

1. With the completeness theorem at the back of our mind, the following arguments are clearly sufficient. Without using it, we would have to argue more carefully why our proof system \vdash really has all the properties implicitly used below.

(a) Let $\psi = (\text{Pr}(x) \rightarrow \varphi)$ and use (i) to obtain an S -sentence π with

$$(\boxtimes) \quad T \vdash (\pi \leftrightarrow \psi(\ulcorner \pi \urcorner)).$$

Now (ii) together with (iv) easily yield with $\chi = \text{Pr}(\ulcorner \psi(\ulcorner \pi \urcorner) \urcorner)$

$$T \vdash (\text{Pr}(\ulcorner \pi \urcorner) \rightarrow \chi).$$

From this together with the fact that $T \vdash (\chi \rightarrow (\text{Pr}(\ulcorner \text{Pr}(\ulcorner \pi \urcorner) \urcorner) \rightarrow \text{Pr}(\ulcorner \varphi \urcorner)))$ by (iv) and $T \vdash (\text{Pr}(\ulcorner \pi \urcorner) \rightarrow \text{Pr}(\ulcorner \text{Pr}(\ulcorner \pi \urcorner) \urcorner))$ by (iii) we can conclude

$$T \vdash (\text{Pr}(\ulcorner \pi \urcorner) \rightarrow \text{Pr}(\ulcorner \varphi \urcorner)).$$

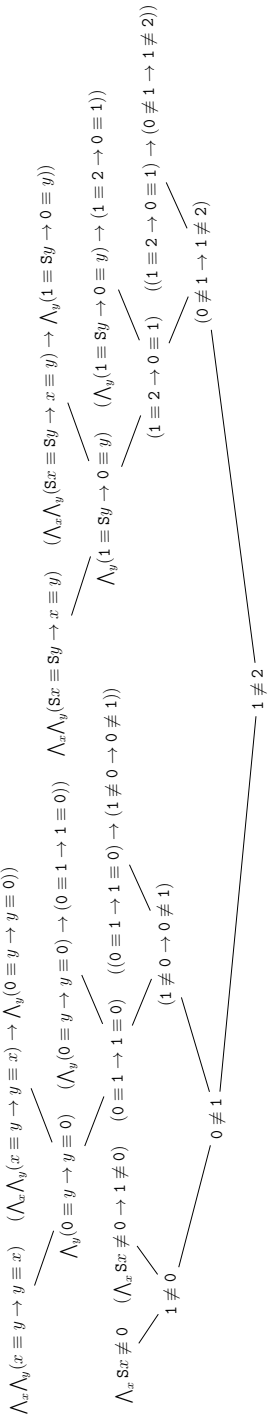
So if we assume $T \vdash (\text{Pr}(\ulcorner \varphi \urcorner) \rightarrow \varphi) = \psi(\ulcorner \varphi \urcorner)$ this leads to

$$(\clubsuit) \quad T \vdash (\text{Pr}(\ulcorner \pi \urcorner) \rightarrow \varphi) = \psi(\ulcorner \pi \urcorner),$$

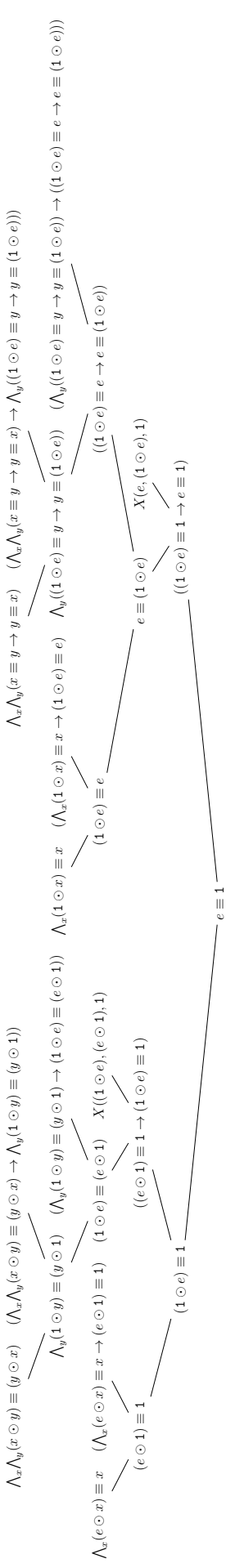
so $T \vdash \pi$ by (\boxtimes) , so $T \vdash \text{Pr}(\ulcorner \pi \urcorner)$ by (ii), so $T \vdash \varphi$ by (\clubsuit) .

(b) Given that $T \not\vdash \perp$, we get $T \not\vdash (\text{Pr}(\ulcorner \perp \urcorner) \rightarrow \perp) = \neg \text{Pr}(\ulcorner \perp \urcorner)$ by (b).

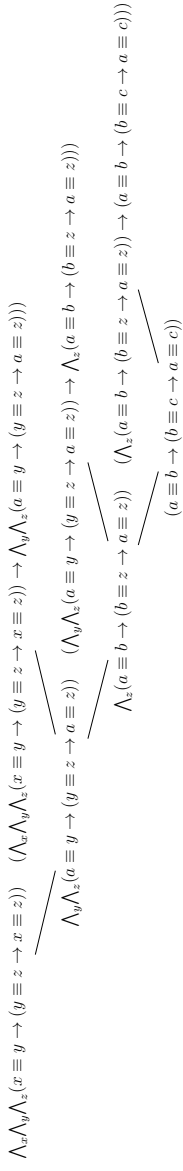
2. (a) A PA-proof of $1 \neq 2$ can be read off the following tree:



(b) Let $\varphi = \Lambda_x(e \odot x) \equiv x$. A $\text{CRT} \cup \{\varphi\}$ -proof of $e \equiv 1$ can be read off the tree



where $X(a, b, c)$ is twice used as a placeholder for the following rooted tree:



With the algorithm described in the proof of the deduction lemma this $\text{CRT} \cup \{\varphi\}$ -proof of $e \equiv 1$ can easily be converted into a CRT-proof of $(\varphi \rightarrow e \equiv 1)$. This would add $32 = 16 \cdot 2 + 0 \cdot 6$ vertices to our tree because we used 16 times the rule of modus ponens and 0 times the rule of generalization.

3. Let $\varepsilon^{-1}(x) = \{v \in X : v < x\}$ for $x \in X$. Then

$$\varepsilon^{-1}(x) = \begin{cases} \{0, 1, \dots, x-1\} & \text{for } X = \mathbb{N}, \\ (-\infty, x) & \text{for } X = \mathbb{R}. \end{cases}$$

Axioms	$X = \mathbb{N}$	$X = \mathbb{R}$
(EXT)	Yes! $x = \varepsilon^{-1}(x) $	Yes! $x = \sup \varepsilon^{-1}(x)$
(EMP)	Yes! $\varepsilon^{-1}(0) = \emptyset$	No! $\varepsilon^{-1}(x) \neq \emptyset$ for all $x \in \mathbb{R}$
(PAI)	No! $\varepsilon^{-1}(z) = \{x\} \Rightarrow z = 1, x = 0$	No! $ \varepsilon^{-1}(z) > 2$ for all $z \in \mathbb{R}$
(UNI)	Yes! $\sqcup x = \begin{cases} 0 & \text{if } x = 0 \\ x-1 & \text{else} \end{cases}$	Yes! $\sqcup x = x$
(POW)	Yes! $P(x) = x + 1$	Yes! $P(x) = x$
(INF)	No! < has no maximum in \mathbb{N}	No! (EMP) already failed
(CHO)	Yes! “vacuously” satisfied	Yes! “vacuously” satisfied
(REG)	Yes! 0 minimal in $\varepsilon^{-1}(x)$ if $x \neq 0$	No! no minimal elements in $\varepsilon^{-1}(x)$

4. Define for $Y \subseteq X$

$$\begin{aligned} X^{f\leq} &= \{x \in X : f(x) \leq x\}, & X_Y &= \{x \in X : x \leq \inf Y\}, \\ X^{\leq f} &= \{x \in X : x \leq f(x)\}, & X^Y &= \{x \in X : \sup Y \leq x\}. \end{aligned}$$

Let's begin with proving the hint:

Lemma. For $Y \subseteq X^{f\leq}$ and $Z \subseteq X^{\leq f}$ we have:

- (1) $\inf Y \in X^{f\leq}$ and $\sup Z \in X^{\leq f}$.
- (2) $\inf X^{f\leq}$ is a least element and $\sup X^{\leq f}$ a greatest element of X^f .
- (3) X_Y and X^Z are complete sublattices of X stable under f .

Proof. In each item we will only prove the first part of the statement given that the second part is dual everywhere. Let $y = \inf Y$.

(1) We have $f(y) \leq f(x) \leq x$ for all $x \in Y$ where the first inequality uses $y \leq x$ and that f is an endomorphism and the second one $Y \subseteq X^{f\leq}$. Hence, $f(y) \leq \inf Y = y$.

(2) Since $f(y) \leq y$ by (1), it follows that $f(y) \in X^{f\leq}$ because f is an endomorphism. If $Y = X^{f\leq}$, we get $y = \inf Y \leq f(y)$, so $y = f(y)$ is a least element of $X^f \subseteq X^{f\leq}$.

(3) Clearly, X_Y has as a least element $\inf X$ and as a greatest element $y = \inf Y$, which readily implies that suprema and infima of subsets of X_Y in X are suprema and infima in X_Y . Consequently, X_Y forms a complete sublattice of X . Moreover, given that $f(y) \leq y$ by (1), it follows that f maps X_Y into itself. \square

The lemma shows that for $Y \subseteq X^f = X^{f\leq} \cap X^{\leq f}$ the set $(X_Y)^f$, which consists of the lower bounds of Y in X^f , has a greatest element. The existence of suprema can be proved dually. It follows that X^f forms a complete lattice.