Logic and Set Theory Universität Bielefeld
Solutions 4 SS 2019

1.

(1) $\Rightarrow$ (2) This was proved in the lecture.

(2) $\Rightarrow$ (3) Let $\alpha \in \beta \in \gamma$. We must show $\alpha \sqsubseteq \beta$, equivalently $\beta \not\sqsubseteq \alpha$ by assumption. If we had $\beta \sqsubseteq \alpha$, then $\alpha \in \alpha$. But this is impossible, since $\alpha \in \gamma$ by transitivity of γ such that $[\alpha]$ must have an $\in$-minimal element by well-foundedness of $\in$ on γ.

(3) $\Rightarrow$ (4) Let us call α and β incomparable if none of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ holds. It is enough to check that $X = [\alpha \in \gamma : \alpha$ and β are incomparable for some $\beta \in \gamma]$ is empty, since the well-foundedness of $\in$ on γ implies that for $\alpha, \beta \in \gamma$ at most one of $\alpha \in \beta$, $\alpha = \beta$, $\beta \in \alpha$ can hold. We will now assume that X is not empty in order to arrive at a contradiction. By the well-foundedness of $\in$ on γ we can first pick an $\in$-minimal δ in X and then an $\in$-minimal ε in $[\beta \in \gamma : \delta$ and β are incomparable$]$. We claim the absurd identity $\delta = \varepsilon$ holds. For the inclusion $\sqsubseteq$ note that each $\sigma \in \delta$ must be comparable with ε, which means $\sigma \in \varepsilon$ because neither $\sigma = \varepsilon$ nor $\varepsilon \in \sigma$, since this would imply $\varepsilon \in \delta$ by the transitivity of δ, is possible. For the inclusion $\sqsupseteq$ note that each $\sigma \in \varepsilon$ must be comparable with δ, which similarly leads to $\sigma \in \delta$.

(4) $\Rightarrow$ (1) By the assumptions it merely remains to verify that $\in$ is transitive on γ. So let $\delta, \alpha, \beta \in \gamma$ with $\delta \in \alpha \in \beta$. We can neither have $\delta = \beta$ nor $\beta \in \delta$, since this would imply that $[\delta, \alpha, \beta]$ has no $\in$-minimal element. So necessarily $\delta \in \beta$.

2. We use results from 4. here, and vice versa. But we avoid circular reasoning.

(i) These hold by definition.

(ii) For $\underline{0} + \alpha = \alpha$ use 4. (a) and the order isomorphism $\underline{0} \dot\sqcup \alpha \twoheadrightarrow \alpha$, $\langle \underline{1}, \gamma \rangle \mapsto \gamma$.

To prove $\underline{0} \cdot \alpha = \underline{0}$ assume by induction $\underline{0} \cdot \varepsilon = \underline{0}$ for all $\varepsilon < \alpha$. For $\alpha = \underline{0}$ use (i). For successor ordinals α we can once again use (i) to get $\underline{0} \cdot \alpha = \underline{0} \cdot (\alpha - \underline{1}) + \underline{0} = \underline{0}$. For limit ordinals α we also have $\underline{0} \cdot \alpha = \bigsqcup_{\varepsilon < \alpha} \underline{0} \cdot \varepsilon = \underline{0}$.

For $\underline{1} \cdot \alpha = \alpha$ use 4. (b) and the order isomorphism $\underline{1} * \alpha \twoheadrightarrow \alpha$, $\langle \underline{0}, \gamma \rangle \mapsto \gamma$.

(iii) The identity $\alpha \cdot \underline{1} = \underline{0} + \alpha = \alpha$ holds by definition and (ii).

The identity $\alpha^{(\underline{1})} = \underline{1} \cdot \alpha = \alpha$ holds by definition and (ii).

For $\underline{1}^{(\alpha)} = \underline{1}$ use 4. (c) and $(\alpha \twoheadrightarrow \underline{1}) = [\alpha * \underline{1}]$.

(iv) Use 4. (c) and $(\alpha \twoheadrightarrow \underline{0}) = \underline{0}$ if $\underline{0} < \alpha$.

(v) Use 4. (a,b) together with the fact that the obvious maps $(\alpha \dot\sqcup \beta) \dot\sqcup \gamma \twoheadrightarrow \alpha \dot\sqcup (\beta \dot\sqcup \gamma)$ and $(\alpha * \beta) * \gamma \twoheadrightarrow \alpha * (\beta * \gamma)$ (with the obvious orders on the respective domain and range) are order isomorphisms.

(vi) Use 4. (a,b) together with the fact that the obvious map $\alpha * (\beta \dot\sqcup \gamma) \twoheadrightarrow (\alpha * \beta) \dot\sqcup (\alpha * \gamma)$ (with the obvious orders on domain and range) is an order isomorphism.

(vii) Use 4. (a,b,c) and the fact that the obvious map $((\beta \dot\sqcup \gamma) \twoheadrightarrow \alpha) \twoheadrightarrow (\beta \twoheadrightarrow \alpha) * (\gamma \twoheadrightarrow \alpha)$ (with the obvious orders on domain and range) is an order isomorphism.

(viii) Use 4. (b,c) and the fact that the obvious map $(\gamma \twoheadrightarrow (\beta \twoheadrightarrow \alpha)) \twoheadrightarrow ((\beta * \gamma) \twoheadrightarrow \alpha)$ (with the obvious orders on domain and range) is an order isomorphism.

(ix) The successor map s on $\mathbb{O}^{\mathcal{M}}$ given by $\varepsilon \mapsto \varepsilon + \underline{1}$ preserves $\leq$ and satisfies $\varepsilon \leq s(\varepsilon)$. Therefore we get $\alpha + \gamma = s^{\gamma}(\alpha) \leq s^{\delta}(\alpha) \leq s^{\delta}(\beta) = \beta + \delta$.

The maps a_σ on $\mathbb{O}^\mathcal{M}$ given by $\varepsilon \mapsto \varepsilon + \sigma$ preserve $\leq$ and satisfy due to (i) and by what has already been shown here $\varepsilon = \varepsilon + \underline{0} \leq a_\alpha(\varepsilon) \leq a_\beta(\varepsilon)$. As a consequence, we get $\alpha \cdot \gamma = a_\alpha^\gamma(\underline{0}) \leq a_\alpha^\delta(\underline{0}) \leq a_\beta^\delta(\underline{0}) = \beta \cdot \delta$.

(x) By (ix) it suffices to show $\beta+\gamma < \beta+\delta$. We have $\varepsilon < s(\varepsilon)$ such that $\mathrm{It}_{s,\beta}$ is normal, so in particular it is order-preserving, hence $\beta + \gamma = s^\gamma(\beta) < s^\delta(\beta) = \beta + \delta$.

(xi) By (ix) it suffices to show $\beta \cdot \gamma < \beta \cdot \delta$. We have $\varepsilon = \varepsilon + \underline{0} < a_\beta(\varepsilon)$ because of $\underline{0} < \beta$ and (i,x) such that $\mathrm{It}_{a_\beta,\underline{0}}$ is normal. Hence, $\beta \cdot \gamma = a_\beta^\gamma(\underline{0}) < a_\beta^\delta(\underline{0}) = \beta \cdot \delta$.

(xii) The maps m_σ on $\mathbb{O}^\mathcal{M}$ given by $\varepsilon \mapsto \varepsilon \cdot \sigma$ preserve $\leq$ and because of $\underline{1} \leq \alpha$ and (iii,ix) satisfy $\varepsilon = \varepsilon \cdot \underline{1} \leq m_\alpha(\varepsilon) \leq m_\beta(\varepsilon)$. Hence, $\alpha^{(\gamma)} = m_\alpha^\gamma(\underline{1}) \leq m_\alpha^\delta(\underline{1}) \leq m_\beta^\delta(\underline{1}) = \beta^{(\delta)}$.

(xiii) By (xii) it suffices to show $\beta^{(\gamma)} < \beta^{(\delta)}$. We have $\varepsilon = \varepsilon \cdot \underline{1} < m_\beta(\varepsilon)$ because of $\underline{1} < \beta$ and (iii,xi) such that $\mathrm{It}_{m_\beta,\underline{1}}$ is normal. Hence, $\beta^{(\gamma)} = m_\beta^\gamma(\underline{1}) < m_\beta^\delta(\underline{1}) = \beta^{(\delta)}$.

(xiv) Use (x).

(xv) Use (xi).

(xvi) Use (xii).

Lemma. *$\beta \leq \underline{2}^{(\beta)}$ for all $\beta \in \mathbb{O}^\mathcal{M}$.*

Proof. By induction we may assume $\alpha \leq \underline{2}^{(\alpha)}$ for all $\alpha < \beta$.

If $\beta = \underline{0}$ we have $\beta \leq \underline{1} = \underline{2}^{(\beta)}$ by (i).

If β is a successor ordinal, we compute $\beta = (\beta - \underline{1}) + \underline{1} \leq \underline{2}^{(\beta-\underline{1})} + \underline{1} \leq \underline{2}^{(\beta)}$ where the first inequality uses (ix) and the last inequality uses the normality of $\mathrm{It}_{m_{\underline{2}},\underline{1}}$.

If β is a non-zero limit ordinal, then $\beta = \bigsqcup_{\alpha<\beta} \alpha \leq \bigsqcup_{\alpha<\beta} \underline{2}^{(\alpha)} = \underline{2}^{(\beta)}$. □

For all $\alpha, \beta \in \mathbb{N}^\mathcal{M}$ induction on β readily yields $\alpha + \beta \in \mathbb{N}^\mathcal{M}$, then also $\alpha \cdot \beta \in \mathbb{N}^\mathcal{M}$, and finally $\alpha^{(\beta)} \in \mathbb{N}^\mathcal{M}$. If $\mathbb{N}^\mathcal{M}$ forms an $\mathcal{M}$-set ω, we have for all $\alpha \in \mathbb{N}^\mathcal{M}$

$$\begin{array}{llllllll} \omega = \underline{0} + \omega & \leq & \alpha + \omega & = & \bigsqcup_{\beta<\omega} (\alpha + \beta) & \leq & \omega & \text{for all } \alpha, \\ \omega = \underline{1} \cdot \omega & \leq & \alpha \cdot \omega & = & \bigsqcup_{\beta<\omega} (\alpha \cdot \beta) & \leq & \omega & \text{for all } \alpha > \underline{0}, \\ \omega \leq \underline{2}^{(\omega)} & \leq & \alpha^{(\omega)} & = & \bigsqcup_{\beta<\omega} \alpha^{(\beta)} & \leq & \omega & \text{for all } \alpha > \underline{1}, \end{array}$$

using the lemma and the normality of $\mathrm{It}_{s,\alpha}$, $\mathrm{It}_{a_\alpha,\underline{0}}$ for $\alpha > \underline{0}$, and $\mathrm{It}_{m_\alpha,\underline{1}}$ for $\alpha > \underline{1}$. Consequently, we have for all natural numbers α in $\mathcal{M}$ the identities

$$\alpha + \omega = \alpha \cdot \omega = \alpha^{(\omega)} = \omega\,.$$

They provide us with the following counterexamples:

(I) $\underline{1} + \omega = \omega < \omega + \underline{1}$

(II) $\underline{2} \cdot \omega = \omega = \omega \cdot \underline{1} < \omega \cdot \underline{2}$

(III) $(\underline{1} + \underline{1}) \cdot \omega = \underline{2} \cdot \omega = \omega = \omega \cdot \underline{1} < \omega \cdot \underline{2} = \omega \cdot \underline{1} + \omega \cdot \underline{1} = \underline{1} \cdot \omega + \underline{1} \cdot \omega$

(IV) $(\underline{2} \cdot \underline{2})^{(\omega)} = \omega = \omega \cdot \underline{1} < \omega \cdot \omega = \underline{2}^{(\omega)} \cdot \underline{2}^{(\omega)}$

(V) $\underline{0} + \omega = \omega = \underline{1} + \omega$

(VI) $\underline{1} \cdot \omega = \omega = \underline{2} \cdot \omega$

(VII) $\underline{2}^{(\omega)} = \omega = \underline{3}^{(\omega)}$

(VIII) See (V).

(IX) See (VI).

(X) See (VII).

Let's first prove the remark from the end of the exercise sheet:

Lemma. *Every non-empty finite $\mathcal{M}$-subset X of an $\mathcal{M}$-ordinal β has a maximum.*

Proof. If not, we could construct with the help of well-ordered recursion a sequence $\langle \alpha_\varepsilon \rangle_{\varepsilon \in \mathbb{N}^\mathcal{M}}$ given by $\alpha_\varepsilon = \min(X \smallsetminus [\alpha_\delta : \delta < \varepsilon])$. It would then follow that $\mathbb{N}^\mathcal{M}$ forms an $\mathcal{M}$-set ω satisfying the impossible $\omega \leq |X| < \omega$. □

3.

(a) The map $f = \mathrm{It}_{s,\alpha}$ is normal with image $\{\gamma \in \mathbb{O}^\mathcal{M} : \gamma \geq \alpha\}$. Take $\beta - \alpha = f^{-1}(\beta)$.

(b) The map $f = \mathrm{It}_{a_\beta,\underline{0}}$ is normal for $\beta > \underline{0}$. Let $\gamma \in \mathbb{O}^\mathcal{M}$ be such that $f(\gamma)$ is the maximum in the image of f that is not greater than α, which exists since $\alpha \geq f(\underline{0})$. Then $\beta \cdot \gamma \leq \alpha < \beta \cdot (\gamma + \underline{1})$, so $\alpha = \beta \cdot \gamma + \delta$ with $\delta = \alpha - \beta \cdot \gamma$. If we had $\delta \geq \beta$, this would give rise to the contradiction

$$\beta \cdot (\gamma + \underline{1}) \overset{\text{(i,ix)}}{\leq} \beta \cdot (\gamma + \underline{1}) + (\delta - \beta) \overset{\text{(v)}}{=} \beta \cdot \gamma + (\beta + (\delta - \beta)) = \alpha\,.$$

Now assume $\beta \cdot \gamma + \delta = \beta \cdot \gamma' + \delta'$ with $\delta' < \beta$. By (xiv) it suffices to check $\gamma = \gamma'$. Assume not, say $\gamma < \gamma'$. Then we get the contradiction

$$\beta \cdot \gamma + \delta \overset{\text{(x)}}{<} \beta \cdot \gamma + \beta = \beta \cdot (\gamma + \underline{1}) \overset{\text{(ix)}}{\leq} \beta \cdot \gamma' \overset{\text{(i,ix)}}{\leq} \beta \cdot \gamma' + \delta'\,.$$

(c) Let γ be a left divisor of α and $\alpha + \beta$. We show that γ is a left divisor of β, too. This is true for $\gamma = \underline{0}$ by (ii). So let's assume $\gamma \neq \underline{0}$. Write $\alpha = \gamma \cdot \delta$ and using (b) $\beta = \gamma \cdot \sigma + \tau$ where $\delta, \sigma, \tau \in \mathbb{O}^\mathcal{M}$ with $\tau < \gamma$. We need $\tau = \underline{0}$. A calculation yields

$$\alpha + \beta \overset{\text{(v)}}{=} (\gamma \cdot \delta + \gamma \cdot \sigma) + \tau \overset{\text{(vi)}}{=} \gamma \cdot (\delta + \sigma) + \tau\,.$$

Since γ is a left divisor of $\alpha + \beta$, the uniqueness in (b) implies $\tau = \underline{0}$.

(d) Let $C \subseteq \mathbb{O}^\mathcal{M}$ be a non-empty $\mathcal{M}$-subclass of $\mathbb{O}^\mathcal{M} \setminus \{\underline{0}\}$.

Greatest common left divisor. With (b), (c), the calculation rules in 3., and recursion, finding the greatest common left divisor $\gcd(C)$ of C is standard.

Firstly, we will verify the existence of $\gcd(\{\alpha, \beta\})$ for all $\underline{0} < \beta \leq \alpha$. The Euclidean algorithm – which works thanks to well-ordered recursion and (b) – yields sequences $\langle \Gamma_\sigma \rangle_{\sigma \in \mathbb{N}^\mathcal{M}}$ and $\langle \Delta_\sigma \rangle_{\sigma \in \mathbb{N}^\mathcal{M}}$ with $\Gamma_{\underline{0}} = \alpha$ and $\Gamma_{\underline{1}} = \beta$ satisfying for $\underline{0} < \sigma \in \mathbb{N}^\mathcal{M}$

$$\begin{aligned} \Gamma_{\sigma - \underline{1}} = \Gamma_\sigma \cdot \Delta_\sigma + \Gamma_{\sigma + \underline{1}} \text{ and } \Gamma_{\sigma+\underline{1}} < \Gamma_\sigma \qquad & \text{if } \Gamma_\sigma \neq \underline{0}, \\ \Gamma_{\sigma + \underline{1}} = \underline{0} \qquad & \text{if } \Gamma_\sigma = \underline{0}. \end{aligned}$$

Clearly, Γ is eventually constant $\underline{0}$, since otherwise it would be strictly decreasing, in contradiction to the fact that $\mathbb{O}^\mathcal{M}$ is well-ordered. So let $\sigma_0 = \min\{\sigma : \Gamma_\sigma = \underline{0}\}$. We claim $\gcd(\{\Gamma_{\sigma-\underline{1}}, \Gamma_\sigma\}) = \gcd(\{\Gamma_\sigma, \Gamma_{\sigma+\underline{1}}\})$ for all $\sigma < \sigma_0$ (in particular for $\sigma = \underline{0}$). Otherwise let σ be maximal such that $\gcd(\{\Gamma_\sigma, \Gamma_{\sigma+\underline{1}}\})$ exists but $\gcd(\{\Gamma_{\sigma-\underline{1}}, \Gamma_\sigma\})$ does not. We prove that this is not possible by showing that $\Gamma_{\sigma-\underline{1}}$ and Γ_σ have the same common left divisors as Γ_σ and $\Gamma_{\sigma+\underline{1}}$. Now, clearly $\Gamma_\sigma \neq \underline{0}$ and by (c) and (v) every common left divisor of $\Gamma_{\sigma-\underline{1}}$ and Γ_σ is a left divisor of $\Gamma_{\sigma+\underline{1}}$. And conversely, by (v,vi) every common left divisor of Γ_σ and $\Gamma_{\sigma+\underline{1}}$ is a left divisor of $\Gamma_{\sigma-\underline{1}}$.

Next, let $\langle X_\sigma \rangle_{\sigma \in \mathbb{N}^\mathcal{M}}$ and $\langle \Gamma_\sigma \rangle_{\sigma \in \mathbb{N}^\mathcal{M}}$ be sequences with $X_{\underline{0}} = \Gamma_{\underline{0}}$ an arbitrary element of C and $X_\sigma = X_{\sigma-\underline{1}} \sqcup [\alpha_\sigma]$ and $\Gamma_\sigma = \gcd(\{\Gamma_{\sigma-\underline{1}}, \alpha_\sigma\})$ for all $0 < \sigma \in \mathbb{N}^\mathcal{M}$ where

$$\alpha_\sigma = \begin{cases} \min C_\sigma & \text{if } C_\sigma = \{\delta \in C : \Gamma_{\sigma-\underline{1}} \text{ is not a left divisor of } \delta\} \neq \emptyset, \\ \underline{0} & \text{otherwise.} \end{cases}$$

Since Γ is decreasing, it takes eventually some constant value γ, so C_σ is empty for large enough σ. Using induction it is easy to see $\Gamma_\sigma = \gcd(\Subset^{-1}(X_\sigma))$ for all $\sigma \in \mathbb{N}^{\mathcal{M}}$. Then for large enough σ we get $\gamma = \Gamma_\sigma = \gcd(C)$ because of $X_\sigma \sqsubseteq C$ and $C_\sigma = \emptyset$.

Greatest common right divisor. By the remark, which was proved above, it is enough to check that for every non-zero $\mathcal{M}$-ordinal α its $\mathcal{M}$-class of right divisors

$$C = \left\{\delta \in \mathbb{O}^{\mathcal{M}} : \alpha = \gamma \cdot \delta \text{ for some } \gamma \in \mathbb{O}^{\mathcal{M}}\right\}$$

forms a finite $\mathcal{M}$-set. If this were not the case, we could use well-ordered recursion to obtain two sequences $\langle \delta_\tau \rangle_{\tau \in \mathbb{N}^{\mathcal{M}}}$ and $\langle \gamma_\tau \rangle_{\tau \in \mathbb{N}^{\mathcal{M}}}$ given by

$$\delta_\tau = \min(C \setminus \{\delta_\sigma : \sigma < \tau\}), \qquad \gamma_\tau = \min\{\gamma \in \mathbb{O}^{\mathcal{M}} : \alpha = \gamma \cdot \delta_\tau\}.$$

For all $\underline{0} < \tau \in \mathbb{N}^{\mathcal{M}}$ we would have $\delta_{\tau-\underline{1}} < \delta_\tau$ and $\gamma_{\tau-\underline{1}} \cdot \delta_{\tau-\underline{1}} = \alpha = \gamma_\tau \cdot \delta_\tau$ such that according to (xi) $\gamma_{\tau-\underline{1}} > \gamma_\tau$, contradicting the fact that $\mathbb{O}^{\mathcal{M}}$ is well-ordered.

4.

(a) The rules $\langle \underline{0}, \delta \rangle \mapsto \delta$ and $\langle \underline{1}, \delta \rangle \mapsto \alpha + \delta$ yield an $\mathcal{M}$-function $h\colon \alpha \,\dot{\sqcup}\, \beta \twoheadrightarrow \alpha + \beta$, which is well-defined because $\delta < \alpha + \beta$ for all $\delta < \alpha$ and $\alpha + \delta < \alpha + \beta$ for all $\delta < \beta$ by (i,ix,x). It is surjective because $\varepsilon = \alpha + (\varepsilon - \alpha)$ for all $\alpha \le \varepsilon < \alpha + \beta$ and it is injective because of (xiv). We endow $\alpha \,\dot{\sqcup}\, \beta$ with the lexicographic order $<$, i.e.

$$\langle \gamma, \delta \rangle < \langle \gamma', \delta' \rangle \;\Leftrightarrow\; \gamma < \gamma' \text{ or } (\gamma = \gamma' \text{ and } \delta < \delta').$$

Then $\langle \gamma, \delta \rangle < \langle \gamma', \delta' \rangle \Leftrightarrow h(\langle \gamma, \delta \rangle) < h(\langle \gamma', \delta' \rangle)$ for $\langle \gamma, \delta \rangle \Subset \alpha \,\dot{\sqcup}\, \beta$. In case $\gamma = \gamma' = \underline{0}$ this is obvious, in case $\gamma = \gamma' = \underline{1}$ it follows from (ix,x), in case $\gamma = \underline{0}$, $\gamma' = \underline{1}$ from $\delta < \alpha \Rightarrow \delta < \alpha + \delta'$ in view of (i,ix), and similarly in case $\gamma = \underline{1}$, $\gamma' = \underline{0}$.

(b) The rule $\langle \gamma, \delta \rangle \mapsto \alpha \cdot \delta + \gamma$ yields an $\mathcal{M}$-function $\alpha * \beta \twoheadrightarrow \alpha \cdot \beta$, which is well-defined because for $\gamma < \alpha$ and $\delta < \beta$

$$\alpha \cdot \delta + \gamma \overset{\text{(x)}}{<} \alpha \cdot \delta + \alpha = \alpha \cdot (\delta + \underline{1}) \overset{\text{(ix)}}{\le} \alpha \cdot \beta.$$

It is bijective since for $\varepsilon < \alpha \cdot \beta$ there are unique $\gamma < \alpha$ and $\delta < \beta$ with $\varepsilon = \alpha \cdot \delta + \gamma$ by 3. (b) and (i,ix) and the identity $\underline{0} \cdot \beta = \underline{0}$ found in (ii). We endow $\alpha * \beta$ with the anti-lexicographic order $<$, i.e.

$$\langle \gamma, \delta \rangle < \langle \gamma', \delta' \rangle \;\Leftrightarrow\; \delta < \delta' \text{ or } (\delta = \delta' \text{ and } \gamma < \gamma').$$

Then $\langle \gamma, \delta \rangle < \langle \gamma', \delta' \rangle \Leftrightarrow \alpha \cdot \delta + \gamma < \alpha \cdot \delta' + \gamma'$ for $\langle \gamma, \delta \rangle \Subset \alpha * \beta$. In case $\delta = \delta'$ this follows from (ix,x), in case $\delta < \delta'$, using $\gamma < \alpha + \gamma'$ for $\gamma < \alpha$ by (i,ix), from

$$\alpha \cdot \delta + \gamma \overset{\text{(x)}}{<} \alpha \cdot \delta + (\alpha + \gamma') \overset{\text{(v)}}{=} \alpha \cdot (\delta + \underline{1}) + \gamma' \overset{\text{(ix)}}{\le} \alpha \cdot \delta' + \gamma',$$

and the case $\delta' < \delta$ is analogous to the case $\delta < \delta'$. Observe that here only the part of (v) was used whose proof depended solely on 4. (a).

(c) We begin with an auxiliary lemma:

Lemma. *Fix $\alpha \in \mathbb{O}^{\mathcal{M}}$ and for all $\beta \in \mathbb{O}^{\mathcal{M}}$ let $C_\beta = \{f \in \{\beta \twoheadrightarrow \alpha\} : \operatorname{supp}(f) \text{ is finite}\}$ where $\operatorname{supp}(f) = [\gamma < \beta : f(\gamma) \neq \underline{0}]$.*

- *(1) For every $f \in C_\beta$ the $\mathcal{M}$-set $\operatorname{supp}(f)$ has a maximum m_f with the notational convention $m_f = -\underline{1}$ in case $\operatorname{supp}(f) = \varnothing$ and $-\underline{1} + \underline{1} = \underline{0}$.*
- *(2) If β is a successor ordinal, then the following is a bijective $\mathcal{M}$-class function:*

$$\begin{aligned} C_\beta &\longrightarrow C_{\beta-\underline{1}} * \alpha \\ f &\longmapsto \langle f|_{\beta-\underline{1}}, f(\beta - \underline{1}) \rangle \end{aligned}$$

(3) If β is a limit ordinal, then the following is a bijective $\mathcal{M}$-class function:

$$C_\beta \longrightarrow \left\{f \in \bigcup_{\gamma<\beta} C_\gamma : \mathrm{dom}(f) = m_f + \underline{1}\right\}$$

$$f \longmapsto f|_{m_f+\underline{1}}$$

(4) C_β forms an $\mathcal{M}$-set $(\beta \twoheadrightarrow \alpha)$.

Proof. (1) Use the remark, which was proved above.

(2) and (3) are easily checked and together with $C_{\underline{0}} = \{\underline{0}\}$ imply (4) by induction. □

For all $\beta \in \mathbb{O}^{\mathcal{M}}$ we now endow $(\beta \twoheadrightarrow \alpha)$ with the $\mathcal{M}$-relation $<$ given by
$f < f' \;\Leftrightarrow\; f \neq f'$ and $f(m_{f,f'}) < f'(m_{f,f'})$ for $m_{f,f'} = \max\left[\gamma < \beta : f(\gamma) \neq f'(\gamma)\right]$.

We will prove by induction that there is a bijective $\mathcal{M}$-function $h_\beta\colon (\beta \twoheadrightarrow \alpha) \twoheadrightarrow \alpha^{(\beta)}$ that is order-preserving with order-preserving inverse.

Explicitly, h_β will be given as $h_\beta(f) = \alpha^{(m_f)} \cdot f(m_f) + h_{m_f}(f|_{m_f})$ if $\beta > \underline{0}$.

In case $\beta = \underline{0}$ we have $(\beta \twoheadrightarrow \alpha) = \underline{1} = \alpha^{(\beta)}$ and everything works out.

In case $\beta \in \mathbb{O}^{\mathcal{M}}_{+\underline{1}}$ let h_β be the composition of the chain of $\mathcal{M}$-bijections

$$(\beta \twoheadrightarrow \alpha) \longrightarrow ((\beta - \underline{1}) \twoheadrightarrow \alpha) * \alpha \longrightarrow \alpha^{(\beta-\underline{1})} * \alpha \longrightarrow \alpha^{(\beta-\underline{1})} \cdot \alpha = \alpha^{(\beta)}$$

where the first map is the $\mathcal{M}$-bijection formed by the $\mathcal{M}$-class function from (2), the second map is given by the rule $\langle f, \gamma\rangle \mapsto \langle h_{\beta-\underline{1}}(f), \gamma\rangle$, and the third map is the one described in (b). It is straightforward to verify that h_β satisfies the explicit formula stated above. Let's finally check that as required $f < f' \Leftrightarrow h_\beta(f) < h_\beta(f')$ for all distinct $f, f' \sqsubseteq (\beta \twoheadrightarrow \alpha)$, which is equivalent to checking

$$f < f' \;\Leftrightarrow\; \langle h_{\beta-\underline{1}}(f), f(\beta - \underline{1})\rangle < \langle h_{\beta-\underline{1}}(f'), f'(\beta - \underline{1})\rangle\,.$$

If $m_{f,f'} < \beta - \underline{1}$, this follows by induction from the corresponding property of $h_{\beta-\underline{1}}$. Otherwise, it is clear by definition of the orders on $(\beta \twoheadrightarrow \alpha)$ and $\alpha^{(\beta-\underline{1})} * \alpha$.

In case $\beta \in \mathbb{O}^{\mathcal{M}}_{\lim}$ let h_β be the composition of the chain of $\mathcal{M}$-functions

$$(\beta \twoheadrightarrow \alpha) \longrightarrow \left[f \sqsubseteq \bigsqcup_{\gamma<\beta}(\gamma \twoheadrightarrow \alpha) : \mathrm{dom}(f) = m_f + \underline{1}\right] \longrightarrow \bigsqcup_{\gamma<\beta} \alpha^{(\gamma)} = \alpha^{(\beta)}$$

where the first map is the $\mathcal{M}$-bijection formed by the $\mathcal{M}$-class function from (3) and the second map is given by $f \mapsto h_{\mathrm{dom}(f)}(f)$. It is invertible with inverse $\varepsilon \mapsto h^{-1}_{\gamma_\varepsilon}(\varepsilon)$ where $\gamma_\varepsilon = \min\left[\gamma < \beta : \varepsilon < \alpha^{(\gamma)}\right]$. Thus h_β is bijective. Again, it is straightforward to verify that h_β satisfies the above formula. Moreover, h_β is an order isomorphism since all h_γ with $\gamma < \beta$ are and for $f, f' \sqsubseteq (\beta \twoheadrightarrow \alpha)$ we have $m_{f,f'} + \underline{1} < \beta$ and

$$f < f' \;\Leftrightarrow f|_{m_{f,f'}+\underline{1}} < f'|_{m_{f,f'}+\underline{1}}\,.$$