**1.** (a) Let $R$ be a boolean ring and $u, v, x, y, z \in R$.

(i) $\wedge = \cdot$ and $+$ are associative by definition of a ring and together with the law of distributivity and the commutativity of $+$ we compute for $\vee$:

$$(x \vee y) \vee z \;=\; x + y + z + xy + xz + yz + xyz \;=\; x \vee (y \vee z)$$

(ii) $\wedge = \cdot$ is idempotent by definition of a boolean ring and using $2 = 0$ we get:

$$x \vee x \;=\; 2x + x^2 \;=\; 0 + x \;=\; x$$

(iii) $+$ is commutative by definition of a ring and $\wedge = \cdot$ by Lemma 1.1.3. This implies the commutativity of $\vee$ and $\leftrightarrow$, which in turn yields that of $\uparrow$ and $\downarrow$.

(iv) Clearly, $x \wedge 0 = 0$, $x \wedge 1 = x, x \vee 0 = x$ and $x \vee 1 = 2x + 1 = 1$, since $2 = 0$.

(v) Using the idempotence of $\wedge = \cdot$ and $2 = 0$:

$$\begin{aligned} x \wedge (x \vee y) &= x(x + y + xy) = x + 2xy = x \\ x \vee (x \wedge y) &= \;x + xy + x^2 y\; = x + 2xy = x \end{aligned}$$

(vi) Using $2 = 0$:

$$\begin{aligned} \neg(x \wedge y) &= xy + 1 + 2(x + y + 1) = \neg x \vee \neg y \\ \neg(x \vee y) &= \qquad x + y + xy + 1 \qquad = \neg x \wedge \neg y \end{aligned}$$

(vii) Using $\neg\neg y = y$ (because of $2 = 0$) and the commutativity of $\vee$:

$$x \to y \;=\; \neg x \vee y \;=\; \neg\neg y \vee \neg x \;=\; \neg y \to \neg x$$

(viii) Using the idempotence of $\wedge = \cdot$ and $2 = 0$ we compute

$$x \leftrightarrow y \;=\; (\neg x \vee y)(\neg y \vee x) \;=\; (xy + x + 1)(xy + y + 1) \;=\; x + y + 1\,,$$

such that $x \leftrightarrow y = 1 \Leftrightarrow x + y = 0 \Leftrightarrow x = y$.

(ix) Using (vi) and the associativity of $\vee$:

$$(x \wedge y) \to z \;=\; \neg(x \wedge y) \vee z \;=\; \neg x \vee \neg y \vee z \;=\; x \to (y \to z)$$

(x) Using the idempotence of $\wedge = \cdot$ and $2 = 0$:

$$\begin{aligned} (x \wedge \neg y) \vee (\neg x \wedge y) &= \;(xy + x) \vee (xy + y)\; = x + y + 6xy = x + y \\ (x \vee y) \wedge \neg(x \wedge y) &= (x + y + xy)(xy + 1) = x + y + 4xy = x + y \end{aligned}$$

(xi) Using $x \to y = xy + x + 1$, $x \wedge y = xy$, $x \vee y = xy + x + y$, $x \wedge \neg y = xy + x$ and $2 = 0$ it is easy to see:

$$x \le y \;\Leftrightarrow\; x \to y = 1 \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \wedge \neg y = 0$$

Clearly, $xy = x \Rightarrow y \mid x$. Conversely, let's assume now $x = uy$ for some $u$. From $(xuy)xy = xuy$ and $(xy)x = xy$ we get $x = xuy \le xy \le x$. Hence, we have $x \leftrightarrow xy = (x \to xy)(xy \to x) = 1$ such that $x = xy$ by (viii), so $x \le y$.

(xii) Let $u \le x$ and $v \le y$. By (xi) this means $u \wedge x = u$ and $v \wedge y = v$ such that $(u \wedge v) \wedge (x \wedge y) = u \wedge v$ by associativity of $\wedge$, i.e. $u \wedge v \le x \wedge y$. Similarly, $u \vee x = x$ and $v \vee y = y$ such that $(u \vee v) \vee (x \vee y) = x \vee y$, i.e. $u \vee v \le x \vee y$.

(xiii) If $x \wedge y = 0$ and $x \vee y = 1$, then $y = x + x \wedge y + x \vee y = x + 1 = \neg x$.

(b) Let $R$ be a boolean ring. Then we have:

(I) (xi) and (iv) yield $0 \leq x \leq 1$.
We have $x \leq x$ because of $x \wedge x = x$ in view of (ii) and (xi).
If $x \leq y$ and $y \leq x$, then $x \leftrightarrow y = 1$, so $x = y$ by (viii).
If $x \leq y \leq z$, then $x = x \vee 0 \leq y \vee z = z$ by (iv), (xii), and (xi).

(II) By (v) and (xi) $x \wedge y$ is a lower bound and $x \vee y$ an upper bound of $\{x, y\}$.
If $z \leq x$ and $z \leq y$, then $z = z \wedge z \leq x \wedge y$ by (xii) and (ii).
If $x \leq z$ and $y \leq z$, then $x \vee y \leq z \vee z = z$ by (xii) and (ii).

(III) Using the idempotence of $\wedge = \cdot$ and $2 = 0$:

$$\begin{aligned} x \wedge (y \vee z) &= xy + xz + xyz = (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= \ x + yz + xyz \ = (x \vee y) \wedge (x \vee z) \end{aligned}$$

(IV) Using the idempotence of $\wedge = \cdot$ and $2 = 0$:

$$\begin{aligned} x \wedge \neg x &= \qquad x(x+1) \qquad = \quad 2x \quad = 0 \\ x \vee \neg x &= x + (x+1) + x(x+1) = 4x + 1 = 1 \end{aligned}$$

(c) Let $(R, \leq, 0, 1, \neg, \wedge, \vee)$ be a boolean algebra and define $\cdot = \wedge$ and $+$ as follows:

$$x + y \quad = \quad (x \wedge \neg y) \vee (\neg x \wedge y)$$

It is clear that $\cdot = \wedge$ and $\vee$ are commutative and idempotent. They are associative, since for $\diamond = \wedge$ resp. $\diamond = \vee$ both $(x \diamond y) \diamond z$ and $x \diamond (y \diamond z)$ are easily checked to be the infimum resp. supremum of $\{x, y, z\}$. We will use the following observations:

**Lemma.**

*(1) If $x \wedge y = 0$ and $x \vee y = 1$, then $y = \neg x$.*
*(2) $\neg(x + y) = (\neg x \wedge \neg y) \vee (x \wedge y)$.*
*(3) $\neg(x \wedge y) = \neg x \vee \neg y$.*

*Proof.* (1) We calculate $y = y \wedge 1 = y \wedge (x \vee \neg x) = (x \wedge y) \vee (y \wedge \neg x) = y \wedge \neg x \leq \neg x$. Interchanging the roles of $y$ and $\neg x$, we get $\neg x \leq y$. Hence, $y = \neg x$.

For (2) choose $z = (\neg x \wedge \neg y) \vee (x \wedge y)$ and $w = x + y$. For (3) choose $z = \neg x \vee \neg y$ and $w = x \wedge y$. In both cases $z \wedge w = 0$ and $z \vee w = 1$ such that we can use (1). □

Now it is easy to see that $(R, +, \cdot)$ is a ring:

- The associativity of $\cdot$ and $1 \cdot x = x = x \cdot 1$ show that $(R, \cdot)$ is a monoid.
- The commutativity of $\wedge$ and $\vee$ yields the commutativity of $+$. Since $\neg 0 = 1$ by (1), we have $x + 0 = (x \wedge 1) \vee (\neg x \wedge 0) = x \vee 0 = x$. By a straightforward calculation using (2) both $(x + y) + z$ and $x + (y + z)$ are equal to

$$(x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z) \vee (x \wedge y \wedge z)\,,$$

  proving the associativity of $+$. Finally, $x + x = 0 \vee 0 = 0$, i.e. each element of $R$ is its own inverse w.r.t. $+$. This shows that $(R, +)$ is an abelian group.
- Using (3) for the first identity we get the distributive law

$$\begin{aligned} xy + xz &= ((x \wedge y) \wedge (\neg x \vee \neg z)) \vee ((\neg x \vee \neg y) \wedge (x \wedge z)) \\ &= \qquad (x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge z) \qquad = x(y + z)\,. \end{aligned}$$

**2.** Clearly, $0, 1 \in R'$. Let $x, y, z \in R'$. The calculation

$$(x-y)^4 \;=\; x - 4xy + 6xy - 4xy + y \;=\; x - 2xy + y \;=\; (x-y)^2$$

shows that $x +' y = (x-y)^2 = x + y - 2xy$ yields an operation $+'$ on $R'$. It has 0 as neutral element, satisfies $x +' x = 0$, is commutative and also associative because of

$$(x +' y) +' z \;=\; x + y + z - 2(xy + xz + yz) + 4xyz \;=\; x +' (y +' z)\,.$$

Thus $(R', +')$ is an abelian group. Given that $\cdot$ is commutative, we see that $x \cdot' y = x \cdot y$ induces an operation $\cdot'$ on $R'$. It only remains to check the distributive law:

$$x \cdot' (y +' z) \;=\; x(y-z)^2 \;=\; (xy - xz)^2 \;=\; x \cdot' y +' x \cdot' z$$

**3.** We have $D_{xy} = D_x \cdot D_y$, since $xy \notin \mathfrak{p} \Leftrightarrow (x \notin \mathfrak{p} \text{ and } y \notin \mathfrak{p})$ for prime ideals $\mathfrak{p}$.

For $D_{x+y} = D_x + D_y$ we have to verify $x + y \notin \mathfrak{p} \Leftrightarrow (x \notin \mathfrak{p}, y \in \mathfrak{p} \text{ or } y \notin \mathfrak{p}, x \in \mathfrak{p})$ for all prime ideals $\mathfrak{p}$. The implication $\Leftarrow$ is true, since $\mathfrak{p}$ is an ideal. To check $\Rightarrow$ assume now $x + y \notin \mathfrak{p}$. If $x \in \mathfrak{p}$, then necessarily $y \notin \mathfrak{p}$ because $\mathfrak{p}$ is an ideal, so we are done. If we had $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, we would have $\neg x, \neg y \in \mathfrak{p}$ by Lemma 1.1.4, since $\mathfrak{p}$ is a prime ideal, and thus the contradiction $x + y = \neg x + \neg y \in \mathfrak{p}$.

Clearly, $D_1 = \operatorname{Spec}(R) = 1$ because prime ideals are proper.

This shows that $D$ is a ring homomorphism. It remains to verify its injectivity. So let $x \in R$ with $x \neq 0$. Then $\neg x$ is a zero divisor (because of $\neg x \wedge x = 0$) and thus contained in some maximal ideal $\mathfrak{p}$ of $R$. By Lemma 1.1.4 we have $x \notin \mathfrak{p}$, so $D_x \neq 0$.

The final statement readily follows from $\mathcal{P}(X) \cong {}^X\mathbb{F}_2$ (see Remark 1.1.9).

**4.** Let us write $\circ$, $-$, $\setminus$ for the functions that associate with subsets $U \subseteq X$ their interior $\circ U = U^\circ$, their closure $-U = \overline{U}$, and their complement $\setminus U = X \setminus U$.

From general topology we have the following properties for all $U, V \subseteq X$:

(i) $\circ$ and $-$ preserve $\subseteq$ and we have $\circ \subseteq -$ pointwise.
(ii) $\circ\circ = \circ$ and $-- = -$ and $\setminus - = \circ \setminus$ and $\setminus \circ = - \setminus$.
(iii) $\circ(U \cap V) = \circ U \cap \circ V$ and $-(U \cup V) = -U \cup -V$.
(iv) $\circ U \cap -V \subseteq -(U \cap V)$.

For the possibly non-standard fact (iv) just observe that for open $M$ with $M \subseteq U$ and closed $N$ with $U \cap V \subseteq N$ the set $\setminus(M \cap \setminus N)$ is closed and contains $V$.

**Lemma.** *For all $U, V \subseteq X$:*

*(1) $U$ regular $\Leftrightarrow U = \neg\neg U$.*
*(2) $\circ \subseteq \neg\neg$.*
*(3) $\neg\neg\neg\circ = \neg\circ$.*
*(4) $\neg\neg\neg\neg = \neg\neg$.*
*(5) $\circ U \cap \neg\neg V \subseteq \neg\neg(U \cap V)$.*
*(6) $\neg\neg(U \cap V) \subseteq \neg\neg U \cap \neg\neg V$ with equality if either $U$ or $V$ is open.*

*Proof.* (1) Since $\neg = \setminus - = \circ \setminus$ we have $\neg\neg = \circ \setminus \setminus - = \circ -$.

(2) Applying $\setminus - \setminus$ on the left to $\circ \subseteq -$ gives $\circ = \circ\circ = \circ \setminus \setminus \circ = \setminus - \setminus \circ \subseteq \setminus - \setminus - = \neg\neg$.

(3) Applying $\neg$ on the left and $\circ$ on the right to (2) we get $\neg\circ \supseteq \neg\neg\neg\circ$, whereas applying $\neg\circ$ on the right to (2) gives $\neg\circ = \circ\neg\circ \subseteq \neg\neg\neg\circ$.

(4) Apply $\setminus$ on the right to (3) and use $\neg = \circ \setminus$.

(5) Applying $\setminus - \setminus$ to (iv) for the last step yields

$$\circ U \cap \neg\neg V \;=\; \setminus(-\setminus \circ U \cup -\neg V) \;=\; \setminus -(\setminus \circ U \cup \neg V)$$
$$= \setminus - \setminus(\circ U \cap - V) \;\subseteq\; \neg\neg(U \cap V)$$

(6) The inclusion is clear. So without loss of generality let's assume that $U$ is open. Then $U = \circ U$ and $\neg\neg V = \circ \neg\neg V$ such that using (5) twice and then (4) gives

$$\neg\neg U \cap \neg\neg V \;\subseteq\; \neg\neg(U \cap \neg\neg V) \;\subseteq\; \neg\neg\neg\neg(U \cap V) \;=\; \neg\neg(U \cap V)\,.$$

□

As a consequence of the lemma we see using (1) that, if $U$ and $V$ are regular, then so is $\neg U$ by (3) and $U \vee V$ by (4) and $U \wedge V$ by (6).

Thus $\neg$, $\wedge$, $\vee$ are indeed well-defined operations on $\mathcal{R}(X)$.

We next verify the defining properties of a complete boolean algebra:

(I) Since the partial order $\leq$ on $\mathcal{R}(X)$ will be characterized by $U \leq V \Leftrightarrow U \wedge V = U$, we must choose $\leq\, = \,\subseteq$. With this choice it is then obvious that $\leq$ is a partial order with $\emptyset$ as a least element and $X$ as a greatest element.

(II) Clearly, $U \wedge V$ is the infimum of $\{U, V\}$. To check the existence of suprema pick $\mathcal{Y} \subseteq \mathcal{R}(X)$. As a union of open sets $Y = \bigcup \mathcal{Y}$ is open. Hence, $Y \subseteq \neg\neg Y$ by (2) and by (1) and (4) $\neg\neg Y$ is a regular upper bound of $\mathcal{Y}$. Actually, $\neg\neg Y$ is the supremum of $\mathcal{Y}$, since for each regular $Z$ with $Y \subseteq Z$ we have $\neg\neg Y \subseteq \neg\neg Z = Z$ by (1).

(III) Both distributive laws for $\wedge$ and $\vee$ follow with a straightforward computation from (1) and (6) and the distributive laws for $\cap$ and $\cup$.

(IV) Clearly, $U \wedge \neg U = U \setminus \overline{U} = \emptyset$. If $U$ is open, then $U \cup \neg U = \setminus \partial U$ is a dense set (otherwise there would be an open $W$ with $W \subseteq \partial U = \overline{U} \setminus U$, which is absurd). So in particular for regular $U$ we have $U \vee \neg U = \neg \setminus -(U \cup \neg U) = \neg \setminus X = \neg\emptyset = X$.

This proves that $\mathcal{R}(X)$ is a complete boolean algebra.