Logic and Set Theory Universität Bielefeld
Problem Set 3 SS 2019

For S-formulas φ with exactly one free variable x and S-terms t let $\varphi(t)$ be $\varphi(x/t)$.

1. This exercise treats the abstract version of Gödel's 2^{nd} Incompleteness Theorem.

Let T be a consistent S-theory such that there exists a function that associates with each S-sentence φ a variable-free S-term $\ulcorner\varphi\urcorner$ with the following two properties:

(i) EXISTENCE OF FIXED POINTS. For every S-formula ψ with exactly one free variable there exists an S-sentence φ with

$$T \vdash (\varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner)) \; .$$

(ii) ENCODABILITY OF PROVABILITY. There exists an S-formula Pr with exactly one free variable such that

$$T \vdash \varphi \;\; \Rightarrow \;\; T \vdash \mathrm{Pr}(\ulcorner\varphi\urcorner) \; .$$

(iii) FORMAL ENCODABILITY OF PROVABILITY. For every S-sentence φ

$$T \vdash (\mathrm{Pr}(\ulcorner\varphi\urcorner) \rightarrow \mathrm{Pr}(\ulcorner \mathrm{Pr}(\ulcorner\varphi\urcorner)\urcorner)) \; .$$

(iv) FORMALIZATION RESPECTS IMPLICATION. For all S-sentences φ and ψ

$$T \vdash (\mathrm{Pr}(\ulcorner(\varphi \rightarrow \psi)\urcorner) \rightarrow (\mathrm{Pr}(\ulcorner\varphi\urcorner) \rightarrow \mathrm{Pr}(\ulcorner\psi\urcorner))) \, .$$

(a) Prove *Löb's Theorem*, which states for every S-sentence φ

$$T \vdash (\mathrm{Pr}(\ulcorner\varphi\urcorner) \rightarrow \varphi) \;\; \Rightarrow \;\; T \vdash \varphi \, .$$

(b) Deduce the *Second Incompleteness Theorem*, which states $T \nvdash \neg\,\mathrm{Pr}(\ulcorner\bot\urcorner)$.

Hint for (b): Apply (i) to $(\mathrm{Pr}(x) \rightarrow \varphi)$.

2. (a) Give a PA-proof of the S^{Peano}-sentence $\mathtt{1} \not\equiv \mathtt{2}$ where $\mathtt{1} = \mathtt{S0}$ and $\mathtt{2} = \mathtt{SS0}$.

(b) Give a CRT-proof of the S^{Ring}-formula $\big(\bigwedge_x (e \odot x) \equiv x \rightarrow e \equiv \mathtt{1}\big)$ with $e \neq x$.

3. Decide for each of the axioms $\varphi \in \mathrm{ZFC} \setminus \mathsf{REP}$ whether $\hat{\mathbb{N}} \vDash \varphi$ and whether $\hat{\mathbb{R}} \vDash \varphi$ with strict posets $(X, <)$ viewed as S^{Set}-structures $\hat{X}$ via $\underline{\hat{X}} = X$ and $\epsilon^{\hat{X}} = <$.

4. Let $(X, \leq)$ be a *complete lattice*, i.e. a partially ordered set in which each subset has both an infimum and a supremum, and let f be an endomorphism of $(X, \leq)$.

Prove the *Knaster–Tarski Theorem*, which states that the set of fixed points of f

$$X^f \; = \; \{x \in X : f(x) = x\}$$

forms itself a complete lattice with respect to $\leq$, so in particular is non-empty.

Hint: Prove as a first step that f has a least fixed point and a greatest fixed point. Then apply this knowledge to suitable complete sublattices of $(X, \leq)$.