Logic and Set Theory Universität Bielefeld
Problem Set 5 SS 2019

Let $\mathcal{M}$ be a $(\mathrm{ZF}^\circ \cup \mathtt{INF})$-universe. Denote by $\sqsubseteq$ the relation on $M = \underline{\mathcal{M}}$ given by $\epsilon^{\mathcal{M}}$ and denote by $\mathbb{O}$ the $\mathcal{M}$-class consisting of all $\mathcal{M}$-ordinals.

A *cumulative $\mathcal{M}$-hierarchy* is by definition a sequence $\langle V_\alpha \rangle_{\alpha \in \mathbb{O}}$ of $\mathcal{M}$-sets with the properties $V_{\beta - \underline{1}} \sqsubseteq V_\beta$ for all $\beta \in \mathbb{O}_{+\underline{1}}$ and $V_\beta = \bigsqcup V[\beta]$ for all $\beta \in \mathbb{O}_{\lim}$.

1. Let $\langle V_\alpha \rangle_{\alpha \in \mathbb{O}}$ be a cumulative $\mathcal{M}$-hierarchy and $\mathbb{V} = \bigcup_{\alpha \in \mathbb{O}} \mathbb{V}_\alpha$ with $\mathbb{V}_\alpha = \sqsubseteq^{-1}(V_\alpha)$.

Prove the *Reflection Principle*, which states that for every S^{Set}-sentence π there is a closed and unbounded $\mathcal{M}$-class C_π in $\mathbb{O}$ such that for every $\alpha \in C_\pi$

$$\mathcal{M}|_{\mathbb{V}} \vDash \pi \ \Leftrightarrow\ \mathcal{M}|_{\mathbb{V}_\alpha} \vDash \pi\,.$$

Hint: Convince yourself in a first step that for every $\mathcal{M}$-class function $f\colon \mathbb{O} \to \mathbb{O}$ the $\mathcal{M}$-class $\{\alpha \in \mathbb{O} : f(\gamma) < \alpha \text{ for all } \gamma < \alpha\}$ is closed and unbounded in $\mathbb{O}$.

2. Let $\prec$ be a set-like $\mathcal{M}$-class relation $\prec$ on C and call $\mathcal{M}$-class functions $r\colon C \to \mathbb{O}$ *$\prec$-ranking* if the strict inequality $r(X) < r(Y)$ holds for all $X, Y \in C$ with $X \prec Y$.

Prove:

(a) $\prec$ is well-founded on C iff there exists a $\prec$-ranking $\mathcal{M}$-class function r on C.

Assume from now on that $\prec$ is well-founded on C. Prove the following:

(b) The rank $\mathrm{rk}_{C,\prec}$ is the unique $\prec$-ranking $\mathcal{M}$-class function r on C such that for all $Y \in C$ and $\alpha < r(Y)$ there exists some $X \sqsubseteq C_{\preceq^\infty Y}$ with $r(X) = \alpha$.

(c) If $\prec$ is transitive on C, then the transitive collapse and rank on C w.r.t $\prec$ coincide, i.e. $t_{C,\prec} = \mathrm{rk}_{C,\prec}$.

Compute transitive collapse $t_{X,\sqsubseteq}$ and rank $\mathrm{rk}_{X^\infty,\sqsubseteq}$ for the following $\mathcal{M}$-sets X:

(d) $X = \langle \underline{2}, \underline{1} \rangle$.

(e) $X = \underline{2} * \underline{1}$.

(f) $X = \mathsf{P}(\underline{3})$.

3. Let $\mathbb{HF}$ be the $\mathcal{M}$-class consisting of all well-founded $\mathcal{M}$-sets X with $\mathrm{rk}(X) \in \mathbb{N}^{\mathcal{M}}$. Prove that $\mathcal{M}|_{\mathbb{HF}}$ is a $(\mathrm{ZF}^\circ \cup \mathtt{POW} \cup \mathtt{CHO} \cup \mathtt{REG})$-universe, i.e. it satisfies all ZFC axioms except Infinity.

4. Prove that every $\mathcal{M}$-set is well-orderable if for every infinite $\mathcal{M}$-set X there is an injective $\mathcal{M}$-function $X * X \rightarrowtail X$. Conclude that ZF-universes $\mathcal{M}$ satisfy Choice if and only if $X * X \simeq X$ for all infinite $\mathcal{M}$-sets X.