Logic and Set Theory Universität Bielefeld
Problem Set 4 SS 2019

Let $\mathcal{M}$ be a ZF°-universe and let $\mathrel{\underline{\in}}$ be the relation on $M = \underline{\mathcal{M}}$ given by $\epsilon^{\mathcal{M}}$.

1. Show that the following properties are all equivalent for transitive $\mathcal{M}$-sets γ on which $\mathrel{\underline{\in}}$ is well-founded:

(1) γ is an $\mathcal{M}$-ordinal.
(2) $\sqsubseteq$ totally orders γ.
(3) β is transitive for all $\beta \mathrel{\underline{\in}} \gamma$.
(4) Exactly one of $\alpha \mathrel{\underline{\in}} \beta$, $\alpha = \beta$, $\beta \mathrel{\underline{\in}} \alpha$ holds for all $\alpha, \beta \mathrel{\underline{\in}} \gamma$.

2. Verify some of the following statements for $\alpha, \beta, \gamma, \delta \in \mathbb{O}^{\mathcal{M}}$ and $\diamond \in \{+, \cdot\}$:

(i) $\alpha + \underline{0} = \alpha \ \& \ \alpha \cdot \underline{0} = \underline{0} \ \& \ \alpha^{(\underline{0})} = \underline{1}$
(ii) $\underline{0} + \alpha = \alpha \ \& \ \underline{0} \cdot \alpha = \underline{0} \ \& \ \underline{1} \cdot \alpha = \alpha$
(iii) $\alpha \cdot \underline{1} = \alpha \ \& \ \alpha^{(\underline{1})} = \alpha \ \& \ \underline{1}^{(\alpha)} = \underline{1}$
(iv) $\underline{0} < \alpha \implies \underline{0}^{(\alpha)} = \underline{0}$
(v) $(\alpha \diamond \beta) \diamond \gamma = \alpha \diamond (\beta \diamond \gamma)$
(vi) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
(vii) $\alpha^{(\beta+\gamma)} = \alpha^{(\beta)} \cdot \alpha^{(\gamma)}$
(viii) $(\alpha^{(\beta)})^{(\gamma)} = \alpha^{(\beta \cdot \gamma)}$
(ix) $\alpha \le \beta \ \& \ \gamma \le \delta \implies \alpha \diamond \gamma \le \beta \diamond \delta$
(x) $\alpha \le \beta \ \& \ \gamma < \delta \implies \alpha + \gamma < \beta + \delta$
(xi) $\underline{0} < \alpha \le \beta \ \& \ \gamma < \delta \implies \alpha \cdot \gamma < \beta \cdot \delta$
(xii) $\underline{0} < \alpha \le \beta \ \& \ \gamma \le \delta \implies \alpha^{(\gamma)} \le \beta^{(\delta)}$
(xiii) $\underline{1} < \alpha \le \beta \ \& \ \gamma < \delta \implies \alpha^{(\gamma)} < \beta^{(\delta)}$
(xiv) $\alpha + \beta = \alpha + \gamma \implies \beta = \gamma$
(xv) $\underline{0} < \alpha \ \& \ \alpha \cdot \beta = \alpha \cdot \gamma \implies \beta = \gamma$
(xvi) $\underline{1} < \alpha \ \& \ \alpha^{(\beta)} = \alpha^{(\gamma)} \implies \beta = \gamma$

Assuming $\mathbb{N}^{\mathcal{M}} \neq \mathbb{O}^{\mathcal{M}}$, give counterexamples for each of the following claims:

(I) $\alpha + \beta = \beta + \alpha$
(II) $\alpha \cdot \beta = \beta \cdot \alpha$
(III) $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$
(IV) $(\alpha \cdot \beta)^{(\gamma)} = \alpha^{(\gamma)} \cdot \beta^{(\gamma)}$
(V) $\alpha < \beta \ \& \ \gamma \le \delta \implies \alpha + \gamma < \beta + \delta$
(VI) $\alpha < \beta \ \& \ \underline{0} < \gamma \le \delta \implies \alpha \cdot \gamma < \beta \cdot \delta$
(VII) $\alpha < \beta \ \& \ \underline{0} < \gamma \le \delta \implies \alpha^{(\gamma)} < \beta^{(\delta)}$
(VIII) $\alpha + \beta = \gamma + \beta \implies \alpha = \gamma$
(IX) $\underline{0} < \beta \ \& \ \alpha \cdot \beta = \gamma \cdot \beta \implies \alpha = \gamma$
(X) $\alpha^{(\beta)} = \gamma^{(\beta)} \implies \alpha = \gamma$

For $\gamma, \delta \in \mathbb{O}^{\mathcal{M}}$ we call γ a *left divisor* and δ a *right divisor* of the $\mathcal{M}$-ordinal $\gamma \cdot \delta$. If an $\mathcal{M}$-ordinal is a left (resp. right) divisor of each element of an $\mathcal{M}$-class $C \subseteq \mathbb{O}^{\mathcal{M}}$, then γ is said to be a *common left* (resp. *right*) *divisor* of C.

3. Prove the following for $\alpha, \beta \in \mathbb{O}^{\mathcal{M}}$:

(a) If $\alpha \le \beta$, there is a unique $\beta - \alpha \in \mathbb{O}^{\mathcal{M}}$ with $\beta = \alpha + (\beta - \alpha)$.
(b) If $\beta \neq \underline{0}$, there are unique $\gamma, \delta \in \mathbb{O}^{\mathcal{M}}$ with $\alpha = \beta \cdot \gamma + \delta$ and $\delta < \beta$.
(c) If an $\mathcal{M}$-ordinal is a left divisor of α and $\alpha + \beta$, then also of β.
(d) Every non-empty $\mathcal{M}$-subclass of $\mathbb{O}^{\mathcal{M}} \setminus \{\underline{0}\}$ has a greatest common left divisor and a greatest common right divisor.

4. Define $\mathcal{M}$-relations $<$ on $\alpha \mathbin{\dot\sqcup} \beta$ and $\alpha * \beta$ and, after proving its existence, also on

$$(\beta \rightarrowtriangle \alpha) = \big[f \in \{\beta \rightarrowtriangle \alpha\} : [\gamma < \beta : f(\gamma) \neq \underline{0}] \text{ is finite} \big]$$

for all $\mathcal{M}$-ordinals α, β such that there exist "natural" order-preserving $\mathcal{M}$-bijections with order-preserving inverses

$$\alpha \mathbin{\dot\sqcup} \beta \rightarrowtriangle \alpha + \beta\,, \qquad \alpha * \beta \rightarrowtriangle \alpha \cdot \beta\,, \qquad (\beta \rightarrowtriangle \alpha) \rightarrowtriangle \alpha^{(\beta)}\,.$$

Remark: You may use in 3. and 4. (without proof) that non-empty finite $\mathcal{M}$-subsets of $\mathcal{M}$-ordinals have maxima.