REPRESENTATION THEORY
EXERCISES 9

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1. Prove the following:

(a) Let k be a field and let Γ and Λ be non-zero finite-dimensional Gorenstein k-algebras. Then the tensor product $\Gamma \otimes_k \Lambda$ is Gorenstein with

$$\operatorname{Gor.dim} \Gamma \otimes_k \Lambda \;=\; \operatorname{Gor.dim} \Gamma + \operatorname{Gor.dim} \Lambda\,.$$

(b) Let Λ be a Gorenstein ring and let M be a Λ-bimodule such that M_Λ and ${}_\Lambda M$ are projective and the tensor algebra $\Gamma = T_\Lambda M$ is noetherian. Then Γ is Gorenstein with

$$\operatorname{Gor.dim} \Lambda \;\leq\; \operatorname{Gor.dim} \Gamma \;\leq\; \operatorname{Gor.dim} \Lambda + 1\,.$$

2. (a) Let $\Lambda = \mathbb{Z}G$ for a finite group G. Prove that Λ is Gorenstein of dimension 1 with

$$\operatorname{Gproj} \Lambda \;=\; \operatorname{mod} \Lambda \cap \operatorname{Gproj} \mathbb{Z} \;=\; \operatorname{mod} \Lambda \cap \operatorname{proj} \mathbb{Z}\,.$$

(b) Let $\Lambda = k\big({}_{\varepsilon}\circlearrowright \bullet \longrightarrow \bullet\big)/(\varepsilon^2)$ where k is a field. Prove that Λ is Gorenstein, determine its Gorenstein dimension and find all Gorenstein projective modules in $\operatorname{mod} \Lambda$.

3. Let Λ be a Gorenstein ring. We wish to investigate the idempotent completeness of the singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ in certain situations.

We begin with a negative example given in Auslander's article "Comments on the Functor Ext" after Proposition 2.8. It is also discussed in Hartshorne's "Coherent Functors" as Example 5.5. Considering $\Lambda = R_{(x,y)}$ where $R = \mathbb{C}[x,y]/(x^2 - y^2(y+1))$, we make the following observations:

(a) Λ is a complete intersection ring, so it is in particular Gorenstein.

(b) Λ is an integral domain with non-local integral closure Γ.

(c) Γ is a finitely generated Gorenstein projective Λ-module.

(d) Γ has no proper direct summands as a Λ-module

(e) $\widehat{\Gamma}$ has (two) proper direct summands as a $\widehat{\Lambda}$-module.

(f) $\underline{\operatorname{End}}_\Lambda(\Gamma)$ has (two) proper idempotents.

(g) $\operatorname{Ext}^1_\Lambda(\Gamma, -)$ has (two) proper direct summands that do not belong to $\operatorname{Ext}^1_\Lambda(\underline{\operatorname{mod}}\,\Lambda, -)$.

(h) $M \mapsto \operatorname{Ext}^1_\Lambda(M, -)$ yields a fully faithful functor from $\underline{\operatorname{mod}}\,\Lambda$.

(i) The stable category $\underline{\operatorname{Gproj}}\,\Lambda$ is not idempotent complete.

(j) The singularity category $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is not idempotent complete.

This finishes the negative example.

On the positive side prove now that $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ is idempotent complete for every artin algebra Λ.

To be handed in via email by June 22, 2020, 2 p.m.