REPRESENTATION THEORY
EXERCISES 2

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1. (a) Let $\mathcal{A}$ be an exact category such that every morphism f in $\mathcal{A}$ admits a factorization $f = gh$ where g is an admissible mono and f is an admissible epi. Prove that $\mathcal{A}$ is abelian.

(b) Let $(\mathcal{A}, \mathcal{E}) \xrightarrow{F} (\mathcal{B}, \mathcal{F})$ be an exact functor and let $\mathcal{F}' \subseteq \mathcal{F}$ be another exact structure on $\mathcal{B}$. Show that $(\mathcal{A}, \mathcal{E} \cap F^{-1}(\mathcal{F}'))$ is an exact category.

2. (a) Let $\mathcal{A}$ be a cocomplete abelian category satisfying (AB5) and let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$ that is closed under coproducts. Prove that $\mathcal{A}/\mathcal{C}$ is cocomplete and satisfies (AB5).

(b) Give an example of a cocomplete abelian category that does not satisfy condition (AB5).

3. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an almost split sequence in an abelian category $\mathcal{A}$.

We denote by $E_X = \operatorname{End}_{\mathcal{A}}(X)$ and $E_Z = \operatorname{End}_{\mathcal{A}}(Z)$ the endomorphism rings of the end terms and consider the effaceable functors $S \in \operatorname{eff} \mathcal{A}$ and $S^\vee \in \operatorname{eff} \mathcal{A}^{\mathrm{op}}$ defined by the exact sequences

$$\operatorname{Hom}_{\mathcal{A}}(-, Y) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(-,g)} \operatorname{Hom}_{\mathcal{A}}(-, Z) \longrightarrow S \longrightarrow 0\,,$$

$$\operatorname{Hom}_{\mathcal{A}}(Y, -) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(f,-)} \operatorname{Hom}_{\mathcal{A}}(X.-) \longrightarrow S^\vee \longrightarrow 0\,.$$

Recall or verify the following facts:

(a) E_X and E_Z are local.

(b) S and $S^\vee$ are simple.

(c) $\operatorname{End} S \cong \operatorname{top} E_Z$ and $\operatorname{End} S^\vee \cong \operatorname{top} E_X^{\mathrm{op}}$.

Conclude that there is a canonical isomorphism $\operatorname{top} E_X \cong \operatorname{top} E_Z$.

To be handed in via email by May 4, 2020, 2 p.m.