Logic and Set Theory Universität Bielefeld
Problem Set 1 SS 2019

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1. Exemplarily verify some of the following statements:

(a) For every boolean ring R and $u, v, x, y, z \in R$ we have:

(i) $\wedge, \vee, +$ are associative.
(ii) $\wedge, \vee$ are idempotent.
(iii) $\wedge, \vee, +, \leftrightarrow, \uparrow, \downarrow$ are commutative.
(iv) $x \wedge 0 = 0$, $x \wedge 1 = x$ and $x \vee 0 = x$, $x \vee 1 = 1$.
(v) $\wedge, \vee$ are *mutually absorptive*: $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.
(vi) *De Morgan's laws hold*: $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$.
(vii) $x \to y = \neg y \to \neg x$.
(viii) $x \leftrightarrow y = 1 \Leftrightarrow x = y$.
(ix) $(x \wedge y) \to z = x \to (y \to z)$.
(x) $(x \wedge \neg y) \vee (\neg x \wedge y) = x + y = (x \vee y) \wedge \neg(x \wedge y)$.
(xi) $x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \wedge \neg y = 0 \;\Leftrightarrow\; y \mid x$.
(xii) $(u \leq x \text{ and } v \leq y) \;\Rightarrow\; (u \wedge v \leq x \wedge y \text{ and } u \vee v \leq x \vee y)$.
(xiii) $(x \wedge y = 0 \text{ and } x \vee y = 1) \;\Rightarrow\; y = \neg x$.

(b) A boolean ring R is a *boolean algebra* (= *complemented distributive lattice*) w.r.t. $(\leq, 0, 1, \neg, \wedge, \vee)$, i.e. for all $x, y, z \in R$:

(I) $\leq$ is a partial order on R with least element 0 and greatest element 1.
(II) $x \wedge y$ is the infimum and $x \vee y$ the supremum of $\{x, y\}$ w.r.t. $\leq$.
(III) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
(IV) $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

(c) Conversely, every boolean algebra $(R, \leq, 0, 1, \neg, \wedge, \vee)$ yields a boolean ring R with addition and multiplication given for $x, y \in R$ by:

$$\begin{aligned} x + y &= (x \wedge \neg y) \vee (\neg x \wedge y) \\ x \cdot y &= x \wedge y \end{aligned}$$

2. Prove that the set R' of idempotent elements in any commutative ring R forms a boolean ring with addition $x +' y = (x - y)^2$ and multiplication $x \cdot' y = x \cdot y$.

3. Let R be a boolean ring and denote by $\operatorname{Spec}(R)$ the set of all prime ideals in R. Prove *Stone's Representation Theorem*, which states that the map

$$\begin{aligned} R &\longrightarrow \mathcal{P}(\operatorname{Spec}(R)) \\ x &\longmapsto D_x = \{\mathfrak{p} \in \operatorname{Spec}(R) : x \notin \mathfrak{p}\} \end{aligned}$$

is an injective homomorphism of boolean rings. Conclude that the boolean rings are precisely the rings that are isomorphic to a unitary subring of ${}^X\mathbb{F}_2$ for some set X.

4. Let X be a topological space. A subset U of X is said to be *regular* if it coincides with the interior of its closure, i.e. formally $U = (\overline{U})^\circ$.

Prove that the set $\mathcal{R}(X)$ of regular subsets of X becomes a complete boolean algebra under the operations $\neg U = X \setminus \overline{U}$ and $U \wedge V = U \cap V$ and $U \vee V = \neg\neg(U \cup V)$.