REPRESENTATION THEORY
EXERCISES 12

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Our goal is to compare the right derived functor $\mathbf{R}(G\circ F)$ of the composite $G\circ F$ of two left exact functors F and G with the composite $\mathbf{R}G\circ\mathbf{R}F$ of their right derived functors. To do this, we begin with an example and then refresh our knowledge about spectral sequences.

1. Let Λ be the path algebra of the quiver $\bullet\to\bullet$ over some field k and as usual $D=\operatorname{Hom}_k(-,k)$. Consider on $\operatorname{Mod}\Lambda$ the endofunctor $F=G=\operatorname{Hom}_\Lambda(D\Lambda,-)$. Show that $\mathbf{R}(G\circ F)\ncong\mathbf{R}G\circ\mathbf{R}F$.

From now on, fix an abelian category $\mathcal{A}$ with countable exact coproducts.
By a *differential object* in $\mathcal{A}$ we mean a pair (X,d) consisting of an object X in $\mathcal{A}$ together with an endomorphism $d\in\operatorname{End}_{\mathcal{A}}(X)$ such that $d^2=0$. Its *cohomology* is defined as $H(X)=\operatorname{Ker}d/\operatorname{Im}d$.
A *spectral sequence* in $\mathcal{A}$ is a sequence $E=(E_r)_{r\in\mathbb{N}_+}$ of differential objects with $E_{r+1}=H(E_r)$.
Given such E, define inductively $B_r=(\varepsilon_r\pi_r)^{-1}(\operatorname{Im}d_r)$ and $Z_r=(\varepsilon_r\pi_r)^{-1}(\operatorname{Ker}d_r)$ to obtain

$$0=B_0\subseteq\cdots\subseteq B_r\subseteq B_{r+1}\subseteq\cdots\subseteq Z_{r+1}\subseteq Z_r\subseteq\cdots\subseteq Z_0=E_1$$

where d_r is the differential of E_r and $Z_{r-1}\xrightarrow{\pi_r}Z_{r-1}/B_{r-1}\xrightarrow{\varepsilon_r}E_r$ are the canonical maps.
Granted existence of $B_\infty=\bigcup_r B_r$ and $Z_\infty=\bigcap_r Z_r$, the *limit* of E is $E_\infty=Z_\infty/B_\infty$.

2. An *exact couple* in $\mathcal{A}$ is a triple $\Gamma=(\alpha,\beta,\gamma)$ where $\cdots\xrightarrow{\gamma}A\xrightarrow{\alpha}A\xrightarrow{\beta}X\xrightarrow{\gamma}\cdots$ is exact in $\mathcal{A}$.
Verify the following facts about exact couples Γ in $\mathcal{A}$:

(a) The pair $X_\Gamma=(X,\beta\gamma)$ is a differential object in $\mathcal{A}$.

(b) Γ gives rise to another exact couple Γ' where $A'=\operatorname{Im}\alpha$ and $X'=H(X_\Gamma)$ and the map α' is induced by α, the map β' by $\beta\alpha^{-1}$ and the map γ' by γ.

(c) Γ gives rise to a spectral sequence E given by $E_r=X_{\Gamma_r}$ where $\Gamma_1=\Gamma$ and $\Gamma_{r+1}=\Gamma'_r$. The differentials d_r of E_r are thus induced by $\beta\alpha^{-r+1}\gamma$.

Check for each exact sequence $\eta\colon 0\to X\xrightarrow{f}X\xrightarrow{g}Y\to 0$ of differential objects in $\mathcal{A}$:

(d) η gives rise to a spectral sequence E_η that is induced by the exact couple $(H(f),H(g),\delta)$ where δ is the connecting morphism in cohomology obtained from the snake lemma.

Now convince yourself of the following facts that hold for each *filtered* differential object X in $\mathcal{A}$, i.e. coming equipped with a filtration of differential subobjects $\cdots\subseteq X^{p+1}\subseteq X^p\subseteq\cdots\subseteq X$:

(e) The spectral sequence E induced by $\bigoplus_p(0\to X^{p+1}\to X^p\to X^p/X^{p+1}\to 0)$ starts with

$$E_1=\bigoplus\nolimits_p H(X^p/X^{p+1}).$$

(f) Whenever for the spectral sequence from (e) we have in each X^p/X^{p+1} the identities

$$\bigcup\nolimits_r d(X^{p-r})\cap X^p=\operatorname{Im}d\cap X^p\quad\text{and}\quad\bigcap\nolimits_r d^{-1}(X^{p+r})\cap X^p=\operatorname{Ker}d\cap X^p,$$

we say E *p-converges* to $H(X)^p$, since in this situation there is a canonical isomorphism

$$E_\infty\cong\bigoplus\nolimits_p H(X)^p/H(X)^{p+1}.$$

To record this fact, we use the common notation $E_r^p\Rightarrow_p H(X)^p$.

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Next, recall that a *double complex* $C = (C, d_\to, d_\uparrow)$ consists of a $(\mathbb{Z}\times\mathbb{Z})$-graded object C in $\mathcal{A}$ with maps $d_\to$ of degree $(1,0)$ and $d_\uparrow$ of degree $(0,1)$ such that $d_\to^2 = d_\uparrow^2 = d_\to d_\uparrow + d_\uparrow d_\to = 0$.

Denote by $\operatorname{Tot} C$ the *total complex*, i.e. the differential object $(\bigoplus_{i,j} C^{i,j}, d_\to + d_\uparrow)$. There are two natural ways to view it as a filtered differential object: as $\operatorname{Tot}_\to C$ and as $\operatorname{Tot}_\uparrow C$ with components

$$\operatorname{Tot}^p_\to C \;=\; \bigoplus\nolimits_{j\geq p} C^{\bullet,j} \quad\text{and}\quad \operatorname{Tot}^p_\uparrow C \;=\; \bigoplus\nolimits_{i\geq p} C^{i,\bullet}\,.$$

Let ${}_\to E$ and ${}_\uparrow E$ be the spectral sequences induced by $\operatorname{Tot}_\to C$ and $\operatorname{Tot}_\uparrow C$, respectively.

Check the statements below:

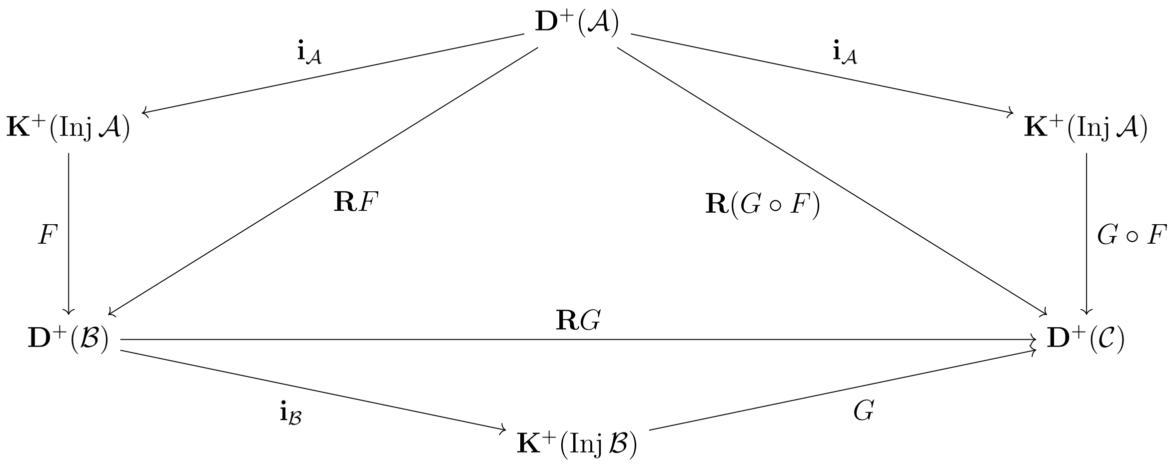
(g) With the bigradings $H(\operatorname{Tot}_\to C)^{p,q} = H^{p+q}(\operatorname{Tot}^p_\to C)$ and $H(\operatorname{Tot}_\uparrow C)^{p,q} = H^{p+q}(\operatorname{Tot}^p_\uparrow C)$ the differentials ${}_\to d_r$ and ${}_\uparrow d_r$ are homogeneous maps of degree $(r, -r+1)$ and we have

$$\begin{aligned} {}_\to E_1^{p,q} \;&\cong\; H^q_\to(C^{\bullet,p}) \qquad\text{and}\qquad {}_\to E_2^{p,q} \;\cong\; H^p_\uparrow(H^q_\to(C^{\bullet,\bullet}))\,,\\ {}_\uparrow E_1^{p,q} \;&\cong\; H^q_\uparrow(C^{p,\bullet}) \qquad\text{and}\qquad {}_\uparrow E_2^{p,q} \;\cong\; H^p_\to(H^q_\uparrow(C^{\bullet,\bullet}))\,. \end{aligned}$$

Moreover, ${}_\to d_1^{p,q}$ and ${}_\uparrow d_1^{p,q}$ identify with $H^q_\to(d_\uparrow^{\bullet,p})$ and $H^q_\uparrow(d_\to^{p,\bullet})$ under the left-hand maps.

(h) If C is *mostly positively* (resp. *mostly negatively*) graded, i.e. there is n such that $C^{i,j}\neq 0$ implies $i,j\geq n$ (resp. $i,j\leq n$), each of ${}_\to E^{p,q}$ and ${}_\uparrow E^{p,q}$ p-converges to $H^{p+q}(\operatorname{Tot} C)$.

3. Let $\mathcal{A}\xrightarrow{F}\mathcal{B}$ and $\mathcal{B}\xrightarrow{G}\mathcal{C}$ be left exact functors between abelian categories with enough injectives and countable exact coproducts. Recall that by definition of the derived functors we have a diagram



where the outer triangles commute and the functors $\mathbf{i}_\mathcal{X}$ are left inverse quasi-inverses of the canonical embeddings $\mathbf{K}^+(\operatorname{Inj}\mathcal{X})\to\mathbf{D}^+(\mathcal{X})$. In particular, there is a canonical natural transformation

$$\mathbf{R}(G\circ F) \longrightarrow \mathbf{R}G\circ\mathbf{R}F\,.$$

Convince yourself of the following facts:

(a) For $X\in\mathcal{A}$ the canonical map $F(X)\to\mathbf{R}F(X)$ is invertible iff $R^iF(X)=0$ for all $i\neq 0$. If this is the case, the object X is said to be *right F-acyclic*.

(b) The map $\mathbf{R}(G\circ F)\to\mathbf{R}G\circ\mathbf{R}F$ is invertible iff $F(\operatorname{Inj}\mathcal{A})$ consists of G-acyclic objects.

(c) Every $X\in\mathbf{C}^+(\mathcal{A})$ admits a *Cartan–Eilenberg resolution*, i.e. a mostly positively $(\mathbb{Z}\times\mathbb{N})$-graded double complex C in $\operatorname{Inj}\mathcal{A}$ together with a map $\iota\colon X\to C^{\bullet,0}$ such that for each p the following diagram commutes and all of its columns are injective resolutions in $\mathcal{A}$:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\operatorname{Ker} d_{\to}^{p,1} & \hookrightarrow & C^{p,1} & \twoheadrightarrow & \operatorname{Im} d_{\to}^{p,1} & & H^p_{\to}(C^{\bullet,1}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\operatorname{Ker} d_{\to}^{p,0} & \hookrightarrow & C^{p,0} & \longrightarrow & \operatorname{Im} d_{\to}^{p,0} & & H^p_{\to}(C^{\bullet,0}) \\
\uparrow & & \uparrow \iota & & \uparrow & & \uparrow \iota \\
\operatorname{Ker} d_X^p & \hookrightarrow & X^p & \twoheadrightarrow & \operatorname{Im} d_X^p & & H^p(X)
\end{array}
$$

Show for any Cartan–Eilenberg resolution (C, ι) of $X \in \mathbf{C}^+(\mathcal{A})$:

(d) The map ι induces an isomorphism $X \to \operatorname{Tot} C$ in $\mathbf{D}^+(\mathcal{A})$.

(e) For the spectral sequences ${}_{\to}E$ and ${}_{\uparrow}E$ of the double complex $F(C)$ we have

$$ {}_{\to}E_2^{p,q} \;\cong\; R^pF(H^q(X)) \qquad \text{and} \qquad {}_{\uparrow}E_1^{p,q} \;\cong\; R^qF(X^p)\,. $$

Both of these spectral sequences p-converge to $R^{p+q}F(X)$.

(f) If $\mathbf{R}(G \circ F) \to \mathbf{R}G \circ \mathbf{R}F$ is invertible, there is a spectral sequence E in $\mathcal{A}$ with

$$ E_2^{p,q} \;=\; R^pG(R^qF(X)) \;\Rightarrow_p\; R^{p+q}(G \circ F)(X)\,. $$

This is known as *Grothendieck's spectral sequence*.

4. Let Λ be a ring. Show that for each left Λ-module M and each complex $X \in \mathbf{C}^-(\operatorname{Mod}\Lambda)$ there exists a spectral sequence E such that

$$ E_2^{p,q} \;=\; \operatorname{Tor}_p^{\Lambda}(H^q(X), M) \;\Rightarrow_p\; H^{p+q}\big(X \otimes_{\Lambda}^{\mathbf{L}} M\big)\,. $$