REPRESENTATION THEORY
EXERCISES 11

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Let k be an infinite field and fix integers $n, d \in \mathbb{N}$. We use the notations from sheets 9 and 10. Moreover, for $\lambda \in \Lambda(n,d)$ write $S_k(n,\lambda)$ for the algebra $e_\lambda S_k(n,d) e_\lambda$.

In case $n \geq d$ denote by $\mathbb{1}$ the sequence $(1,\dots,1) \in \Lambda(n,d)$ of length d and for each $\sigma \in \mathfrak{S}_d$ write ω_σ for the element $\omega \in \Omega$ with $(i_{\mathbb{1}}\sigma, i_{\mathbb{1}}) \in \omega$.

As usual, the canonical map $k\mathrm{GL}(n,k) \to S_k(n,d)$ is used for restriction of scalars.

1. The *weight space* of weight $\lambda \in \Lambda(n,d)$ for a left $S_k(n,d)$-module M is by definition

$$M_\lambda \;=\; \left\{ m \in M : xm = x_1^{\lambda_1} \cdots x_n^{\lambda_n} m \text{ for all } \mathrm{diag}(x_1,\dots,x_n) \in \mathrm{GL}(n,k) \right\}.$$

Verify directly the weight-space decomposition $M = \bigoplus_{\lambda \in \Lambda(n,d)} M_\lambda$ and prove $M_\lambda = e_\lambda M$.

2. The *character* of a finite-dimensional left $S_k(n,d)$-module M is the integer polynomial

$$\chi_M \;=\; \sum_{\lambda \in \Lambda(n,d)} \dim_k M_\lambda \cdot X_1^{\lambda_1} \cdots X_n^{\lambda_n}.$$

Prove that characters are …

(a) *symmetric* in the sense that $\chi_M \in \mathbb{Z}[X_1,\dots,X_n]^{\mathfrak{S}_n}$,

(b) *additive* in the sense that $\chi_{M \oplus N} = \chi_M + \chi_N$,

(c) *multiplicative* in the sense that $\chi_{M \otimes_k N} = \chi_M \cdot \chi_N$.

One can show that $\chi_M(x_1,\dots,x_n) = \mathrm{tr}_k(M \xrightarrow{g\cdot} M)$ for all $g \in \mathrm{GL}(n,k)$ where $x_1,\dots,x_n \in \overline{k}$ are the eigenvalues of g listed with multiplicity (see Green's *Polynomial Representations of* GL_n).

3. Assuming $n \geq d$ show the following:

(a) $\sigma \mapsto e_{\omega_\sigma}$ defines an isomorphism $k\mathfrak{S}_d \to S_k(n,\mathbb{1})$ of k-algebras.

(b) $1 \mapsto e_{i_{\mathbb{1}}}$ defines an isomorphism $k\mathfrak{S}_d \to V^{\mathbb{1}}$ of right $k\mathfrak{S}_d$-modules.

(c) $f \mapsto f(e_{i_{\mathbb{1}}})$ defines an isomorphism $S_k(n,d)e_{\mathbb{1}} \to V^{\otimes d}$ of left $S_k(n,d)$-modules.

Deduce that there is an induced isomorphism $\mathrm{End}_{S_k(n,d)}(V^{\otimes d}) \cong k\mathfrak{S}_d$ of k-algebras.

4. Let $n \geq d$ and regard $e_{\mathbb{1}} S_k(n,d)$ as a left $k\mathfrak{S}_d$-module via 3. (a). Prove that the Schur functor

$$F = \mathrm{Hom}_{S_k(n,d)}\left(V^{\otimes d}, -\right) : \quad S_k(n,d)\,\mathrm{Mod} \longrightarrow k\mathfrak{S}_d\,\mathrm{Mod}$$

is isomorphic to $e_{\mathbb{1}} S_k(n,d) \otimes_{S_k(n,d)} -$.

Verify that F maps each projective left module $P_\lambda = S_k(n,d)e_\lambda$ to the dual of the corresponding permutation module V^λ and, restricted to the subcategories spanned by these modules, induces an equivalence, i.e. the map $\mathrm{Hom}_{S_k(n,d)}(P_\lambda, P_\mu) \to \mathrm{Hom}_{k\mathfrak{S}_d}((V^\lambda)^\vee, (V^\mu)^\vee)$ induced by F is bijective.

To be handed in by January 27, 2020, 2 p.m. into post box 30.