REPRESENTATION THEORY EXERCISES 8

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1. Let Λ be a ring and M a Λ-bimodule. Recall that the *trivial extension* of Λ by M is the ring

$$T = \Lambda \ltimes M = \begin{pmatrix} \Lambda & M \\ 0 & \Lambda \end{pmatrix}.$$

It contains Λ as a subring via the diagonal embedding $\Lambda \to T$ and can be regarded as a $\mathbb{Z}$-graded ring with $T^0 = \Lambda$ and $T^1 = \left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$. Clearly, also the quotient T/T^+ is isomorphic to Λ.

Verify the following facts:

(a) T is canonically isomorphic to the quotient of the tensor algebra of M by the ideal $M^{\otimes 2}$.

(b) $T^\times = \Lambda^\times \ltimes M$ and $J(T) = J(\Lambda) \ltimes M$.

(c) $\operatorname{Mod} T$ is canonically isomorphic to a category with class of objects

$$\left\{ (X, x) \;\middle|\; \begin{array}{l} X \in \operatorname{Mod}\Lambda, \\ x \in \operatorname{Hom}_\Lambda(X \otimes_\Lambda M, X) \end{array} \text{ with } x \circ (x \otimes \mathrm{id}_M) = 0 \right\}.$$

(d) $\operatorname{GrMod} T$ is canonically isomorphic to a category with class of objects

$$\left\{ (X^n, x^n)_{n\in\mathbb{Z}} \;\middle|\; \begin{array}{l} X^n \in \operatorname{Mod}\Lambda, \\ x^n \in \operatorname{Hom}_\Lambda(X^n \otimes_\Lambda M, X^{n+1}) \end{array} \text{ with } x^n \circ (x^{n-1} \otimes \mathrm{id}_M) = 0 \right\}.$$

(e) $\operatorname{GrMod} T$ is equivalent to $\operatorname{Mod} R$ for the *repetitive algebra*

$$R = \begin{pmatrix} \ddots & \ddots & & \\ & \Lambda & M & \\ & & \Lambda & M \\ & & & \ddots & \ddots \end{pmatrix}$$

where multiplication of off-diagonal elements is defined as zero.

Fix now a commutative artinian ring k and denote by $D = \operatorname{Hom}_k(-, E)$ the Matlis duality over k. We assume in what follows that Λ is an artin algebra over k.

(f) If M is finitely generated over k, then T is an artin algebra over k.

Finally, we restrict to the important special case $M = D\Lambda$.

(g) T is a *symmetric algebra*, i.e. the T-bimodules T and DT are isomorphic.

(h) $\operatorname{grmod} T$ is a Frobenius category with $\operatorname{grproj} T = \operatorname{add}\{T(n) : n \in \mathbb{Z}\}$.

We finish with some explicit examples where k is assumed to be symmetric.

(i) For $\Lambda = k$ we have $T \cong k[x]/(x^2)$ with $\deg x = 1$.

(j) For $\Lambda = k[x]/(x^2)$ we have $T \cong k[x, y]/(x^2, y^2)$ with $\deg x = 0$ and $\deg y = 1$.

(k) For $\Lambda = k\big(\bullet \xrightarrow{x} \bullet\big)$ we have $T \cong k\big(\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet\big)/(xyx, yxy)$ with $\deg x = 0$ and $\deg y = 1$.

What can you say about $D^b(\operatorname{mod}\Lambda)$ in these examples?

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2. Let Λ be a semilocal ring, $\overline{\Lambda} = \Lambda/J(\Lambda)$ and $M = J(\Lambda)/J^2(\Lambda)$. Consider the *separated algebra*

$$\Sigma \;=\; \left(\overline{\Lambda} \times \overline{\Lambda}\right) \ltimes M$$

where $x \in \overline{\Lambda} \times \overline{\Lambda}$ acts on $m \in M$ as $xm = x_1 m$ on the left and as $mx = mx_2$ on the right. Prove the following:

(a) Σ is hereditary with $J^2(\Sigma) = 0$.

(b) $\operatorname{Mod}\Sigma$ is canonically isomorphic to a category with class of objects

$$\left\{ (X', X'', \phi) \;\middle|\; \begin{array}{l} X', X'' \in \operatorname{Mod}\overline{\Lambda}, \\ \qquad \phi \in \operatorname{Hom}_{\overline{\Lambda}}(X' \otimes_{\overline{\Lambda}} M, X'') \end{array} \right\}.$$

(c) There is a functor $\operatorname{Mod}\Lambda \xrightarrow{T} \operatorname{Mod}\Sigma$ given on objects as

$$Y \;\mapsto\; (Y/\operatorname{rad}Y, \operatorname{rad}Y/\operatorname{rad}^2 Y, \phi_Y)$$

where the map ϕ_Y is induced by multiplication, i.e. $\phi_Y(\overline{y} \otimes m) = ym$.

We consider now the trivial extension $\widetilde{\Lambda} = \overline{\Lambda} \ltimes M$.

(d) $\widetilde{\Lambda}$ identifies with a subring of Σ via the diagonal embedding $\overline{\Lambda} \to \overline{\Lambda} \times \overline{\Lambda}$.

(e) Restriction of scalars $\operatorname{Mod}\Sigma \xrightarrow{S} \operatorname{Mod}\widetilde{\Lambda}$ acts on objects as

$$(X', X'', \phi) \;\mapsto\; \left(X' \oplus X'', \left(\begin{smallmatrix} 0 & 0 \\ \phi & 0 \end{smallmatrix}\right)\right) \;.$$

In the following, we assume $J^2(\Lambda) = 0$ and that the projection $\Lambda \to \overline{\Lambda}$ admits a right inverse σ.

(f) The assignment $(\overline{x}, m) \mapsto \sigma(\overline{x}) + m$ defines a ring isomorphism $\widetilde{\Lambda} \xrightarrow{\sigma_*} \Lambda$.

It can be shown (see Auslander–Reiten–Smalø's *Representation Theory of Artin Algebras*, Ch. X.2) that, if Λ is an artin algebra, then T induces an equivalence $\underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Sigma$ of stable categories. Discuss the special cases $\Lambda = k[x]/(x^2)$ and $\Lambda = k[x,y]/(x^2, y^2)$ where k is a field.

3. Let Λ be a commutative noetherian ring. Denote by $\mathfrak{S}$ the class of Serre subcategories of $\operatorname{mod}\Lambda$ and by $\mathfrak{T}$ the class of thick subcategories of $\mathbf{D}^{\mathrm{per}}(\Lambda)$. Prove that the assignment

$$\mathcal{C} \;\mapsto\; \mathcal{T}_{\mathcal{C}} \;=\; \left\{ X \in \mathbf{D}^{\mathrm{per}}(\Lambda) : H^i(X) \in \mathcal{C} \text{ for all } i \in \mathbb{Z} \right\}$$

establishes a bijection $\mathfrak{S} \to \mathfrak{T}$.

Hint: Use the *classification theorem of Hopkins and Neeman*, which states that $\mathfrak{T}$ is in bijection with the set of specialization-closed subsets of $\operatorname{Spec}\Lambda$ via $\mathcal{T} \mapsto \operatorname{Supp}\mathcal{T} = \bigcup_{X \in \mathcal{T},\, i \in \mathbb{Z}} \operatorname{Supp} H^i(X)$.

Furthermore, observe that every thick subcategory of $\operatorname{mod}\Lambda$ is a Serre subcategory.