REPRESENTATION THEORY EXERCISES 10

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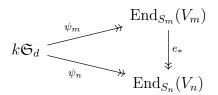
Let k be a commutative ring and fix integers $n, d \in \mathbb{N}$. We will use notation from sheet 9.

- **1.** Consider $V_n = (k^n)^{\otimes d}$ as a left module over the Schur algebra $S_n = S_k(n, d)$.
 - (a) Convince yourself that elements $f \in \operatorname{End}_{S_n}(V_n)$ correspond to tuples $(f_{\lambda})_{\lambda \in \Lambda(n,d)}$ of endomorphisms $f_{\lambda} \in \operatorname{End}_k(V^{\lambda})$ making the following diagrams commute for all $\omega \in \Omega(n, d)$:

(b) Prove that the k-algebra homomorphism $k\mathfrak{S}_d \xrightarrow{\psi_n} \operatorname{End}_{S_n}(V_n)$ is surjective.

For $m \in \mathbb{N}$ with $m \ge n$ we consider now the idempotent $e = \sum_{\lambda \in \Lambda(n,d)} e_{\lambda}$ in S_m .

(c) Verify the identities $eS_m e = S_n$ and $eV_m = V_n$. Deduce that e yields a commuting triangle:



Next, we wish to see that e_* is a map between free k-modules with free kernel.

To describe a k-basis of the kernel of e_* , we introduce some more notation.

Write $\Lambda^+(n, d) \subseteq \Lambda(n, d)$ for the subset of all partitions. It is sometimes convenient to visualize indices $i \in \underline{n}^d$ as tableaux (i.e. fillings) |i| of the Young diagram corresponding to i^* such that |i| records at position (s, t) the position of the t-th s in i. Formally, this means

$$|i|(s,t) = \min\left\{r \in \underline{d} : \#i^{-1}(s) \cap \underline{r} = t\right\}.$$

An index $i \in \underline{n}^{\underline{d}}$ is called *standard* if |i| is a standard Young tableau. Set $I_{\lambda} = \{i \in \underline{n}^{\underline{d}} : i^* = \lambda\}$ for compositions $\lambda \in \Lambda(n, d)$ and let $I_{\lambda}^+ \subseteq I_{\lambda}$ be the subset of all standard indices.

For $\lambda \in \Lambda(n, d)$ we denote by i_{λ} the unique increasing function in I_{λ} . Observe that i_{λ} is standard. For each index $i \in I_{\lambda}$ let σ_i be the unique permutation $\sigma \in \mathfrak{S}_d$ such that $\sigma|i| = |i_{\lambda}|$.

Define
$$m_{\lambda} = \sum_{\sigma \in \mathfrak{S}_{\lambda}} \operatorname{sgn}(\sigma) \sigma \in k \mathfrak{S}_d$$
 and for $i, j \in I_{\lambda}$ define $m_{ij} = \sigma_i m_{\lambda} \sigma_j^{-1}$.

Let
$$M_n = \bigcup_{\lambda \in \Lambda^+(n,d)} M_{\lambda'}$$
 where $M_\lambda = \{m_{ij} : i, j \in I_\lambda^+\}$.

- (d) Prove that M_d is a k-basis of $k\mathfrak{S}_d$. It is known as Murphy's standard basis.
- (e) Prove that ψ_m identifies $M_m \setminus M_n$ with a basis of Ker e_* .

To be handed in by January 9, 2020, 2 p.m. into post box 30.

Hint for (b): It is straightforward to check that ψ_m is an isomorphism for all $m \ge d$. If it helps, assume that k is a field, use Murphy's basis and the fact that Ker ψ_n has $M_d \setminus M_n$ as a k-basis.

2. Let e_1, \ldots, e_n be the standard basis of $V = k^n$ and let $e_{ii} \in \text{End}_k(V)$ be given by $e_{ii}(e_j) = \delta_{ji}e_i$.

Show that the elements $(e_{\lambda})_{\lambda \in \Lambda(n,d)}$ with $e_{\lambda} := \prod_{i=1}^{n} e_{ii}^{\otimes \lambda_i}$ form a complete set of orthogonal idempotents of $\Gamma^d E$, where E is the subalgebra of $\operatorname{End}_k(V)$ generated by the idempotents e_{ii} .

3. Show that the tensor product

 $-\otimes -: \operatorname{Rep} \Gamma_k^d \times \operatorname{Rep} \Gamma_k^e \longrightarrow \operatorname{Rep} \Gamma_k^{d+e}$

is right exact and preserves direct sums in both arguments.

4. Let $\lambda \in \Lambda(n, d)$ and $W = k^r$. Find an explicit formula for the rank of the k-module $\Gamma^{\lambda}W$.

FROHE WEIHNACHTEN UND EIN GUTES NEUES JAHR!