Logic and Set Theory Universität Bielefeld
Problem Set 6 SS 2019

Let $\mathcal{M}$ be a ZFC^{-}-universe and denote by $\sqsubseteq$ the relation on $M = \underline{\mathcal{M}}$ given by $\epsilon^{\mathcal{M}}$ and abbreviate as usual $\mathbb{W} = \mathbb{W}^{\mathcal{M}}$ and $\omega = \omega^{\mathcal{M}}$. Moreover, for every $\mathcal{M}$-cardinal κ, write $\mathbb{H}_\kappa$ for the $\mathcal{M}$-class consisting of all well-founded $\mathcal{M}$-sets X with $|X^\infty| < \kappa$.

Define $\beth = \mathrm{It}_{\mu \mapsto \underline{2}^\mu, \omega}$ and $\lambda^{<\kappa} = \bigsqcup_{|\mu| = \mu < \kappa} \lambda^\mu$ for $\mathcal{M}$-cardinals λ and κ..

1. Let $\kappa > \omega$ be an $\mathcal{M}$-cardinal. Prove the following:

(a) $\mathbb{H}_\kappa$ forms an $\mathcal{M}$-set of cardinality $\underline{2}^{<\kappa}$.

(b) $\mathbb{W}_\kappa$ forms an $\mathcal{M}$-set of cardinality $\beth_\kappa$.

(c) $\mathbb{H}_\kappa \subseteq \mathbb{W}_\kappa$ with equality if and only if $\kappa = \beth_\kappa$.

(d) $\mathcal{M}|_{\mathbb{H}_\kappa} \vDash \mathtt{EXT} \cup \mathtt{EMP} \cup \mathtt{PAI} \cup \mathtt{UNI} \cup \mathtt{INF} \cup \mathtt{CHO} \cup \mathtt{REG} \cup \mathtt{REP}$ if κ is regular.

(e) $\mathcal{M}|_{\mathbb{W}_\kappa} \vDash \mathtt{EXT} \cup \mathtt{EMP} \cup \mathtt{PAI} \cup \mathtt{UNI} \cup \mathtt{INF} \cup \mathtt{CHO} \cup \mathtt{REG} \cup \mathtt{POW}$.

2. Recursively define for all $n \in \mathbb{N}$ the *n-th derivative* of a normal $\mathcal{M}$-class function f on the $\mathcal{M}$-ordinals as $f^{(n)} = \left(f^{(n-1)}\right)'$ for $n > 0$ and $f^{(0)} = f$.

Furthermore, let us call an $\mathcal{M}$-class C a *Grothendieck universe* in $\mathcal{M}$ if it satisfies

(i) C is transitive,

(ii) $[X, Y] \in C$ for all $X, Y \in C$,

(iii) $\mathsf{P}(X) \in C$ for all $X \in C$,

(iv) $\bigsqcup_{i \sqsubseteq I} X_i$ for all families $\langle X_i \rangle_{i \sqsubseteq I}$ of elements of C with $I \in C$.

(a) Prove for regular $\mathcal{M}$-cardinals $\kappa > \omega$ the equivalence of the following statements:

(1) $\underline{2}^\lambda < \kappa$ for all $\mathcal{M}$-cardinals $\lambda < \kappa$.

(2) $\mu^\lambda < \kappa$ for all $\mathcal{M}$-cardinals $\mu, \lambda < \kappa$.

(3) $\kappa = \beth_\kappa$.

(4) $\kappa = \beth_\kappa^{(n)}$ for all $n \in \mathbb{N}$.

(5) $\mathbb{H}_\kappa = \mathbb{W}_\kappa$.

(6) $\mathbb{W}_\kappa$ is a Grothendieck universe in $\mathcal{M}$.

Regular $\mathcal{M}$-cardinals $\kappa > \omega$ that satisfy these conditions are called *inaccessible*.

(b) Conclude that there exist ZFC-universes without inaccessible cardinals.

3. Show the following:

(a) $\kappa \leq \underline{2}^{<\kappa} \leq (\underline{2}^{<\kappa})^{\mathrm{cof}(\kappa)} = \underline{2}^\kappa$ for every infinite $\mathcal{M}$-cardinal κ.

(b) $\underline{2}^{<\kappa} = \kappa^{<\kappa}$ for every infinite successor $\mathcal{M}$-cardinal κ.

(c) $\mathcal{M}$ satisfies GCH iff $\underline{2}^{<\kappa} = \kappa$ for every infinite $\mathcal{M}$-cardinal κ.

(d) If κ is a singular $\mathcal{M}$-cardinal such that there is some $\mathcal{M}$-cardinal $\lambda < \kappa$ with the property $\underline{2}^\mu = \underline{2}^\lambda$ for all $\mathcal{M}$-cardinals μ with $\lambda \leq \mu < \kappa$, then $\underline{2}^\kappa = \underline{2}^\lambda$.

4. Prove that for every infinite $\mathcal{M}$-cardinal κ there exists an $\mathcal{M}$-set $X \sqsubseteq \mathsf{P}(\kappa)$ with $|X| = \kappa^+$ such that $|U| = |V| = \kappa$ and $|U \sqcap V| < \kappa$ for all $U, V \sqsubseteq X$ with $U \neq V$.