REPRESENTATION THEORY
EXERCISES 1

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1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with a fully faithful right adjoint and set

$$S \;=\; \{\sigma \in \operatorname{Mor}\mathcal{C} : F\sigma \text{ invertible}\}\,.$$

Prove that S admits a calculus of left fractions.

2. Let A be a ring and S a multiplicatively closed subset of A. Recall that a *right ring of fractions* of A with respect to S is a ring AS^{-1} together with a ring homomorphism $\varphi\colon A \to AS^{-1}$ satisfying:

(1) $\varphi(S) \subseteq (AS^{-1})^{\times}$.
(2) $AS^{-1} = \{\varphi(a)\varphi(s)^{-1} : a \in A, s \in S\}$.
(3) $\operatorname{Ker}\varphi = \{a \in A : as = 0 \text{ for some } s \in S\}$.

Show that the right ring of fractions AS^{-1} exists if and only if the following two properties hold:

(i) *Right Ore condition:* For all $s \in S$ and $a \in A$ there exists $s' \in S$ and $a' \in A$ with $as' = sa'$.
(ii) *Right reversibility:* If $sa = 0$ for some $s \in S$ and $a \in A$, there exists $s' \in S$ with $as' = 0$.

In this case AS^{-1} satisfies the universal property of the localization $A[S^{-1}]$.

3. (a) Find an example of a ring A and a multiplicatively closed subset S in A such that the right ring of fractions AS^{-1} exists but the left ring of fractions $S^{-1}A$ does not.
(b) Let S be the set of non-zero divisors in the group algebra $A = K[G]$ of a finite group G over a field K. Show that the canonical map $A \to A[S^{-1}]$ is an isomorphism.

To be handed in via email by April 27, 2020, 2 p.m.