REPRESENTATION THEORY
EXERCISES 4

HENNING KRAUSE
JAN GEUENICH

1. Let $\mathcal{T}$ be an additive category with shift functor Σ. *Candidate triangles* are triangles (α, β, γ) such that $(\Sigma^{-1}\gamma, \alpha, \beta, \gamma, \Sigma\alpha)$ forms a complex. A morphism between two candidate triangles in $\mathcal{T}$

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

is *null-homotopic* if there are maps $\Phi_1 \colon Y \to X'$, $\Phi_2 \colon Z \to Y'$, $\Phi_3 \colon \Sigma X \to Z'$ in $\mathcal{T}$ satisfying:

$$\begin{aligned} \phi_1 &= \Sigma^{-1}(\gamma' \circ \Phi_3) + \Phi_1 \circ \alpha \\ \phi_2 &= \alpha' \circ \Phi_1 + \Phi_2 \circ \beta \\ \phi_3 &= \beta' \circ \Phi_2 + \Phi_3 \circ \gamma \end{aligned}$$

A candidate triangle in $\mathcal{T}$ is said to be *contractible* if its identity morphism is null-homotopic.

Verify that the following statements hold in every triangulated category:

(a) Every contractible triangle is an exact triangle.

(b) Every exact triangle of the form $X \to Y \to Z \xrightarrow{0} \Sigma X$ *splits*, i.e. it is isomorphic to

$$X \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} X \oplus Z \xrightarrow{(0\,1)} Z \xrightarrow{0} \Sigma X\,.$$

(c) Triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ are exact iff their sum $(\alpha \oplus \alpha', \beta \oplus \beta', \gamma \oplus \gamma')$ is exact.

2. A triangulated category $\mathcal{T}$ is said to be *algebraic* if there is an exact equivalence between $\mathcal{T}$ and the stable category $\operatorname{St}\mathcal{A}$ of a Frobenius category $\mathcal{A}$ with the induced triangulated structure.

Prove the following:

(a) For every exact triangle of the form

$$X \xrightarrow{2\cdot \mathrm{id}_X} X \longrightarrow Z \longrightarrow \Sigma X$$

in an algebraic triangulated category we have $2 \cdot \mathrm{id}_Z = 0$.

(b) The category $\mathcal{T}$ of finitely generated projective modules over the ring $R = \mathbb{Z}/4\mathbb{Z}$ with shift $\Sigma = \mathrm{id}_{\mathcal{T}}$ can be endowed with a triangulated structure such that the exact triangles are the triangles isomorphic to finite direct sums of contractible triangles and the triangle

$$R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} \Sigma R = R\,.$$

(c) The triangulated category $\mathcal{T}$ in (b) is not algebraic.

To be handed in via email by May 18, 2020, 2 p.m.

3. Let $\mathcal{A}$ be an exact category and denote for objects $X, Z \in \mathcal{A}$ by $\mathrm{Ext}^1(Z,X)$ the collection of all isomorphism classes of admissible exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

Recall or verify the following facts:

(a) There are well-defined bilinear maps

$$\begin{aligned} \mathrm{Ext}^1(Z,X) \times \mathrm{Hom}(Z',Z) &\longrightarrow \mathrm{Ext}^1(Z',X)\,, \qquad (\eta, g) \longmapsto \eta \cdot g\,, \\ \mathrm{Hom}(X,'X) \times \mathrm{Ext}^1(Z,X) &\longrightarrow \mathrm{Ext}^1(Z,'X)\,, \qquad (f,\eta) \longmapsto f \cdot \eta\,, \end{aligned}$$

satisfying $f \cdot (\eta \cdot g) = (f \cdot \eta) \cdot g$ induced by commutative diagrams in $\mathcal{A}$ as follows:

$$\begin{array}{rccccccccc}
\eta \cdot g\colon & 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & & \Vert & & \downarrow & & \downarrow{\scriptstyle g} & & \\
\eta\ : & 0 & \longrightarrow & X & \xrightarrow{\ \alpha\ } & Y & \xrightarrow{\ \beta\ } & Z & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f} & & \downarrow & & \Vert & & \\
f \cdot \eta\colon & 0 & \longrightarrow & 'X & \longrightarrow & 'Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

Namely, $\eta \cdot g$ is obtained by pulling back β along g and $f \cdot \eta$ by pushing out α along f.

(b) For every morphism of admissible exact sequences in $\mathcal{A}$

$$\begin{array}{rccccccccc}
\xi\colon & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\
\xi'\colon & 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0
\end{array}$$

the identity $\phi_1 \cdot \xi = \xi' \cdot \phi_3$ holds in $\mathrm{Ext}^1(Z,X')$, i.e. we can factor $\phi_2 = \phi_2''\phi_2'$ such that there is a commutative diagram as drawn below:

$$\begin{array}{rccccccccc}
\xi\colon & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2'} & & \Vert & & \\
 & 0 & \longrightarrow & X' & \longrightarrow & \overline{Y} & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & \Vert & & \downarrow{\scriptstyle \phi_2''} & & \downarrow{\scriptstyle \phi_3} & & \\
\xi'\colon & 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0
\end{array}$$