Logic and Set Theory Universität Bielefeld
Solutions 6 SS 2019

1.

(a) Pick a well-order $<$ on $Z = \bigsqcup_{|\mu|=\mu<\kappa} \mathtt{P}(\mu * \mu)$ and consider the $\mathcal{M}$-class function $f\colon \mathbb{H}_\kappa \to \underline{\in}^{-1}(Z)$ that maps $\mathcal{M}$-sets $X \in \mathbb{H}_\kappa$ to the $<$-minimal $\prec\, \underline{\in}\, \mathtt{P}(|X^\infty| * |X^\infty|)$ such that there exists an isomorphism $\langle |X^\infty|, \prec\rangle \mathrel{-\!\!\triangleright} \langle X^\infty, \underline{\in}\rangle$, which is then uniquely determined by Theorem 2.5.9. Because X is well-founded, X can be recovered as the $\underline{\in}$-maximal element of X^∞. This shows that f is injective and so $\mathbb{H}_\kappa$ forms an $\mathcal{M}$-set H_κ of cardinality $|H_\kappa| \le |Z| \le \underline{2}^{<\kappa}$, where the last inequality holds since $|\mathtt{P}(\mu * \mu)| = \underline{2}^{|\mu*\mu|} = \underline{2}^\mu \le \underline{2}^{<\kappa}$ for all $\mathcal{M}$-cardinals $\omega \le \mu < \kappa$ and $\kappa \le \underline{2}^{<\kappa}$ by 3. (a).

Conversely, $\underline{2}^{<\kappa} \le |H_\kappa|$ because of $\underline{2}^\mu = |\mathtt{P}(\mu)| \le |H_\kappa|$ for every $\mathcal{M}$-cardinal $\mu < \kappa$, since for each $X \,\underline{\in}\, \mathtt{P}(\mu)$ we have $X^\infty \sqsubseteq \mu^\infty$, so $|X^\infty| \le |\mu^\infty| = \mu < \kappa$, so $X \,\underline{\in}\, H_\kappa$.

(b) We prove $|W_{\omega+\beta}| = \beth_\beta$ for $\beta \in \mathbb{O}$. By induction assume $|W_{\omega+\alpha}| = \beth_\alpha$ for $\alpha < \beta$.

If β is a successor, then $|W_{\omega+\beta}| = |\mathtt{P}(W_{\omega+\beta-\underline{1}})| = \underline{2}^{|W_{\omega+\beta-\underline{1}}|} = \underline{2}^{\beth_{\beta-\underline{1}}} = \beth_\beta$.

If β is a limit $> \underline{0}$, then $W_{\omega+\alpha} \sqsubseteq W_{\omega+\beta}$ for $\alpha < \beta$, so $\beth_\alpha = |W_{\omega+\alpha}| \le |W_{\omega+\beta}|$ and

$$\beth_\beta \;\le\; |W_{\omega+\beta}| \;=\; \left|\bigsqcup\nolimits_{\alpha<\beta} W_{\omega+\alpha}\right| \;\le\; \max\left[\beta, \bigsqcup\nolimits_{\alpha<\beta} |W_{\omega+\alpha}|\right] \;=\; \max\left[\beta, \beth_\beta\right] \;=\; \beth_\beta\,.$$

Finally, $|W_{\omega+\underline{0}}| = \omega = \beth_{\underline{0}}$ because of $\omega \sqsubseteq W_\omega$ and $|W_\alpha| < \omega$ for all $\alpha < \omega$.

It is thus enough to check $\omega + \kappa = \kappa$ for $\mathcal{M}$-cardinals $\kappa > \omega$. But for all $\gamma < \kappa$ it is $|\omega + \gamma| = \omega \oplus |\gamma| \le \max[\omega, \gamma] < \kappa$, so $\omega + \gamma < \kappa$ and $\kappa \le \omega + \kappa = \bigsqcup_{\gamma<\kappa}(\omega + \gamma) \le \kappa$.

(c) To prove the inclusion let $Y \in \mathbb{H}_\kappa$. We know $[\mathrm{rk}(X) : X \,\underline{\in}\, Y^\infty] = \mathrm{rk}(Y) + \underline{1}$ from 2. (b) on Problem Set 5, so $|\mathrm{rk}(Y)| = |\mathrm{rk}(Y) + \underline{1}| \le |Y^\infty| < \kappa$, so indeed $\mathrm{rk}(Y) < \kappa$.

Assume now $\kappa = \beth_\kappa$ and let $Y \in \mathbb{W}_\kappa$. Because κ is a limit ordinal greater than ω we have $Y \in \mathbb{W}_{\omega+\alpha}$ for some $\mathcal{M}$-ordinal α with $\omega+\alpha < \kappa$. But then we get $Y^\infty \sqsubseteq W_{\omega+\alpha}$ and therefore $|Y^\infty| \le |W_{\omega+\alpha}| = \beth_\alpha < \beth_\kappa = \kappa$, which means that $Y \in \mathbb{H}_\kappa$.

Assume now that $\mathbb{H}_\kappa = \mathbb{W}_\kappa$. Then for every $\alpha < \kappa$ it is also $\omega + \alpha + \underline{1} < \kappa$ such that $W_{\omega+\alpha} \,\underline{\in}\, W_{\omega+\alpha+\underline{1}} \sqsubseteq W_\kappa = H_\kappa$, so $\beth_\alpha = |W_{\omega+\alpha}| \le |W^\infty_{\omega+\alpha}| < \kappa$, so $\beth_\kappa = \bigsqcup_{\alpha<\kappa} \beth_\alpha \le \kappa$. The inequality $\kappa \le \beth_\kappa$ holds anyway because $\beth$ is normal.

(d,e) We will apply the lemma stated in the solution of Exercise 3 on Problem Set 5.

Either $T \in \{\mathbb{H}_\kappa, \mathbb{W}_\kappa\}$ is a transitive $\mathcal{M}$-class with $T \subseteq \mathbb{W}$ and $\boxslash, \bigsqcup X, Z \sqcap X \in T$ for all $X, Y \in T$ and $\mathcal{M}$-sets Z. Furthermore, $[X, Y] \in T$ because κ is a limit ordinal and $\omega \in T$ because of $\kappa > \omega$. Hence, $\mathcal{M}|_T \vDash \mathtt{EXT} \cup \mathtt{EMP} \cup \mathtt{PAI} \cup \mathtt{UNI} \cup \mathtt{INF} \cup \mathtt{CHO} \cup \mathtt{REG}$.

(d) It remains to check $\mathcal{M}|_{\mathbb{H}_\kappa} \vDash \mathtt{REP}$. For this, it is sufficient to show that $f[X] \in \mathbb{H}_\kappa$ for each partial $\mathcal{M}$-class function $f\colon \underline{\in}^{-1}(X) \nrightarrow \mathbb{H}_\kappa$ with $X \in \mathbb{H}_\kappa$. Choose a bijective $g\colon \lambda = |f[X]| \mathrel{-\!\!\triangleright} f[X]$ and set $\mu_\gamma = |g(\gamma)^\infty|$. Then $\lambda < \kappa$ and $\mu_\gamma < \kappa$ for all $\gamma < \lambda$. Using the fact that $f[X]^\infty = [f[X]] \sqcup \bigsqcup_{Y \,\underline{\in}\, f[X]} Y^\infty$ we conclude $f[X] \in \mathbb{H}_\kappa$ from

$$|f[X]^\infty| \;\le\; \underline{1} \oplus \bigoplus\nolimits_{\gamma<\lambda} \mu_\gamma \;<\; \kappa\,,$$

where the strict inequality is due to $\lambda < \kappa = \mathrm{cof}(\kappa)$ and Theorem 2.10.24.

(e) It only remains to check $\mathcal{M}|_{\mathbb{W}_\kappa} \vDash \mathtt{POW}$. This follows from $\mathtt{P}(X) \sqcap \mathbb{W}_\kappa = \mathtt{P}(X) \in \mathbb{W}_\kappa$ for all $X \in \mathbb{W}_\kappa$, which holds because κ is a limit ordinal.

2.

(a) We frequently use below that κ as an infinite $\mathcal{M}$-cardinal is a limit $\mathcal{M}$-ordinal.

(1) $\Rightarrow$ (2) For all $\mathcal{M}$-cardinals $\mu, \lambda < \kappa$ use that $\mu^\lambda \le (\underline{2}^\mu)^\lambda = \underline{2}^{\mu \otimes \lambda} = \max\left[\underline{2}^\mu, \underline{2}^\lambda\right]$ by Lemma 2.10.9 and Theorem 2.10.7.

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are clear.

(3) $\Leftrightarrow$ (5) holds by 1. (c).

(1) $\Rightarrow$ (3) Because $\beth$ is normal and κ is a limit ordinal, we have $\kappa \le \beth_\kappa = \bigsqcup_{\alpha<\kappa} \beth_\alpha$. It thus is sufficient to prove $\beth_\alpha < \kappa$ for all $\alpha < \kappa$.

Firstly, $\beth_{\underline{0}} = \omega < \kappa$ by assumption. Secondly, if α is a successor and we inductively assume $\beth_{\alpha-\underline{1}} < \kappa$, then $\beth_\alpha = \underline{2}^{\beth_{\alpha-\underline{1}}} < \kappa$ by (1). Thirdly, if α is a non-zero limit and by induction $\beth_\gamma < \kappa$ for all $\gamma < \alpha$, then $\beth_\alpha = \bigsqcup_{\gamma<\alpha} \beth_\gamma < \kappa$ because $\alpha < \kappa = \mathrm{cof}(\kappa)$.

(3) $\Rightarrow$ (4) Assuming we know $\kappa = f(\kappa)$ for $f = \beth^{(n)}$ and some $n \in \mathbb{N}$, then $\kappa = f'(\alpha)$ for some $\mathcal{M}$-ordinal α. Now $\alpha \le \kappa$ because f' is normal. If α were a successor ordinal, then $\kappa = f^\omega(f'(\alpha - \underline{1}) + \underline{1})$ would lead to the contradiction $\kappa = \mathrm{cof}(\kappa) \le \omega$. So α is a limit ordinal. We then have $\kappa = \bigsqcup f'[\alpha]$ and thus also $\kappa = \mathrm{cof}(\kappa) \le \alpha$.

(5) $\Rightarrow$ (6) If $\mathbb{H}_\kappa = \mathbb{W}_\kappa$, then $\mathbb{W}_\kappa$ is a Grothendieck universe because $\mathbb{W}_\kappa$ clearly is a transitive $\mathcal{M}$-class and by (the proof of) 1. (d,e) $\mathcal{M}|_{\mathbb{W}_\kappa}$ is a ZFC-universe with

$$[X,Y]^{\mathcal{M}|_{\mathbb{W}_\kappa}} = [X,Y] \quad \text{and} \quad \mathtt{P}^{\mathcal{M}|_{\mathbb{W}_\kappa}}(X) = \mathtt{P}(X) \quad \text{and} \quad \bigsqcup\nolimits_{i \in I}^{\mathcal{M}|_{\mathbb{W}_\kappa}} X_i = \bigsqcup\nolimits_{i\in I} X_i$$

for all $\mathcal{M}$-sets $X, Y \in \mathbb{W}_\kappa$ and families $\langle X_i \rangle_{i \in I}$ of $\mathcal{M}$-sets in $\mathbb{W}_\kappa$ with $I \in \mathbb{W}_\kappa$.

(6) $\Rightarrow$ (1) For $\lambda < \kappa$ we have $\lambda \in \mathbb{W}_\kappa$, so $\mathtt{P}(\lambda) \in \mathbb{W}_\kappa$ by (iii). Because of $\underline{2}^\lambda = |\mathtt{P}(\lambda)|$ and $\mathbb{O} \cap \mathbb{W}_\kappa = \mathbb{O}_{<\kappa}$ it suffices to prove the following lemma:

Lemma. *If $\mathbb{W}_\kappa$ is a Grothendieck universe in $\mathcal{M}$, then $|X| \in \mathbb{W}_\kappa$ for every $X \in \mathbb{W}_\kappa$.*

Proof. Assume there is some $X \in \mathbb{W}_\kappa$ with $|X| \notin \mathbb{W}_\kappa$. Then we must have $\kappa \le |X|$. Choose any bijective $f\colon X \twoheadrightarrow |X|$ and let $g\colon |X| \twoheadrightarrow \mathbb{W}_\kappa$ be given by $g|_\kappa = \mathrm{id}_\kappa$ and $g(\alpha) = \underline{0}$ for all $\kappa \le \alpha < |X|$. Using (iv) in the last step, we get the contradiction

$$\kappa = \bigsqcup \kappa = \bigsqcup |X| \sqcap \kappa = \bigsqcup f[X] \sqcap \kappa = \bigsqcup (g \circ f)[X] \in \mathbb{W}_\kappa .$$

□

(b) By Theorem 2.7.4 we can assume that $\mathcal{M}$ is a ZFC-universe.

If $\mathcal{M}$ has no inaccessible cardinals, we are done. Otherwise, let κ be the smallest inaccessible $\mathcal{M}$-cardinal. By (a) and 1. (d,e) $\mathcal{M}|_{\mathbb{W}_\kappa}$ is a ZFC-universe where $\mathbb{W}_\kappa$ is a Grothendieck universe in $\mathcal{M}$. It is clearly enough to show that $\mathbb{K}^{\mathcal{M}|_{\mathbb{W}_\kappa}} = \mathbb{K} \cap \mathbb{W}_\kappa$ and that moreover an $\mathcal{M}|_{\mathbb{W}_\kappa}$-cardinal is inaccessible if and only if it is inaccessible as an $\mathcal{M}$-cardinal. Now (ii,iii,iv) imply $[X \twoheadrightarrow Y]^{\mathcal{M}|_{\mathbb{W}_\kappa}} = [X \twoheadrightarrow Y]$ for all $X, Y \in \mathbb{W}_\kappa$ and together with the lemma proved in (a) we obtain $\mathbb{K}^{\mathcal{M}|_{\mathbb{W}_\kappa}} = \mathbb{K} \cap \mathbb{W}_\kappa$ as desired. But this also shows that the λ-th power of $\underline{2}$ computed in $\mathcal{M}|_{\mathbb{W}_\kappa}$ simply is

$$\left|[\lambda \twoheadrightarrow \underline{2}]^{\mathcal{M}|_{\mathbb{W}_\kappa}}\right|^{\mathcal{M}|_{\mathbb{W}_\kappa}} = \left|[\lambda \twoheadrightarrow \underline{2}]\right| = \underline{2}^\lambda .$$

Hence, using the characterization (1) from (a), the inaccessible $\mathcal{M}|_{\mathbb{W}_\kappa}$-cardinals are nothing but the inaccessible $\mathcal{M}$-cardinals that lie in $\mathbb{W}_\kappa$. Because of $\mathbb{K}^{\mathcal{M}|_{\mathbb{W}_\kappa}} = \mathbb{K}_{<\kappa}$ and the minimal choice of κ we can conclude that $\mathcal{M}|_{\mathbb{W}_\kappa}$ has no inaccessible cardinals.

3.

(a) In case $\kappa = \lambda^+$ we have $\kappa = \lambda^+ \leq \underline{2}^\lambda = \underline{2}^{<\kappa}$ and in case κ is a limit cardinal $\kappa = \bigsqcup_{|\mu|=\mu<\kappa} \mu \leq \underline{2}^{<\kappa}$ in view of Lemma 2.10.9. The other inequality is obvious.

To prove the equality let $\lambda = \operatorname{cof}(\kappa)$. Then by Theorem 2.10.24 there exists a family $\langle \mu_\gamma \rangle_{\gamma<\lambda}$ of $\mathcal{M}$-cardinals with $\mu_\gamma < \kappa$ for all $\gamma < \lambda$ and $\kappa = \bigoplus_{\gamma<\lambda} \mu_\gamma$. We compute

$$\underline{2}^\kappa \;=\; \bigotimes_{\gamma<\lambda} \underline{2}^{\mu_\gamma} \;\leq\; \bigotimes_{\gamma<\lambda} \underline{2}^{<\kappa} \;=\; (\underline{2}^{<\kappa})^\lambda \;\leq\; (\underline{2}^\kappa)^\lambda \;=\; \underline{2}^{\kappa\otimes\lambda} \;=\; \underline{2}^\kappa\,.$$

(b) $\underline{2} \leq \kappa \leq \underline{2}^\lambda$ for infinite $\kappa = \lambda^+$ by Lemma 2.10.9 and so $\underline{2}^{<\kappa} = \underline{2}^\lambda = \kappa^\lambda = \kappa^{<\kappa}$ according to Theorem 2.10.7.

(c) For $\Rightarrow$ assume $\underline{2}^\mu = \mu^+$ for all infinite $|\mu| = \mu < \kappa$ to see $\underline{2}^{<\kappa} = \bigsqcup_{|\mu|=\mu<\kappa} \mu^+ = \kappa$. For $\Leftarrow$ note that $\underline{2}^\kappa = \underline{2}^{<\kappa^+}$ and use the assumption $\underline{2}^{<\kappa^+} = \kappa^+$.

(d) Using (a) in the first step and $\operatorname{cof}(\kappa) < \kappa$ in the last step we compute

$$\underline{2}^\kappa \;=\; (\underline{2}^{<\kappa})^{\operatorname{cof}(\kappa)} \;\leq\; (\underline{2}^\lambda)^{\operatorname{cof}(\kappa)} \;=\; \underline{2}^{\lambda\otimes\operatorname{cof}(\kappa)} \;=\; \max\left[\underline{2}^\lambda, \underline{2}^{\operatorname{cof}(\kappa)}\right] \;=\; \underline{2}^\lambda\,.$$

4. Finding a sequence $\langle U_\gamma \rangle_{\gamma<\kappa^+}$ of $\mathcal{M}$-functions $\kappa \multimap \kappa$ such that for all $\gamma, \gamma' < \kappa^+$ there is some $\varepsilon < \kappa$ with $U_{\gamma,\gamma'} = [\tau < \kappa : U_\gamma(\tau) = U_{\gamma'}(\tau)] \sqsubseteq \varepsilon$ will prove the exercise.

Indeed, taking $X' = [U_\gamma : \gamma < \kappa^+]$ we then have $|X'| = \kappa^+$, since the U_γ are pairwise distinct, and of course for all functions $U, V : \kappa \multimap \kappa$ with $[\tau < \kappa : U(\tau) = V(\tau)] \sqsubseteq \varepsilon$ for some $\varepsilon < \kappa$ we have $|U| = |V| = \kappa$ and $|U \sqcap V| \leq \varepsilon < \kappa$. Choosing a bijective $\mathcal{M}$-function $f : \kappa * \kappa \multimap \kappa$ we can finally take $X = [f[U_\gamma] : \gamma < \kappa^+]$.

Let's now turn to the construction of the sequence $\langle U_\gamma \rangle_{\gamma<\kappa^+}$.

Before we begin with the work, we use Choice for the existence of a family $\langle f_\delta \rangle_{\delta<\kappa^+}$ of surjective $f_\delta : \kappa \multimap \delta$ and for the existence of a well-order $\prec$ on $[\kappa \multimap \kappa]$.

Recursively, assume that $\langle U_\gamma \rangle_{\gamma<\delta}$ is given for some $\delta < \kappa^+$ such that for all $\gamma, \gamma' < \delta$ there is some $\varepsilon < \kappa$ with $U_{\gamma,\gamma'} \sqsubseteq \varepsilon$. Define U_δ to be the $\prec$-least element in the $\mathcal{M}$-set $\mathop{\ast}_{\tau<\kappa}(\kappa \setminus [U_{f_\delta(\sigma)}(\tau) : \sigma < \tau])$, which is non-empty since $\left|[U_{f_\delta(\sigma)}(\tau) : \sigma < \tau]\right| \leq \tau < \kappa$ for all $\tau < \kappa$ and thanks to Choice. To conclude, it merely remains to observe that for all $\gamma < \delta$ we have $U_{\gamma,\delta} \sqsubseteq \sigma + \underline{1}$ for any $\sigma < \kappa$ with $f_\delta(\sigma) = \gamma$.