Logic and Set Theory Universität Bielefeld
Solutions 3 SS 2019

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1. With the completeness theorem at the back of our mind, the following arguments are clearly sufficient. Without using it, we would have to argue more carefully why our proof system $\vdash$ really has all the properties implicitly used below.

(a) Let $\psi = (\mathrm{Pr}(x) \to \varphi)$ and use (i) to obtain an S-sentence π with

$$(\maltese) \qquad T \vdash (\pi \leftrightarrow \psi(\ulcorner\pi\urcorner))\,.$$

Now (ii) together with (iv) easily yield with $\chi = \mathrm{Pr}(\ulcorner\psi(\ulcorner\pi\urcorner)\urcorner)$

$$T \vdash (\mathrm{Pr}(\ulcorner\pi\urcorner) \to \chi)\ .$$

From this together with the fact that $T \vdash (\chi \to (\mathrm{Pr}(\ulcorner\mathrm{Pr}(\ulcorner\pi\urcorner)\urcorner) \to \mathrm{Pr}(\ulcorner\varphi\urcorner)))$ by (iv) and $T \vdash (\mathrm{Pr}(\ulcorner\pi\urcorner) \to \mathrm{Pr}(\ulcorner\mathrm{Pr}(\ulcorner\pi\urcorner)\urcorner))$ by (iii) we can conclude

$$T \vdash (\mathrm{Pr}(\ulcorner\pi\urcorner) \to \mathrm{Pr}(\ulcorner\varphi\urcorner))\ .$$

So if we assume $T \vdash (\mathrm{Pr}(\ulcorner\varphi\urcorner) \to \varphi) = \psi(\ulcorner\varphi\urcorner)$ this leads to

$$(\clubsuit) \qquad T \vdash (\mathrm{Pr}(\ulcorner\pi\urcorner) \to \varphi) = \psi(\ulcorner\pi\urcorner)\,,$$

so $T \vdash \pi$ by ($\maltese$), so $T \vdash \mathrm{Pr}(\ulcorner\pi\urcorner)$ by (ii), so $T \vdash \varphi$ by ($\clubsuit$).

(b) Given that $T \nvdash \bot$, we get $T \nvdash (\mathrm{Pr}(\ulcorner\bot\urcorner) \to \bot) = \neg\,\mathrm{Pr}(\ulcorner\bot\urcorner)$ by (b).

2. (a) A PA-proof of $1 \not\equiv 2$ can be read off the following tree:

$$\bigwedge_x\bigwedge_y(x \equiv y \to y \equiv x) \quad (\bigwedge_x\bigwedge_y(x \equiv y \to y \equiv x) \to \bigwedge_y(0 \equiv y \to y \equiv 0))$$

$$\bigwedge_y(0 \equiv y \to y \equiv 0) \quad (\bigwedge_y(0 \equiv y \to y \equiv 0) \to (0 \equiv 1 \to 1 \equiv 0)) \qquad \bigwedge_x\bigwedge_y(\mathtt{S}x \equiv \mathtt{S}y \to x \equiv y) \quad (\bigwedge_x\bigwedge_y(\mathtt{S}x \equiv \mathtt{S}y \to x \equiv y) \to \bigwedge_y(1 \equiv \mathtt{S}y \to 0 \equiv y))$$

$$\bigwedge_x \mathtt{S}x \not\equiv 0 \quad (\bigwedge_x \mathtt{S}x \not\equiv 0 \to 1 \not\equiv 0) \quad (0 \equiv 1 \to 1 \equiv 0) \quad ((0 \equiv 1 \to 1 \equiv 0) \to (1 \not\equiv 0 \to 0 \not\equiv 1)) \qquad \bigwedge_y(1 \equiv \mathtt{S}y \to 0 \equiv y) \quad (\bigwedge_y(1 \equiv \mathtt{S}y \to 0 \equiv y) \to (1 \equiv 2 \to 0 \equiv 1))$$

$$1 \not\equiv 0 \qquad (1 \not\equiv 0 \to 0 \not\equiv 1) \qquad (1 \equiv 2 \to 0 \equiv 1) \quad ((1 \equiv 2 \to 0 \equiv 1) \to (0 \not\equiv 1 \to 1 \not\equiv 2))$$

$$0 \not\equiv 1 \qquad (0 \not\equiv 1 \to 1 \not\equiv 2)$$

$$1 \not\equiv 2$$

(b) Let $\varphi = \bigwedge_x(e \odot x) \equiv x$. A $\mathrm{CRT} \cup \{\varphi\}$-proof of $e \equiv 1$ can be read off the tree

$$\bigwedge_x\bigwedge_y(x \odot y) \equiv (y \odot x) \quad (\bigwedge_x\bigwedge_y(x \odot y) \equiv (y \odot x) \to \bigwedge_y(1 \odot y) \equiv (y \odot 1)) \qquad \bigwedge_x\bigwedge_y(x \equiv y \to y \equiv x) \quad (\bigwedge_x\bigwedge_y(x \equiv y \to y \equiv x) \to \bigwedge_y((1 \odot e) \equiv y \to y \equiv (1 \odot e)))$$

$$\bigwedge_y(1 \odot y) \equiv (y \odot 1) \quad (\bigwedge_y(1 \odot y) \equiv (y \odot 1) \to (1 \odot e) \equiv (e \odot 1)) \qquad \bigwedge_x(1 \odot x) \equiv x \quad (\bigwedge_x(1 \odot x) \equiv x \to (1 \odot e) \equiv e) \qquad \bigwedge_y((1 \odot e) \equiv y \to y \equiv (1 \odot e)) \quad (\bigwedge_y((1 \odot e) \equiv y \to y \equiv (1 \odot e)) \to ((1 \odot e) \equiv e \to e \equiv (1 \odot e)))$$

$$\bigwedge_x(e \odot x) \equiv x \quad (\bigwedge_x(e \odot x) \equiv x \to (e \odot 1) \equiv 1) \quad (1 \odot e) \equiv (e \odot 1) \quad X((1 \odot e), (e \odot 1), 1) \qquad (1 \odot e) \equiv e \qquad ((1 \odot e) \equiv e \to e \equiv (1 \odot e))$$

$$(e \odot 1) \equiv 1 \quad ((e \odot 1) \equiv 1 \to (1 \odot e) \equiv 1) \qquad e \equiv (1 \odot e) \quad X(e, (1 \odot e), 1)$$

$$(1 \odot e) \equiv 1 \qquad ((1 \odot e) \equiv 1 \to e \equiv 1)$$

$$e \equiv 1$$

where $X(a,b,c)$ is twice used as a placeholder for the following rooted tree:

$$\bigwedge_x\bigwedge_y\bigwedge_z(x \equiv y \to (y \equiv z \to x \equiv z)) \quad (\bigwedge_x\bigwedge_y\bigwedge_z(x \equiv y \to (y \equiv z \to x \equiv z)) \to \bigwedge_y\bigwedge_z(a \equiv y \to (y \equiv z \to a \equiv z)))$$

$$\bigwedge_y\bigwedge_z(a \equiv y \to (y \equiv z \to a \equiv z)) \quad (\bigwedge_y\bigwedge_z(a \equiv y \to (y \equiv z \to a \equiv z)) \to \bigwedge_z(a \equiv b \to (b \equiv z \to a \equiv z)))$$

$$\bigwedge_z(a \equiv b \to (b \equiv z \to a \equiv z)) \quad (\bigwedge_z(a \equiv b \to (b \equiv z \to a \equiv z)) \to (a \equiv b \to (b \equiv c \to a \equiv c)))$$

$$(a \equiv b \to (b \equiv c \to a \equiv c))$$

With the algorithm described in the proof of the deduction lemma this $\mathrm{CRT} \cup \{\varphi\}$-proof of $e \equiv 1$ can easily be converted into a CRT-proof of $(\varphi \to e \equiv 1)$. This would add $32 = 16 \cdot 2 + 0 \cdot 6$ vertices to our tree because we used 16 times the rule of modus ponens and 0 times the rule of generalization.

3. Let $\sqsubseteq^{-1}(x) = \{v \in X : v < x\}$ for $x \in X$. Then

$$\sqsubseteq^{-1}(x) \;=\; \begin{cases} \{0, 1, \ldots, x-1\} & \text{for } X = \mathbb{N}, \\ (-\infty, x) & \text{for } X = \mathbb{R}. \end{cases}$$

Axioms	$X = \mathbb{N}$	$X = \mathbb{R}$
(`EXT`)	Yes! $x = \lvert\sqsubseteq^{-1}(x)\rvert$	Yes! $x = \sup \sqsubseteq^{-1}(x)$
(`EMP`)	Yes! $\sqsubseteq^{-1}(0) = \emptyset$	No! $\sqsubseteq^{-1}(x) \neq \emptyset$ for all $x \in \mathbb{R}$
(`PAI`)	No! $\sqsubseteq^{-1}(z) = \{x\} \Rightarrow z = 1, x = 0$	No! $\lvert\sqsubseteq^{-1}(z)\rvert > 2$ for all $z \in \mathbb{R}$
(`UNI`)	Yes! $\bigsqcup x = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{else} \end{cases}$	Yes! $\bigsqcup x = x$
(`POW`)	Yes! $\mathtt{P}(x) = x + 1$	Yes! $\mathtt{P}(x) = x$
(`INF`)	No! $<$ has no maximum in $\mathbb{N}$	No! (`EMP`) already failed
(`CHO`)	Yes! "vacuously" satisfied	Yes! "vacuously" satisfied
(`REG`)	Yes! 0 minimal in $\sqsubseteq^{-1}(x)$ if $x \neq 0$	No! no minimal elements in $\sqsubseteq^{-1}(x)$

4. Define for $Y \subseteq X$

$$X^{f\leq} = \{x \in X : f(x) \leq x\}, \qquad X_Y = \{x \in X : x \leq \inf Y\},$$
$$X^{\leq f} = \{x \in X : x \leq f(x)\}, \qquad X^Y = \{x \in X : \sup Y \leq x\}.$$

Let's begin with proving the hint:

Lemma. *For $Y \subseteq X^{f\leq}$ and $Z \subseteq X^{\leq f}$ we have:*

(1) $\inf Y \in X^{f\leq}$ *and* $\sup Z \in X^{\leq f}$.

(2) $\inf X^{f\leq}$ *is a least element and* $\sup X^{\leq f}$ *a greatest element of* X^f.

(3) X_Y *and* X^Z *are complete sublattices of* X *stable under* f.

Proof. In each item we will only prove the first part of the statement given that the second part is dual everywhere. Let $y = \inf Y$.

(1) We have $f(y) \leq f(x) \leq x$ for all $x \in Y$ where the first inequality uses $y \leq x$ and that f is an endomorphism and the second one $Y \subseteq X^{f\leq}$. Hence, $f(y) \leq \inf Y = y$.

(2) Since $f(y) \leq y$ by (1), it follows that $f(y) \in X^{f\leq}$ because f is an endomorphism. If $Y = X^{f\leq}$, we get $y = \inf Y \leq f(y)$, so $y = f(y)$ is a least element of $X^f \subseteq X^{f\leq}$.

(3) Clearly, X_Y has as a least element $\inf X$ and as a greatest element $y = \inf Y$, which readily implies that suprema and infima of subsets of X_Y in X are suprema and infima in X_Y. Consequently, X_Y forms a complete sublattice of X. Moreover, given that $f(y) \leq y$ by (1), it follows that f maps X_Y into itself. □

The lemma shows that for $Y \subseteq X^f = X^{f\leq} \cap X^{\leq f}$ the set $(X_Y)^f$, which consists of the lower bounds of Y in X^f, has a greatest element. The existence of suprema can be proved dually. It follows that X^f forms a complete lattice.