REPRESENTATION THEORY
EXERCISES 5

HENNING KRAUSE
JAN GEUENICH

1. We use the same notation as in the lecture for recollements of abelian categories, namely

$$\mathcal{C} \underset{i^!}{\overset{i^*}{\twoheadleftarrow}} \!\!\xrightarrow{i_*}\!\! \mathcal{A} \underset{j_*}{\overset{j_!}{\leftarrow}} \!\!\xrightarrow{j^*}\!\! \mathcal{B}\,.$$

Verify the following claims:

(a) For a functor $\mathcal{A} \xrightarrow{j^*} \mathcal{B}$ of abelian categories with both a left adjoint $j_!$ and a right adjoint j_* we have that $j_!$ is fully faithful iff j_* is fully faithful iff j^* fits into a recollement.

(b) In a recollement, the sequence $\mathcal{B} \xrightarrow{j_\wedge} \mathcal{A} \xrightarrow{i^\vee} \mathcal{C}$ is exact iff $i^\vee$ is exact iff both $i^\vee$ and $j_\wedge$ are exact, where $(\vee, \wedge)$ stands for either $(*, \,!\,)$ or $(\,!\,, *)$. In this situation, $\mathcal{C} \simeq \mathcal{A}\,/\,j_\wedge\mathcal{B}$.

(c) Each recollement induces a natural exact sequence $0 \to i_*i^!j_! \to j_! \xrightarrow{\nu} j_* \to i_*i^*j_* \to 0$ of functors $\mathcal{B} \to \mathcal{A}$. The functor $j_{!*} = \operatorname{Im}\nu$ is called the *intermediate extension functor*.

(d) i_* and $j_{!*}$ map simple objects to simple objects. On the level of isomorphism classes, these assignments are injective and every simple object of $\mathcal{A}$ lies in the image of either i_* or $j_{!*}$.

2. Let Λ be a ring.

Write $\mathfrak{I}(\Lambda)$ for the set of ideals and $\mathfrak{I}^2(\Lambda)$ for the set of idempotent ideals in Λ. Denote by $\mathfrak{C}(\Lambda)$ the class of full subcategories of $\operatorname{Mod}\Lambda$ that are closed under submodules, factor modules, direct sums and products. Finally, let $\mathfrak{S}(\Lambda)$ be the subclass of $\mathfrak{C}(\Lambda)$ consisting of Serre subcategories.

Prove the following:

(a) The map $\mathfrak{I}(\Lambda) \to \mathfrak{C}(\Lambda)$, $I \mapsto \{M \in \operatorname{Mod}\Lambda : MI = 0\}$, is bijective.

(b) The map in (a) restricts to a bijection $\mathfrak{I}^2(\Lambda) \to \mathfrak{S}(\Lambda)$.
In particular, for right artinian Λ the Serre subcategories of $\operatorname{mod}\Lambda$ correspond naturally to the Serre subcategories of $\operatorname{Mod}\Lambda$ that are closed under direct sums and products.

Hint for (a): Any $\mathcal{C} \in \mathfrak{C}(\Lambda)$ is the closure of the set (i.e. not a proper class) $\{\Lambda/I : I \in \mathfrak{I}(\Lambda)\} \cap \mathcal{C}$ under factor modules and direct sums.

3. Let I be an idempotent ideal of a ring Λ and $\operatorname{Mod}\Lambda/I \xrightarrow{i_*} \operatorname{Mod}\Lambda$ the canonical embedding. In the lecture you saw that i_* is homological if I_Λ is projective. Show that the converse is false.

Hint: Consider a suitable quotient Λ of the path algebra $\mathbb{C}Q$ of the quiver

$$Q \;=\; 1 \underset{a}{\overset{b}{\leftrightarrows}} 2\;.$$

To be handed in by November 25, 2019, 2 p.m. into post box 30.

Recall that a *presheaf* $\mathcal{F}$ on a topological space X is a contravariant functor whose source is the poset of open sets in X ordered by inclusion. Assuming the target of $\mathcal{F}$ admits products, $\mathcal{F}$ is said to be a *sheaf* if for any collection $\mathcal{U}$ of open sets in X the canonical map $\mathcal{F}(\bigcup_{U\in\mathcal{U}} U) \to \prod_{U\in\mathcal{U}} \mathcal{F}(U)$ is an equalizer of the canonical pair of parallel maps $\prod_{U\in\mathcal{U}} \mathcal{F}(U) \rightrightarrows \prod_{V,W\in\mathcal{U}} \mathcal{F}(V\cap W)$.

We denote by $\mathrm{PSh}(X)$ and $\mathrm{Sh}(X)$ the category of presheaves and sheaves on X, respectively, whose target is the category of abelian groups. Clearly, $\mathrm{PSh}(X)$ is abelian and $\mathrm{Sh}(X)$ is additive.

4. Verify the following for each topological space X:

(a) *(Sheafification)* The canonical inclusion $\mathrm{Sh}(X) \xrightarrow{\iota_X} \mathrm{PSh}(X)$ has an exact left adjoint σ_X. From this it follows that $\mathrm{Sh}(X)$ is abelian and limits in $\mathrm{Sh}(X)$ can be computed in $\mathrm{PSh}(X)$.

For every continuous map $X \xrightarrow{j} Y$ show:

(b) *(Direct image)* There is an exact functor $\mathrm{PSh}(X) \xrightarrow{j_\flat} \mathrm{PSh}(Y)$, which is given on objects as $j_\flat\mathcal{F}(U) = \mathcal{F}(j^{-1}(U))$. It restricts to a (not necessarily exact) functor $\mathrm{Sh}(X) \xrightarrow{j_*} \mathrm{Sh}(Y)$.

(c) *(Inverse image)* $j_\flat$ and j_* admit exact left adjoints $j^\flat$ and $j^* = \sigma_X \circ j^\flat \circ \iota_Y$, respectively. The inverse image $(\{x\} \subseteq X)^*\mathcal{F}$ is known as the *stalk* $\mathcal{F}_x$ of the sheaf $\mathcal{F}$ at x.

Assume now that j is the open embedding of a subset of Y and let $Z \xrightarrow{i} Y$ be the closed embedding of the complement $Z = Y \setminus X$.

(d) (*Restriction*) $j^\flat$ and j^* are isomorphic to the respective restriction functor $\mathcal{F} \mapsto \mathcal{F}\circ j$.

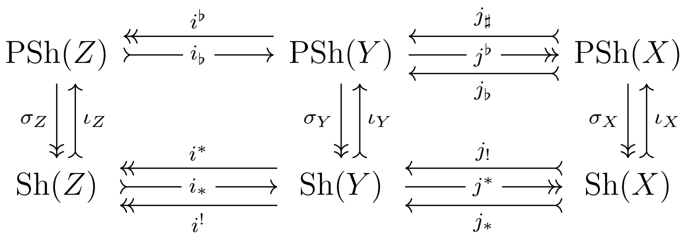
(e) (*Extension by zero*) $j^\flat$ and j^* admit left adjoints $j_\sharp$ and $j_! = \sigma_Y \circ j_\sharp \circ \iota_X$, respectively.

(f) $j_\sharp$ and $j_!$ and hence $j_\flat$ and j_* are fully faithful.

(g) (*Vanishing on* X) i_* admits a right adjoint $i^!$ given as $i^!\mathcal{F}(U) = \mathrm{Ker}(\mathcal{F}(U\cup X) \to \mathcal{F}(X))$.

(h) $i_\flat$ and i_* are fully faithful with $\mathrm{Im}(i_*) = \mathrm{Ker}(j^*)$.

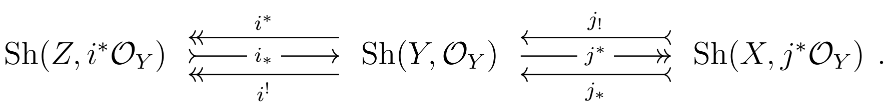
(i) Summarizing, we have the following diagram where the bottom row is a recollement:



The top and the middle row of this recollement form exact sequences of abelian categories. In particular, we have $\mathrm{Sh}(X) \simeq \mathrm{Sh}(Y)\,/\,i_*\,\mathrm{Sh}(Z)$ and $\mathrm{Sh}(Z) \simeq \mathrm{Sh}(Y)\,/\,j_!\,\mathrm{Sh}(X)$.

(j) What are the intermediate extension functors $j_{\sharp\flat}$ and $j_{!*}$?

There is a generalization for ringed spaces. Namely, let $\mathcal{O}_Y$ be a sheaf of commutative rings on Y. Then the categories of sheaves of modules over the induced structure sheaves fit into a recollement



How does the recollement between the sheaf categories in the exercise arise as a special case?