1. By the axiom of choice $\underline{\mathcal{M}} = \prod_{i\in I} \underline{\mathcal{M}_i}$ is non-empty such that $\mathcal{M}$ is well-defined.

(a) For tuples $\vec{x} = (x_1, \dots, x_n)$ of distinct variable symbols in S_V and $\vec{a} = (a^1, \dots, a^n)$ of elements in $\underline{\mathcal{M}}$ we easily get by structural induction for all S-terms t

$$(\maltese) \qquad t^{\mathcal{M}^{\vec{x}}_{\vec{a}}} = \left(t^{(\mathcal{M}_i)^{\vec{x}}_{\vec{a}_i}}\right)_{i\in I}$$

where $\vec{a}_i = (a^1_i, \dots, a^n_i)$. We also get by structural induction for all S-formulas π

$$(\star) \qquad \pi^{\mathcal{M}^{\vec{x}}_{\vec{a}}} = \left(\pi^{(\mathcal{M}_i)^{\vec{x}}_{\vec{a}_i}}\right)_{i\in I}.$$

In particular, $\pi^{\mathcal{M}} = \left(\pi^{\mathcal{M}_i}\right)_{i\in I}$. Therefore, to obtain Łoś Theorem it suffices to show that $\mathcal{M}$ has witnesses since this will imply $\mathcal{M}/\mathfrak{m} \vDash \pi \Leftrightarrow \pi^{\mathcal{M}} \notin \mathfrak{m}$ by Lemma 1.3.9. To do this, assume $\pi = \bigwedge_x \varphi$. Since all $\mathcal{M}_i$ have witnesses, there exists $b \in \underline{\mathcal{M}}$ with

$$\pi^{\mathcal{M}^{\vec{x}}_{\vec{a}}} = \left(\varphi^{(\mathcal{M}_i)^{(\vec{x},x)}_{(\vec{a}_i,b_i)}}\right)_{i\in I} = \varphi^{\mathcal{M}^{(\vec{x},x)}_{(\vec{a},b)}}$$

which finishes the proof.

We'll need for (b) the following fact:

Lemma. *Every subset Y of a boolean ring R with the* finite-join property*, i.e. with $y_1 \vee \dots \vee y_n \neq 1$ for all $y_1, \dots, y_n \in Y$, is contained in a maximal ideal of R.*

Proof. From Exercise 1.1.5 (xii) and Lemma 1.1.4 it follows that the ideal generated by Y is $\{z \in R : z \leq y_1 \vee \dots \vee y_n$ for some $y_1, \dots, y_n \in Y\}$. The finite-join property guarantees that it is proper and therefore contained in a maximal ideal. □

(b) The "only if" part is obvious. So let's assume all finite subsets of T are satisfiable.

Let I be the set of finite subsets of T and consider $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i$ with unassigned models $\mathcal{M}_i$ of $i \in I$, which exist by assumption.

Let $i^* \in \mathcal{R}^{\mathcal{M}} = {}^I\mathbb{F}_2$ be the characteristic function of $\{k \in I : i \subseteq k\}$ and let $i_* = \neg i^*$. The subset $I_* = \{i_* : i \in I\}$ in $R^{\mathcal{M}}$ has the finite-join property, since $i \cup j \in I$ is not in the support of $i_* \vee j_* = (i \cup j)_*$ for all $i, j \in I$. By the lemma there is a maximal ideal $\mathfrak{m}$ in $R^{\mathcal{M}}$ that contains I_*. Then $i^* \notin \mathfrak{m}$ for all $i \in I$ by Lemma 1.1.4.

Given that $\mathcal{M}_i \vDash i$, we then have $\{\pi\}^* \leq \left(\pi^{\mathcal{M}_i}\right)_{i\in I}$ for every $\pi \in T$, so $\left(\pi^{\mathcal{M}_i}\right)_{i\in I} \notin \mathfrak{m}$ since $\{\pi\}^* \notin \mathfrak{m}$ and $\mathfrak{m}$ is downward closed. Thus $\mathcal{M}/\mathfrak{m} \vDash T$ by (a).

(c) We have the following chain of equivalences

$$\begin{aligned} T \nvDash \varphi &\Leftrightarrow && T \cup \{\neg\varphi\} \text{ satisfiable} \\ &\Leftrightarrow \text{ for all finite subsets } T' \text{ of } T\text{:} && T' \cup \{\neg\varphi\} \text{ satisfiable} \\ &\Leftrightarrow \text{ for all finite subsets } T' \text{ of } T\text{:} && T' \nvDash \varphi \end{aligned}$$

where the middle one due to (b).

2. Without loss of generality we can assume that X contains no symbol of S. Let S' be the vocabulary agreeing with S except that $S'_C = S_C \cup X$ and $T' = T \cup T_X$ with

$$T_X \;=\; \{x \not\equiv y : x, y \in X, x \neq y\}\,.$$

Every finite subset T'' of T' has a model: Indeed, let X'' be the subset of X consisting of the symbols that occur in the sentences in $T'' \cap T_X$. Given an infinite $\equiv$-respecting model $\mathcal{M}$ of T, we can choose an injective function $f\colon X'' \to \underline{\mathcal{M}}$ and extend $\mathcal{M}$ to an S'-structure $\mathcal{M}''$ with $x^{\mathcal{M}''} = f(x)$ for $x \in X''$, which will then be a model of T''.

Hence, by the Compactness Theorem T' admits a model and then by Lemma 1.4.2 even a model $\mathcal{M}'$ that respects $\equiv$. Because of $T' \supseteq T$ it is clear that $\mathcal{M}'$ is a model of T and because of $T' \supseteq T_X$ the rule $x \mapsto x^{\mathcal{M}'}$ yields an embedding $X \to \underline{\mathcal{M}'}$.

3. (a) Let $\mathrm{Th}(\mathbb{Z})$ be the set of S^{Ring}-sentences π with $\tilde{\mathbb{Z}} \vDash \pi$ and S' be a vocabulary agreeing with S^{Ring} except that $S'_C \setminus S^{\mathrm{Ring}}_C = \{c\}$ where c is not a symbol of S.

Let $\lceil 1 \rceil = \mathtt{1}$ and $\lceil n \rceil = \lceil n-1 \rceil \oplus \mathtt{1}$ for $n > 1$. Define $T = \mathrm{Th}(\mathbb{Z}) \cup T_c$ with

$$T_c \;=\; \{c \not\equiv \mathtt{0}\} \cup \{\varphi_n = \textstyle\bigvee_z\, (\lceil 1 \rceil \odot \cdots \odot \lceil n \rceil \odot z) \equiv c \,:\, n \in \mathbb{N}_+\}\,.$$

Every finite subset T' of T admits a model: Indeed, let $m = \max\{n \in \mathbb{N}_+ : \varphi_n \in T'\}$. Then $\tilde{\mathbb{Z}}$ can be extended to an S'-structure $\tilde{\mathbb{Z}}'$ with $c^{\tilde{\mathbb{Z}}'} = m!$, which is a model of T'.

Due to the Compactness Theorem the S'-theory T admits model $\mathcal{R}$, which can be assumed to respect $\equiv$ by Lemma 1.4.2. Since $\mathrm{Th}(\mathbb{Z})$ contains S^{Ring}-sentences characterizing commutative integral domains, $\mathcal{R}$ yields a commutative integral domain $R = \underline{\mathcal{R}}$ with $+ = \oplus^{\mathcal{R}}$, $\cdot = \odot^{\mathcal{R}}$ and $\tilde{R} = \mathcal{R} \vDash \pi$ for all S^{Ring}-sentences π with $\tilde{\mathbb{Z}} \vDash \pi$.

Let $x = c^{\tilde{R}}$. Then $x \neq 0$ because of $\mathcal{R} \vDash c \not\equiv \mathtt{0}$ and for $n > 1$ there are $x_n \in R \setminus \{0\}$ with $n! \cdot x_n = x$ because of $\mathcal{R} \vDash \varphi_n$. Hence, canceling in

$$(n-1)! \cdot x_{n-1} \;=\; x \;=\; n! \cdot x_n$$

gives $x_{n-1} = n \cdot x_n$ such that with $I_n = (x_n)$ we get an increasing chain of ideals

$$I_1 \subseteq I_2 \subseteq \cdots\,.$$

To prove that R is non-noetherian it is enough to check that all these inclusions are proper. If this were not the case, say $I_{n-1} = I_n$, then there would be a $y \in R$ with

$$x_n \;=\; y \cdot x_{n-1} \;=\; y \cdot n \cdot x_n\,,$$

so $1 = y \cdot n$ in contradiction with $\bigwedge_u \mathtt{1} \not\equiv (u \odot \lceil n \rceil) \in \mathrm{Th}(\mathbb{Z})$ for $n > 1$.

(b) If there were an S^{Ring}-sentence φ characterizing noetherianity, then $\varphi \in \mathrm{Th}(\mathbb{Z})$ and the non-noetherian ring R in (a) would yield the contradiction $\tilde{R} \vDash \varphi$.

4. It's easy to see that there is an S^{Ring}-theory ACF $\supseteq$ CRT such that rings R satisfy $\tilde{R} \models \text{ACF}$ iff they are algebraically closed fields. Then the $\equiv$-respecting models of

$$\text{ACF}_0 \;=\; \text{ACF} \cup \{\psi_n = \lceil n \rceil \not\equiv \mathsf{0} \;:\; n \in \mathbb{N}_+\}$$

correspond to algebraically closed fields of characteristic 0.

(2) $\Rightarrow$ (3): By the assumption we obtain $\text{ACF}_0 \models \varphi$ and by Exercise 1 (c) there is a finite subset T' of ACF_0 with $T' \models \varphi$. But then every algebraically closed field K of characteristic greater than $\max\{n \in \mathbb{N}_+ : \psi_n \in T'\}$ satisfies $\tilde{K} \models T'$, so $\tilde{K} \models \varphi$.

(3) $\Rightarrow$ (1): Let $\mathcal{M} = \prod_{i \in I} \tilde{K}_i$ where I and K_i are chosen as indicated in the hint.

The characteristic functions of the finite subsets of I form an ideal in the boolean ring $R^{\mathcal{M}} = {}^{I}\mathbb{F}_2$ by Lemma 1.1.4. Since I is infinite, this ideal is proper and thus contained in a maximal ideal $\mathfrak{m}$ of $R^{\mathcal{M}}$.

Define $\mathcal{K} = (\mathcal{M}/\mathfrak{m})/_{\equiv}$. Then we have by Lemma 1.4.2 and Exercise 1 (a)

$$\mathcal{K} \models \pi \;\Leftrightarrow\; \mathcal{M}/\mathfrak{m} \models \pi \;\Leftrightarrow\; \left(\pi^{\tilde{K}_i}\right)_{i \in I} \notin \mathfrak{m}$$

for all S^{Ring}-sentences π. In particular, $\mathcal{K} \models \text{ACF} \cup \{\varphi\}$ because of $\tilde{K}_i \models \text{ACF} \cup \{\varphi\}$ for all $i \in I$. Consequently, $\mathcal{K} = \tilde{K}$ for an algebraically closed field K.

To conclude the proof it suffices to show that K has characteristic 0, i.e. for every prime p we must show $\mathcal{K} \not\models \lceil p \rceil \equiv \mathsf{0}$, which is equivalent to $x_p = \left((\lceil p \rceil \equiv \mathsf{0})^{\tilde{K}_i}\right)_{i \in I} \in \mathfrak{m}$. But this is true, since $x_p = 0 \in \mathfrak{m}$ if $p \notin I$ and $x_p = \chi_p \in \mathfrak{m}$ if $p \in I$.