q}})$ by [Proposition 5.7](#).

Definition 5.12. We fix a positive integer D_0 divisible by p , with the following properties:

- (a) $\Lambda_{B, \mathbb{Z}[1/D_0]} := \Lambda_B \otimes_{\mathbb{Z}} \mathbb{Z}[1/D_0]$ is torsion-free.
- (b) The schematic closure $\mathcal{G} \subset \mathrm{GL}(\Lambda_{B, \mathbb{Z}[1/D_0]})$ of $\mathbf{G}$ is a reductive group scheme over $\mathbb{Z}[1/D_0]$.
- (c) Every prime $\ell \nmid D_0$ satisfies $\ell > b_1(X)$ when X is an abelian variety and $\ell > \max_i b_i(X)$ when X is a hyper-Kähler variety.
- (d) $q - 1 \in \mathbb{Z}[1/D_0]^\times$ and for every geometric Frobenius lift $\mathrm{Frob}_q \in W_{F_v}$, any two distinct eigenvalues a, b of $r_v(\mathrm{Frob}_q)$ on $H_B^i(X_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ satisfy $a - b \in \overline{\mathbb{Z}}[1/D_0]^\times$, where $i = 1$ if X is an abelian variety and i is arbitrary if X is a hyper-Kähler variety.
- (e) ρ_v extends to a $\mathcal{G}$ -valued Weil–Deligne representation of F_v over $\overline{\mathbb{Z}}[1/D_0]$:

$$\rho_v^{\mathrm{int}} = (r_v^{\mathrm{int}}, N_v^{\mathrm{int}}).$$

Here $\overline{\mathbb{Z}}$ is the ring of algebraic integers in $\overline{\mathbb{Q}}$.

These conditions can be achieved by taking D_0 sufficiently divisible.

For a prime $\ell \nmid D_0$, the representation ϕ_ℓ^{MT} takes values in $\mathcal{G}(\mathbb{Z}_\ell)$. By [Example 3.4](#) and [Proposition 3.5](#), we obtain the associated $\mathcal{G}$ -valued Weil–Deligne representation

$$\rho_{\ell, v}^{\mathrm{MT}, \mathrm{int}} = (r_{\ell, v}^{\mathrm{MT}, \mathrm{int}}, N_{\ell, v}^{\mathrm{MT}, \mathrm{int}})$$

of F_v over $\overline{\mathbb{Z}}_\ell$, whose equivalence class is independent of all choices in the construction. Its generic fiber is equivalent to $\rho_{\ell, v}^{\mathrm{MT}}$.

Proposition 5.13. *Let $\ell \nmid D_0$ be a prime. The representations $(r_v^{\mathrm{int}})_{\overline{\mathbb{Z}}_\ell}$ and $r_{\ell, v}^{\mathrm{MT}, \mathrm{int}}$ are $\mathcal{G}(\overline{\mathbb{Z}}_\ell)$ -conjugate. Here $(r_v^{\mathrm{int}})_{\overline{\mathbb{Z}}_\ell}$ denotes the base change of r_v^{int} along the map $\overline{\mathbb{Z}}[1/D_0] \rightarrow \overline{\mathbb{Z}}_\ell$ induced by ι_ℓ . Moreover, their scheme-theoretic centralizers in $\mathcal{G}_{\overline{\mathbb{Z}}_\ell}$ are smooth over $\overline{\mathbb{Z}}_\ell$.*

Proof. Fix a geometric Frobenius lift $\mathrm{Frob}_q \in W_{F_v}$, and put

$$x := (r_v^{\mathrm{int}})_{\overline{\mathbb{Z}}_\ell}(\mathrm{Frob}_q), \quad y := r_{\ell, v}^{\mathrm{MT}, \mathrm{int}}(\mathrm{Frob}_q).$$

Their generic fibers are semisimple and $\mathbf{G}(\overline{\mathbb{Q}}_\ell)$ -conjugate by rational strong compatibility. The centralizers of x and y in $\mathcal{G}_{\overline{\mathbb{Z}}_\ell}$ are smooth and contain maximal tori T_x and T_y of $\mathcal{G}_{\overline{\mathbb{Z}}_\ell}$ by [Lemma 3.6](#). Since a maximal torus is its own centralizer, we have $x \in T_x(\overline{\mathbb{Z}}_\ell)$ and $y \in T_y(\overline{\mathbb{Z}}_\ell)$. By [[Con14](#), Theorem 3.2.6], these tori are $\mathcal{G}(\overline{\mathbb{Z}}_\ell)$ -conjugate, so we may assume that $x, y \in T(\overline{\mathbb{Z}}_\ell)$ for a common maximal torus T . Their generic fibers then lie in the same orbit with respect to the action of the Weyl group associated with T . The corresponding element in the Weyl group lifts to an element $g \in N_{\mathcal{G}_{\overline{\mathbb{Z}}_\ell}}(T)(\overline{\mathbb{Z}}_\ell)$ since the Weyl group (scheme) is finite étale and $\overline{\mathbb{Z}}_\ell$ is strictly henselian. It follows that $gxg^{-1} = y$.

The images of the inertia subgroup for both representations are central, and rational strong compatibility implies that their restrictions to the inertia subgroup are the same. It follows that $(r_v^{\mathrm{int}})_{\overline{\mathbb{Z}}_\ell}$ and $r_{\ell, v}^{\mathrm{MT}, \mathrm{int}}$ are conjugate by the element g . Their centralizers equal those of x and y , respectively, and hence are smooth over $\overline{\mathbb{Z}}_\ell$. $\square$

We shall prove that there is a positive integer D , divisible by D_0 , such that for every prime $\ell \nmid D$, both $(\rho_v^{\mathrm{int}})_{\overline{\mathbb{Z}}_\ell}$ and $\rho_{\ell, v}^{\mathrm{MT}, \mathrm{int}}$ are pure in the sense of [Definition 3.12](#). The key ingredient for this is the following torsion-freeness result.

Proposition 5.14. *Let K be a finite extension of $\mathbb{Q}_p$, and let X be an abelian variety or a hyper-Kähler variety over K . Fix integers $i, m, n \geq 0$, and consider the G_K -module*

$$H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Z}_\ell)^{\otimes m} \otimes_{\mathbb{Z}_\ell} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Z}_\ell)^{\vee, \otimes n}.$$

For all but finitely many $\ell \neq p$, the monodromy operator N_ℓ of the Weil–Deligne representation $\rho_\ell = (r_\ell, N_\ell)$ of K over $\overline{\mathbb{Z}_\ell}$ associated with this G_K -module (Example 3.4) has the following property (t-f): For all $j \geq 0$, the cokernel of $(N_\ell)^j$ is torsion-free.

Proof. The property (t-f) is preserved by duals and tensor products after excluding finitely many further primes. Indeed, by [Ito21, Lemma 2.10] and Nakayama’s lemma, (t-f) is equivalent to compatibility of the monodromy filtration with reduction to $\overline{\mathbb{F}_\ell}$. This compatibility is preserved by duals and tensor products, since the formulas given in [Del80, Proposition (1.6.9)] remain valid over $\overline{\mathbb{F}_\ell}$ outside a finite set of primes depending only on the ranks of the underlying modules. Thus it suffices to consider the Weil–Deligne representation $\rho_\ell = (r_\ell, N_\ell)$ of K over $\overline{\mathbb{Z}_\ell}$ associated with $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Z}_\ell)$. If X is an abelian variety, then the assertion is proved in [Ito21]; more precisely, this follows from [Ito21, Theorem 3.7 and Proposition 3.10(ii)]. We assume that X is a hyper-Kähler variety over K . If $i = 2$, then this again follows from [Ito21]. We shall deduce the general case from this case.

We choose an embedding $K \hookrightarrow \mathbb{C}$ and let $\mathbf{G} := \text{MT}(X_{\mathbb{C}})$. We may assume that $K = K^c$ and obtain the representation $\phi_\ell^{\text{MT}}: G_K \rightarrow \mathbf{G}(\mathbb{Q}_\ell)$ for all ℓ as in Section 4.1. By Lemma 4.2, $H_B^i(X_{\mathbb{C}}, \mathbb{Q})^{\otimes 2}$ is a direct summand of a finite direct sum of $H_B^2(X_{\mathbb{C}}, \mathbb{Q})^{\otimes a} \otimes H_B^2(X_{\mathbb{C}}, \mathbb{Q})^{\vee, \otimes b}$ for some a, b as a $\mathbf{G}$ -representation. From this, we see that $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Z}_\ell)^{\otimes 2}$ is a direct summand of a finite direct sum of $H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Z}_\ell)^{\otimes a} \otimes H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Z}_\ell)^{\vee, \otimes b}$ as a G_K -module for all but finitely many $\ell \neq p$. It follows that $\rho_\ell^{\otimes 2}$ satisfies property (t-f). Finally, property (t-f) can be detected on tensor squares for sufficiently large ℓ . Indeed, the monodromy filtration on a tensor square is the tensor product filtration, both in characteristic zero and, for all but finitely many ℓ , after reduction to $\overline{\mathbb{F}_\ell}$. Since a filtration is determined by its tensor square, compatibility of the monodromy filtration with reduction for $\rho_\ell^{\otimes 2}$ implies the same compatibility for ρ_ℓ . Thus ρ_ℓ also has property (t-f). This completes the proof. $\square$

Remark 5.15. The torsion analogue of the weight–monodromy conjecture formulated in [Ito21, Conjecture 3.5] also holds for hyper-Kähler varieties over K in every degree. Indeed, this follows from Theorem 4.4 and Proposition 5.14 together with [Ito21, Proposition 3.10(iii)].

We return to the notation and assumptions of this subsection. Recall that $\mu: \mathbb{G}_m \rightarrow \mathbf{G}$ is the weight cocharacter. This extends to a cocharacter of $\mathcal{G}$, which we denote by the same notation μ .

Proposition 5.16. *There exists a positive integer D divisible by D_0 , such that, for every prime $\ell \nmid D$, both*

$$(\rho_v^{\text{int}})_{\overline{\mathbb{Z}_\ell}} \quad \text{and} \quad \rho_{\ell, v}^{\text{MT, int}}$$

are pure with respect to μ in the sense of Definition 3.12.

Proof. We want to apply Lemma 3.14. By the preceding results (and the corresponding known results for abelian varieties), the generic fibers are Frobenius semisimple and pure with respect to μ . Condition (d) implies that the Frobenius-weight cocharacters are defined over $\overline{\mathbb{Z}_\ell}$ for every $\ell \nmid D_0$. Moreover, the scheme-theoretic centralizers of the corresponding actions of W_{F_v} are smooth over $\overline{\mathbb{Z}_\ell}$ by Proposition 5.13. Thus conditions (1)–(3) of Definition 3.12 hold.

Let $\mathfrak{g} := \text{Lie}(\mathcal{G})$. Since ρ_v is defined over $\overline{\mathbb{Q}}$, it is easy to see that there exists a positive integer D divisible by D_0 , such that, for every prime $\ell \nmid D$, the isotypic decomposition of $\mathfrak{g}_{\overline{\mathbb{Q}_\ell}}$ for the action of $r_v^{\text{int}}(W_{F_v})$ is defined over $\overline{\mathbb{Z}_\ell}$. By Proposition 5.13, the same holds for $r_{\ell, v}^{\text{MT, int}}$.

It remains to show that for all but finitely many ℓ , the cokernels of

$$d_1: \mathfrak{g}_{\overline{\mathbb{Z}}_\ell} \longrightarrow \mathfrak{g}_{\overline{\mathbb{Z}}_\ell}, \quad Y \mapsto [Y, N_v^{\text{int}}]$$

and

$$d_2: \mathfrak{g}_{\overline{\mathbb{Z}}_\ell} \longrightarrow \mathfrak{g}_{\overline{\mathbb{Z}}_\ell}, \quad Y \mapsto [Y, N_{\ell,v}^{\text{MT,int}}]$$

are torsion-free. Since ρ_v is defined over $\overline{\mathbb{Q}}$, one can easily check the assertion for d_1 . For d_2 , we note that the natural $\mathbf{G}$ -equivariant inclusion

$$\mathfrak{g}_{\mathbb{Q}} \hookrightarrow \bigoplus_i \text{End}(H_B^i(X_{\mathbb{C}}, \mathbb{Q}))$$

splits since $\mathbf{G}$ is reductive. After enlarging D , we may assume that the inclusion and splitting extend over $\mathbb{Z}[1/D]$. Under the étale–Betti comparison, this splitting is also G_{F_v} -equivariant. Now [Proposition 5.14](#) implies that after enlarging D further, the cokernel of d_2 is torsion-free for every $\ell \nmid D$.

All the hypotheses of [Lemma 3.14](#) are now satisfied for both representations. $\square$

Theorem 5.17. *There exists a positive integer D divisible by D_0 , such that, for every prime $\ell \nmid D$, we have*

$$(\rho_v^{\text{int}})_{\overline{\mathbb{Z}}_\ell} \sim_{\mathcal{G}(\overline{\mathbb{Z}}_\ell)} \rho_{\ell,v}^{\text{MT,int}}.$$

Proof. By [Proposition 5.16](#), we may choose D divisible by D_0 such that both representations are pure with respect to μ for every $\ell \nmid D$. Their generic fibers are Frobenius semisimple, and the representations $(r_v^{\text{int}})_{\overline{\mathbb{Z}}_\ell}$ and $r_{\ell,v}^{\text{MT,int}}$ are $\mathcal{G}(\overline{\mathbb{Z}}_\ell)$ -conjugate by [Proposition 5.13](#). The assertion therefore follows from [Proposition 3.13](#). $\square$

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AI declarations. This project began in early 2026, and most of the results presented here, including the weight–monodromy conjecture and the strong compatibility, were obtained before August 2026 without any assistance from generative AI tools. Thereafter, Codex and Claude Code were used to assist with exposition, literature searches, statement simplification, and proof audits. The uses described below bear on the mathematical content.

The second author arrived at the statement of [Lemma 2.1](#) after completing his project with Zhichao Tang [\[TZ26\]](#), and used it to rule out the case in which the unipotent radical is isomorphic to the additive group $\mathbb{G}_a$; at the time he did not realize that a full proof of semisimplicity was within reach. The reduction of a general unipotent quotient to the case of $\mathbb{G}_a$, given in [Proposition 2.2](#), was pointed out by ChatGPT 5.6 Sol in August 2026, while he was asking about an unrelated general fact on algebraic groups during the Algebraic Geometry Summer School at SCMS, Shanghai. With the Mumford–Tate conjecture then available in full, rather than only in its semisimple version, several technical workarounds could be dropped. In preparing [Section 5](#), the first author used ChatGPT 6 Astra to help clarify how to control the finite extensions of base fields required in the argument. In particular, the proof of [Proposition 5.7](#) and the example in [Example 5.11](#) were found with its help.

All AI outputs bearing on the mathematical content were verified independently by the authors.

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