Logic and Set Theory — Universität Bielefeld
Problem Set 2 — SS 2019

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1. The goal of this exercise is to give an alternative, direct proof of the Compactness Theorem that does not rely on Gödel's Completeness Theorem.

The *product* $\prod_{i\in I}\mathcal{M}_i$ of a non-empty family $(\mathcal{M}_i)_{i\in I}$ of boolean-valued S-structures with common set of assigned variables is the boolean-valued S-structure $\mathcal{M}$ with the same set of assigned variables and $\underline{\mathcal{M}} = \prod_{i\in I}\underline{\mathcal{M}_i}$ and $R^{\mathcal{M}} = \prod_{i\in I} R^{\mathcal{M}_i}$ and

$$\sharp^{\mathcal{M}}(m^1,\dots,m^r) = \big(\sharp^{\mathcal{M}_i}(m_i^1,\dots,m_i^r)\big)_{i\in I}.$$

(a) Assume that $\mathcal{M}_i$ has witnesses for every $i\in I$. Prove *Łoś Theorem*, which states that for all maximal ideals $\mathfrak{m}$ in $R^{\mathcal{M}}$ and S-sentences π

$$\mathcal{M}/\mathfrak{m}\vDash\pi \;\Leftrightarrow\; \big(\pi^{\mathcal{M}_i}\big)_{i\in I}\notin\mathfrak{m}\,.$$

(b) Deduce the *Compactness Theorem*, which states that an S-theory T is satisfiable if and only if every finite subset of T is satisfiable.

(c) Conclude for all S-theories T and S-formulas φ that $T\vDash\varphi$ if and only if $T'\vDash\varphi$ for some finite subset T' of T.

Hint for (b): Take for I the set of all finite subsets of T.

2. Let T be an S-theory with equality $\equiv$ that admits an infinite $\equiv$-respecting model. Show that for every set X there is a $\equiv$-respecting model $\mathcal{M}$ of T with $X\subseteq\underline{\mathcal{M}}$.

Hint: Apply the Compactness Theorem to an S'-theory extending T where S' is a vocabulary obtained from S by adding for each element of X a constant symbol.

For the next two exercises, we view rings R as $\equiv$-respecting S^{Ring}-structures $\tilde{R}$ with

$$\mathtt{0}^{\tilde{R}} = 0\,,\quad \mathtt{1}^{\tilde{R}} = 1\,,\quad \oplus^{\tilde{R}} = +\,,\quad \odot^{\tilde{R}} = \,\cdot\,.$$

3. We want to see that being noetherian is not a *first-order property* for rings.

(a) Show that there is a non-noetherian commutative ring R with the property $\tilde{R}\vDash\varphi$ for all S^{Ring}-sentences φ with $\tilde{\mathbb{Z}}\vDash\varphi$.

(b) Conclude that there cannot exist an S^{Ring}-sentence φ with the property that for all commutative rings R we have $\tilde{R}\vDash\varphi$ if and only if R is noetherian.

Hint for (a): Use the Compactness Theorem to obtain such a ring R with a non-zero element that for all n is divisible by the product of the first n positive integers.

4. The *Lefschetz principle* for S^{Ring}-sentences φ says the following are equivalent:

(1) $\tilde{K}\vDash\varphi$ for some algebraically closed field K of characteristic 0.
(2) $\tilde{K}\vDash\varphi$ for every algebraically closed field K of characteristic 0.
(3) There exist infinitely many primes p such that there is an algebraically closed field K of characteristic p with $\tilde{K}\vDash\varphi$.

Prove the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Hint for (3) $\Rightarrow$ (1): Take an infinite set I of primes such that there exist algebraically closed fields K_i of characteristic $i\in I$ with $\tilde{K}_i\vDash\varphi$. Then choose a maximal ideal $\mathfrak{m}$ in ${}^I\mathbb{F}_2$ that contains the characteristic functions χ_i of all elements i of I.