REPRESENTATION THEORY
EXERCISES 10

HENNING KRAUSE
JAN GEUENICH

Let k be a commutative ring and fix integers $n, d \in \mathbb{N}$. We will use notation from sheet 9.

1. Consider $V_n = (k^n)^{\otimes d}$ as a left module over the Schur algebra $S_n = S_k(n, d)$.

(a) Convince yourself that elements $f \in \operatorname{End}_{S_n}(V_n)$ correspond to tuples $(f_\lambda)_{\lambda \in \Lambda(n,d)}$ of endomorphisms $f_\lambda \in \operatorname{End}_k(V^\lambda)$ making the following diagrams commute for all $\omega \in \Omega(n, d)$:

$$\begin{tikzcd} V^{s(\omega)} \arrow[r, "f_{s(\omega)}"] \arrow[d, "e_\omega"'] & V^{s(\omega)} \arrow[d, "e_\omega"] \\ V^{t(\omega)} \arrow[r, "f_{t(\omega)}"'] & V^{t(\omega)} \end{tikzcd}$$

(b) Prove that the k-algebra homomorphism $k\mathfrak{S}_d \xrightarrow{\psi_n} \operatorname{End}_{S_n}(V_n)$ is surjective.

For $m \in \mathbb{N}$ with $m \geq n$ we consider now the idempotent $e = \sum_{\lambda \in \Lambda(n,d)} e_\lambda$ in S_m.

(c) Verify the identities $eS_me = S_n$ and $eV_m = V_n$. Deduce that e yields a commuting triangle:

$$\begin{tikzcd} & \operatorname{End}_{S_m}(V_m) \arrow[d, "e_*"] \\ k\mathfrak{S}_d \arrow[ur, two heads, "\psi_m"] \arrow[r, two heads, "\psi_n"'] & \operatorname{End}_{S_n}(V_n) \end{tikzcd}$$

Next, we wish to see that e_* is a map between free k-modules with free kernel.

To describe a k-basis of the kernel of e_*, we introduce some more notation.

Write $\Lambda^+(n, d) \subseteq \Lambda(n, d)$ for the subset of all partitions. It is sometimes convenient to visualize indices $i \in \underline{n}^{\underline{d}}$ as tableaux (i.e. fillings) $|i|$ of the Young diagram corresponding to i^* such that $|i|$ records at position (s, t) the position of the t-th s in i. Formally, this means

$$|i|(s, t) \;=\; \min\left\{r \in \underline{d} : \#i^{-1}(s) \cap \underline{r} = t\right\}.$$

An index $i \in \underline{n}^{\underline{d}}$ is called *standard* if $|i|$ is a standard Young tableau. Set $I_\lambda = \{i \in \underline{n}^{\underline{d}} : i^* = \lambda\}$ for compositions $\lambda \in \Lambda(n, d)$ and let $I_\lambda^+ \subseteq I_\lambda$ be the subset of all standard indices.

For $\lambda \in \Lambda(n, d)$ we denote by i_λ the unique increasing function in I_λ. Observe that i_λ is standard. For each index $i \in I_\lambda$ let σ_i be the unique permutation $\sigma \in \mathfrak{S}_d$ such that $\sigma|i| = |i_\lambda|$.

Define $m_\lambda = \sum_{\sigma \in \mathfrak{S}_\lambda} \operatorname{sgn}(\sigma)\sigma \in k\mathfrak{S}_d$ and for $i, j \in I_\lambda$ define $m_{ij} = \sigma_i m_\lambda \sigma_j^{-1}$.

Let $M_n = \bigcup_{\lambda \in \Lambda^+(n,d)} M_{\lambda'}$ where $M_\lambda = \left\{m_{ij} : i, j \in I_\lambda^+\right\}$.

(d) Prove that M_d is a k-basis of $k\mathfrak{S}_d$. It is known as *Murphy's standard basis*.

(e) Prove that ψ_m identifies $M_m \setminus M_n$ with a basis of $\operatorname{Ker} e_*$.

To be handed in by January 9, 2020, 2 p.m. into post box 30.

Hint for (b): It is straightforward to check that ψ_m is an isomorphism for all $m \geq d$. If it helps, assume that k is a field, use Murphy's basis and the fact that $\operatorname{Ker} \psi_n$ has $M_d \setminus M_n$ as a k-basis.

2. Let $e_1, \ldots, e_n$ be the standard basis of $V = k^n$ and let $e_{ii} \in \operatorname{End}_k(V)$ be given by $e_{ii}(e_j) = \delta_{ji} e_i$.

Show that the elements $(e_\lambda)_{\lambda \in \Lambda(n,d)}$ with $e_\lambda := \prod_{i=1}^n e_{ii}^{\otimes \lambda_i}$ form a complete set of orthogonal idempotents of $\Gamma^d E$, where E is the subalgebra of $\operatorname{End}_k(V)$ generated by the idempotents e_{ii}.

3. Show that the tensor product

$$- \otimes - \colon \operatorname{Rep} \Gamma_k^d \times \operatorname{Rep} \Gamma_k^e \longrightarrow \operatorname{Rep} \Gamma_k^{d+e}$$

is right exact and preserves direct sums in both arguments.

4. Let $\lambda \in \Lambda(n,d)$ and $W = k^r$. Find an explicit formula for the rank of the k-module $\Gamma^\lambda W$.

FROHE WEIHNACHTEN UND EIN GUTES NEUES JAHR!