SOLUTIONS TO EXERCISES 5 – REPRESENTATION THEORY – WS 2019/20

We will repeatedly use for adjoint pairs (L, R) with counit $LR \xrightarrow{\varepsilon} 1$ and unit $1 \xrightarrow{\eta} RL$:

$$
\begin{array}{lclcl}
L \text{ fully faithful} & \Leftrightarrow & \eta \text{ invertible} & \Rightarrow & L\eta \text{ and } \eta R \text{ have inverses } \varepsilon L \text{ and } R\varepsilon \text{, repsectively.} \\
R \text{ fully faithful} & \Leftrightarrow & \varepsilon \text{ invertible} & \Rightarrow & R\varepsilon \text{ and } \varepsilon L \text{ have inverses } \eta R \text{ and } L\eta \text{, respectively.}
\end{array}
$$

We use the same notation as in the lecture for recollements of abelian categories, namely

$$
\mathcal{C} \;
\begin{array}{c}
\xleftarrow{\quad i^* \quad} \\
\xrightarrow{\quad i_* \quad} \\
\xleftarrow[\quad i^! \quad]{}
\end{array}
\; \mathcal{A} \;
\begin{array}{c}
\xleftarrow{\quad j_! \quad} \\
\xrightarrow{\quad j^* \quad} \\
\xleftarrow[\quad j_* \quad]{}
\end{array}
\; \mathcal{B}
$$

and denote the respective units and counits as follows:

$$
i^*i_* \xrightarrow{\varepsilon^*} 1_{\mathcal{C}} \xrightarrow{\eta^!} i^!i_*
$$

$$
j_!j^* \xrightarrow{\varepsilon_!} 1_{\mathcal{A}} \xrightarrow{\eta^*} i_*i^* \qquad i_*i^! \xrightarrow{\varepsilon^!} 1_{\mathcal{A}} \xrightarrow{\eta_*} j_*j^*
$$

$$
j^*j_* \xrightarrow{\varepsilon_*} 1_{\mathcal{B}} \xrightarrow{\eta_!} j^*j_!
$$

Exercise 5.1 (a).

For adjoint triples $(j_!, j^*, j_*)$ we have the chain of equivalences

$$
j_! \text{ fully faithful} \quad \Leftrightarrow \quad j^*j_! \cong 1_{\mathcal{B}} \quad \overset{(\star)}{\Leftrightarrow} \quad j^*j_* \cong 1_{\mathcal{B}} \quad \Leftrightarrow \quad j_* \text{ fully faithful}
$$

where $(\star)$ holds since $(j^*j_!, j^*j_*)$ is an adjoint pair and $1_{\mathcal{B}}$ is fully faithful.

More explicitly, $\eta_!$ is invertible iff ε_* is invertible because the following square commutes:

$$
\begin{array}{ccc}
j^*j_* & \xrightarrow{\quad \varepsilon_* \quad} & 1_{\mathcal{B}} \\
{\scriptstyle j^*j_*\eta_!}\downarrow & & \downarrow{\scriptstyle \eta_!} \\
j^*j_*j^*j_! & \xrightarrow[\quad \varepsilon_*j^*j_! \quad]{} & j^*j_!
\end{array}
$$

The functors $j_!$ and j_* in a recollement are of course fully faithful. So all that remains to show is that every adjoint triple $(j_!, j^*, j_*)$ with fully faithful $j_!$ and j_* can be completed to a recollement.

Assume we have such a triple.

First of all, j^* is exact because it has a left and a right adjoint.

Thus $\mathcal{C} = \operatorname{Ker}(j^*)$ is a Serre subcategory of $\mathcal{A}$ and the canonical embedding $\mathcal{C} \xrightarrow{i_*} \mathcal{A}$ is exact.

To define η^* and $\varepsilon^!$ we consider exact sequences:

$$
j_!j^* \xrightarrow{\varepsilon_!} 1_{\mathcal{A}} \xrightarrow{\eta^*} i^* \longrightarrow 0
$$

$$
0 \longrightarrow i^! \xrightarrow{\varepsilon^!} 1_{\mathcal{A}} \xrightarrow{\eta_*} j_*j^*
$$

In particular, $i^* = \operatorname{Coker}(\varepsilon_!)$ and $i^! = \operatorname{Ker}(\eta_*)$. These are a priori merely endofunctors of $\mathcal{A}$. But the exactness of j^* and the invertibility of $j^*\varepsilon_!$ and $j^*\eta_*$ make sure they have image in $\mathcal{C}$:

$$\begin{aligned} j^* \operatorname{Coker}(\varepsilon_!) &\cong \operatorname{Coker}(j^*\varepsilon_!) = 0 \\ j^* \operatorname{Ker}(\eta_*) &\cong \operatorname{Ker}(j^*\eta_*) = 0 \end{aligned}$$

Let $\varepsilon^* = (\eta^* i_*)^{-1}$ and $\eta^! = (\varepsilon^! i_*)^{-1}$. Note that $\eta^* i_*$ and $\varepsilon^! i_*$ are invertible because of $j^* i_* = 0$.

By the universal property of cokernels and kernels the following maps are identities:

$$i^* i_* i^* \xrightarrow{\varepsilon^* i^*} i^* \xrightarrow{i^*\eta^*} i^* i_* i^* \qquad\qquad i_* i^! i_* \xrightarrow{\varepsilon^! i_*} i_* \xrightarrow{i_*\eta^!} i_* i^! i_*$$

$$i_* i^* i_* \xrightarrow{i_*\varepsilon^*} i_* \xrightarrow{\eta^* i_*} i_* i^* i_* \qquad\qquad i^! i_* i^! \xrightarrow{i^!\varepsilon^!} i^! \xrightarrow{\eta^! i^!} i^! i_* i^!$$

Hence, (i^*, i_*) and $(i_*, i^!)$ are adjoint pairs with counits ε^*, $\varepsilon^!$ and units η^*, $\eta^!$, respectively.

All in all, we can conlude that the functors $i^*, i_*, i^!, j_!, j^*, j_*$ fit into a recollement as claimed.

Exercise 5.1 (b).

We consider the case $(\wedge, \vee) = (*, !)$, the other case being analogous.

Recall that the sequence $\mathcal{B} \overset{j_*}{\rightarrowtail} \mathcal{A} \overset{i^!}{\twoheadrightarrow} \mathcal{C}$ is said to be *exact* if $i^!$ is a weak quotient functor and j_* is an exact embedding whose essential image is the kernel of $i^!$.

Obviously, the exactness of the sequence implies that $i^!$ and j_* are exact functors. We must show that the exactness of the sequence follows from the exactness of the functor $i^!$ in the recollement.

So let's assume from now on that $i^!$ is exact.

Since the exact functor $i^!$ has a faithful adjoint, it is a weak quotient functor by Exercise 3.3.

Clearly, $\operatorname{Im}(j_*) \subseteq \operatorname{Ker}(i^!)$ because of $\operatorname{Hom}_{\mathcal{C}}(-, i^! j_*) \cong \operatorname{Hom}_{\mathcal{B}}(j^* i_*, -) = 0$.

To verify $\operatorname{Ker}(i^!) \subseteq \operatorname{Im}(j_*)$ take $A \in \mathcal{A}$ with $i^! A = 0$. By (a) $A \xrightarrow{\eta_* A} j_* j^* A$ is a monomorphism. The invertibility of $j^*\eta_*$ and the exactness of j^* imply that $\operatorname{Coker}(\eta_* A) \in \operatorname{Ker}(j^*) = \operatorname{Im}(i_*)$. Hence, there exists an exact sequence

$$0 \longrightarrow A \xrightarrow{\eta_* A} j_* j^* A \longrightarrow i_* C \longrightarrow 0\,.$$

Applying the exact functor $i^!$ to it, we get $i^! i_* C = 0$. Because of $i^! i_* \cong 1_{\mathcal{C}}$ this means $C = 0$ and thus $i_* C = 0$. Consequently, $\eta_* A$ is an isomorphism, which shows that A belongs to $\operatorname{Im}(j_*)$.

It only remains to prove that j_* is exact. To do this, let $0 \to B' \xrightarrow{f} B$ be an exact sequence in $\mathcal{B}$. As a right adjoint, the functor j_* is left exact, so we get an exact sequence

$$0 \longrightarrow j_* B' \xrightarrow{j_* f} j_* B\,.$$

Since $\operatorname{Im}(j_*) = \operatorname{Ker}(i^!)$ as a Serre subcategory of $\mathcal{A}$ is closed under quotient objects and j_* is fully faithful, we can complete it to a short exact sequence of the form

$$0 \longrightarrow j_* B' \xrightarrow{j_* f} j_* B \xrightarrow{j_* g} j_* B'' \longrightarrow 0\,.$$

Applying the exact functor j^* and using $j^* j_* \cong 1_{\mathcal{B}}$, the map g turns out to be a cokernel of f. This finishes the proof of the exactness of j_*.

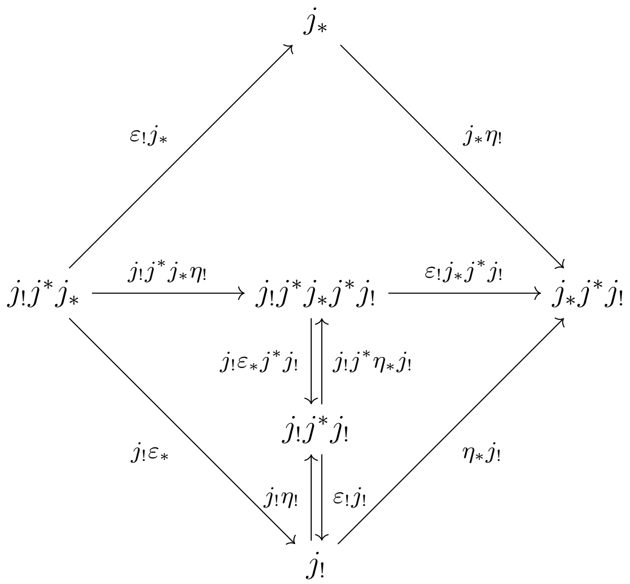
Putting everything together, $\mathcal{B} \overset{j_*}{\rightarrowtail} \mathcal{A} \overset{i^!}{\twoheadrightarrow} \mathcal{C}$ is an exact sequence of abelian categories.

Exercise 5.1 (c).

We know from (a) the exact sequences appearing in the rows of the following diagram:

$$\begin{array}{ccccccccc} & & & & j_!j^*j_* & \xrightarrow{\varepsilon_! j_*} & j_* & \xrightarrow{\eta^* j_*} & i_*i^*j_* \longrightarrow 0 \\ & & & & {\scriptstyle j_!\varepsilon_*}\downarrow\cong & & \cong\downarrow{\scriptstyle j_*\eta_!} & & \\ 0 & \longrightarrow & i_*i^!j_! & \xrightarrow[\varepsilon^! j_!]{} & j_! & \xrightarrow[\eta_* j_!]{} & j_*j^*j_! & & \end{array}$$

Therefore it is enough to prove that the square above commutes. To see that it does, consider the commuting diagram below, where the parallel vertical arrows are mutually inverse to each other by the fully-faithfulness of $j_!$ and j_* in the adjoint triple $(j_!, j^*, j_*)$:



Exercise 5.1 (d).

Given that $\mathrm{Im}(i_*)$ as a Serre subcategory of $\mathcal{A}$ is closed under subobjects and i_* is fully faithful and left exact, the subobjects of S in $\mathcal{C}$ bijectively correspond to the subobjects of i_*S in $\mathcal{A}$ via

$$C \xhookrightarrow{f} S \qquad \longmapsto \qquad i_*C \xhookrightarrow{i_*f} i_*S\,.$$

Since i_* is fully faithful, f is zero / invertible iff i_*f is zero / invertible. Hence, the bijection:

$$\{\text{simples in } \mathcal{C}\}/_{\cong} \xrightarrow{i_*} \{\text{simples in } \mathcal{A} \text{ belonging to } \mathrm{Im}(i_*)\}/_{\cong}$$

To prove that $j_{!*}S$ is simple in $\mathcal{A}$ for each simple object S in $\mathcal{B}$, we have to show $j_{!*}S \neq 0$ and for every short exact sequence $\xi\colon 0 \to A' \to j_{!*}S \to A'' \to 0$ in $\mathcal{A}$ that $A' = 0$ or $A'' = 0$. Applying the exact functor j^* firstly to the exact sequence in (c) yields $j^*j_{!*}S \cong S \neq 0$ and applying it secondly to ξ we obtain an exact sequence $0 \to j^*A' \to S \to j^*A'' \to 0$. Since by assumption S is simple, we can conlude that either A' or A'' must belong to $\mathrm{Ker}(j^*) = \mathrm{Im}(i_*)$. But this means that either $A' = 0$ or $A'' = 0$ because of the following:

$$\mathrm{Hom}_{\mathcal{A}}(i_*, j_{!*}) \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(i_*, j_*) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(j^*i_*, -) \;=\; 0$$

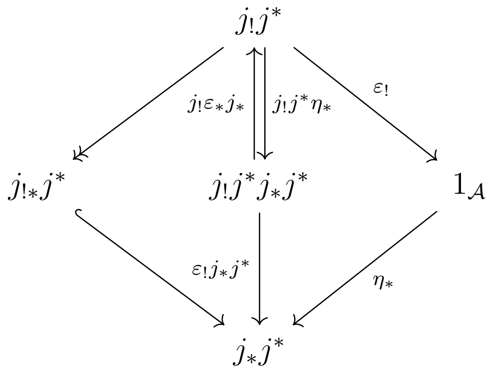
$$\mathrm{Hom}_{\mathcal{A}}(j_{!*}, i_*) \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(j_!, i_*) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(-, j^*j_*) \;=\; 0$$

By the consideration above we have an injective map

$$\{\text{simples in } \mathcal{B}\}/_{\cong} \xrightarrow{j_{!*}} \{\text{simples in } \mathcal{A} \text{ not belonging to } \operatorname{Ker}(j^*)\}/_{\cong}$$

where (restricted to its image) the left inverse is induced by j^*. We now verify its bijectivity.

Observe that according to (c) we have a commuting diagram:



As a consequence, there is an induced map $j_{!*} j^* \hookrightarrow \operatorname{Im}(\eta_*)$.

Thus, for simples S in $\mathcal{A}$ with $j^* S \neq 0$, it follows $0 \neq j_{!*} j^* S \hookrightarrow \operatorname{Im}(\eta_* S) \twoheadleftarrow S$, so $j_{!*} j^* S \cong S$. Furthermore, $j^* S$ is simple in $\mathcal{B}$. Indeed, since $j_{!*}$ preserves monomorphisms thanks to the left exactness of j_*, any subobject $\alpha\colon B \hookrightarrow j^* S$ gives rise to a subobject $j_{!*}\alpha\colon j_{!*} B \hookrightarrow j_{!*} j^* S \cong S$, which is zero or invertible. Applying j^* to $j_{!*}\alpha$ and using (c), we see that α is zero or invertible.

All things considered, this proves that j^* induces the inverse of the above map induced by $j_{!*}$.

Before addressing Exercise 5.4, we begin with a few simple observations.

By definition a category $\mathcal{D}$ *has colimits* or *limits*, respectively, if for every small J the diagonal functor $\mathcal{D} \to \mathcal{D}^J$, $D \mapsto (D)_{j\in J}$, has a left adjoint colim or a right adjoint $\lim$, respectively.

Evidently, a product category $\mathcal{D} = \prod_i \mathcal{D}_i$ has (co)limits iff all $\mathcal{D}_i$ have (co)limits. This follows immediately from the canonical isomorphism $\mathcal{D}^J \cong \prod_i \mathcal{D}_i^J$. In this case, we have for $F \in \mathcal{D}^J$

$$\text{(co)}\lim F \;=\; \big(\text{(co)}\lim F_i\big)_i\,.$$

We can apply these considerations to the product category $\mathrm{PSh}(X) = \mathrm{Ab}^{\mathcal{T}(X)^{\mathrm{op}}}$, where $\mathcal{T}(X)$ is the poset of open sets in X ordered by inclusion and Ab is the category of abelian groups.

Recall that in Ab (co)limits are given explicitly as follows for $F \in \mathrm{Ab}^J$:

$$\operatorname{colim} F \;=\; \bigoplus\nolimits_{j\in J} F(j) \Big/ \Big(e_i x - e_j \alpha(x) : i \xrightarrow{\alpha} j \text{ in } J,\, x \in F(i)\Big)$$

$$\lim F \;=\; \Big\{x \in \prod\nolimits_{j\in J} F(j) : x_j = \alpha(x_i) \text{ for all } i \xrightarrow{\alpha} j \text{ in } J\Big\}$$

This shows that the category $\mathrm{PSh}(X)$ has (co)limits and also how to canonically compute them. As a consequence, it is now clear why $\mathrm{PSh}(X)$ is abelian and why $\mathrm{Sh}(X)$ is additive.

Exercise 5.4 (a).

For collections $\mathcal{U}$ of open sets in X let

$$f_{\mathcal{U}} \;=\; (f^V_{V\cap W} - f^W_{V\cap W})_{V,W\in\mathcal{U}} \colon \prod\nolimits_{U\in\mathcal{U}} \mathcal{F}(U) \longrightarrow \prod\nolimits_{V,W\in\mathcal{U}} \mathcal{F}(V\cap W)$$

be the morphism of abelian groups given by the canonical maps

$$f^V_{V\cap W} \colon \prod\nolimits_{U\in\mathcal{U}} \mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V) \longrightarrow \mathcal{F}(V\cap W)\,.$$

We obtain an endofunctor ϵ of $\mathrm{PSh}(X)$ given on objects as $\epsilon\mathcal{F}(M) = \operatorname{colim}_{\mathcal{U}\in\mathrm{Cov}(M)} \operatorname{Ker}(f_{\mathcal{U}})$, where $\mathrm{Cov}(M)$ is the (opposite of the) poset of open coverings of M ordered by refinement.

We claim that $\sigma_X = \epsilon^2$ has image in $\mathrm{Sh}(X)$ and is left adjoint to ι_X.

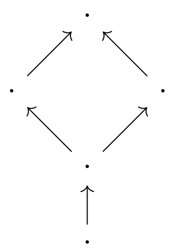
TO BE CONTINUED . . .

Exercise 5.4 (b).

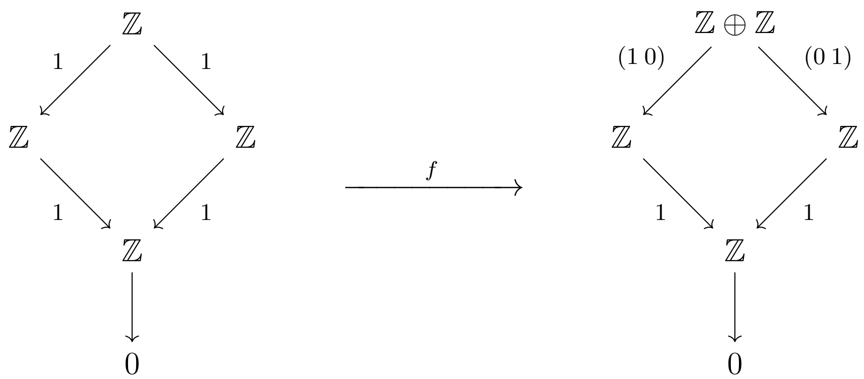
The existence and exactness of $j_\flat$ is clear. It restricts to a functor j_* between the subcategories of sheaves because kernels of sheaves can be computed in the category of presheaves by (a).

Let's now look at an example that illustrates why j_* does not have to be (right) exact.

Namely, consider a topological space X whose poset of opens has the following Hasse diagram:



We have a morphism of presheaves on X induced by the global section $\mathbb{Z} \xrightarrow{\binom{1}{1}} \mathbb{Z} \oplus \mathbb{Z}$:



Taking now for X the topological space $\{0, 1, 2\}$ with subbase $\{\{0, 1\}, \{0, 2\}\}$, the map f is an epimorphism in $\mathrm{Sh}(X)$ because it is surjective on stalks. However, it is clearly not surjective on global sections, i.e. the direct image $(X \to \{\bullet\})_* f$ is not an epimorphism.

Exercise 5.4 (c).

We can take for $j^\flat$ the functor that is given on objects as $j^\flat \mathcal{F}(U) = \operatorname{colim} \mathcal{F}_{j(U)}$ where $\mathcal{F}_{j(U)}$ is the restriction of $\mathcal{F}$ to the set of all neighborhoods of $j(U)$ in Y.

By the universal property of colimits we then have mutually inverse maps

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(j^\flat \mathcal{F}, \mathcal{G}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{\mathrm{PSh}(Y)}(\mathcal{F}, j_\flat \mathcal{G}) \,.$$

Indeed, $\alpha(f)(V)$ simply is $f(j^{-1}(V))$ precomposed with the canonical map $\mathcal{F}(V) \to j^\flat \mathcal{F}(V)$.

Contrariwise, let $\mathcal{F} \xrightarrow{g} j_\flat \mathcal{G}$ and let U be open in X. Then for any neighborhood V of $j(U)$ in Y we can postcompose $g(V)$ with the map $\mathcal{G}(j^{-1}(V)) \to \mathcal{G}(U)$ to obtain a map $\mathcal{F}(V) \to \mathcal{G}(U)$. The universal property of the colimit applied to these latter maps induces $\beta(g)(U)$.

Consequently, $(j^\flat, j_\flat)$ is an adjoint pair and hence so is $(j^* = \sigma_X \circ j^\flat \circ \iota_Y, \sigma_Y \circ j_\flat \circ \iota_X)$ by (a). It only remains to observe that $\sigma_Y \circ j_\flat \circ \iota_X = \sigma_Y \circ \iota_Y \circ j_* \cong j_*$ since ι_Y is fully faithful.

Exercise 5.4 (d).

If $j(U)$ is open in Y, then the subposet of $\mathcal{T}(Y)$ consisting of all neighborhoods of $j(U)$ in Y has $j(U)$ as its least element, so we get an induced isomorphism $\operatorname{colim} \mathcal{F}_{j(U)} \to \mathcal{F}(j(U))$.

Hence, if j is an open embedding, $j^\flat$ is canonically isomorphic to the restriction functor $j^\natural$.

Again, since kernels in the category of sheaves can be computed in the category of presheaves, $j^\flat$ restricts to a functor $j^\maltese$ between sheaf categories. Hence, $j^\maltese \cong \sigma_X \circ \iota_X \circ j^\maltese = \sigma_X \circ j^\natural \circ \iota_Y \cong j^*$.

Exercise 5.4 (e).

We can take for $j_\sharp$ the functor that is given on objects as

$$j_\sharp \mathcal{F}(U) \;=\; \begin{cases} \mathcal{F}(U) & \text{if } U \subseteq X, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, there are canonical mutually inverse maps

$$\operatorname{Hom}_{\mathrm{PSh}(Y)}(j_\sharp \mathcal{F}, \mathcal{G}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \operatorname{Hom}_{\mathrm{PSh}(X)}(\mathcal{F}, j^\flat \mathcal{G})\,.$$

Indeed, $\alpha(f)(U) = f(U)$ and $\beta(g)(U) = g(U)$ for $U \subseteq X$ and $\beta(g)(U) = 0$ for $U \not\subseteq X$.

Consequently, $(j_\sharp, j^\flat)$ is an adjoint pair and hence so is $(j_! = \sigma_Y \circ j_\sharp \circ \iota_X, j^*)$ by (a).

Exercise 5.4 (f).

Similarly as for j_* in (b) and for $j^\flat$ in (d), we see that $j_\sharp$ restricts to a functor $j_?$ between sheaf categories and we can then conclude $j_? \cong j_!$.

Now, denoting by $(\mathcal{C}, \wedge, \vee)$ either of $(\mathrm{PSh}(X), \flat, \sharp)$ or $(\mathrm{Sh}(X), \maltese, ?)$ the unit $1_{\mathcal{C}} \xrightarrow{\eta} j^\wedge j_\vee$ of the adjoint pair $(j_\vee, j^\wedge)$ is invertible because of $\eta\mathcal{F}(U) = \mathrm{id}_{\mathcal{F}(U)}$. Hence, $j_\vee$ is fully faithful.

The fully faithfulness of $j_\flat$ and j_* is then a consequence of Exercise 5.1 (a).

Exercise 5.4 (g).

The pair $(i_*, i^!)$ is adjoint because there are canonical mutually inverse maps

$$\operatorname{Hom}_{\mathrm{Sh}(Y)}(i_* \mathcal{F}, \mathcal{G}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \operatorname{Hom}_{\mathrm{Sh}(Z)}(\mathcal{F}, i^! \mathcal{G})\,.$$

On the one hand, take $i_* \mathcal{F} \xrightarrow{f} \mathcal{G}$. Then, for each open set U in Y, we have a commuting diagram

$$\begin{array}{ccccc}
 & & \operatorname{Ker}(\mathcal{G}^U_{U \cap X}) & \xleftarrow[\cong]{\kappa_{\mathcal{G},U}} & \operatorname{Ker}(\mathcal{G}^{U \cup X}_X) \\
 & \overset{\widetilde{f(U)}}{\dashrightarrow} & \downarrow & \swarrow \widetilde{\kappa}_{\mathcal{G},U} & \downarrow \\
\mathcal{F}(U \cap Z) & \xrightarrow{f(U)} & \mathcal{G}(U) & \longleftarrow & \mathcal{G}(U \cup X) \\
\downarrow & & \downarrow \mathcal{G}^U_{U \cap X} & & \downarrow \mathcal{G}^{U \cup X}_X \\
\mathcal{F}(\emptyset) = 0 & \xrightarrow{f(U \cap X)} & \mathcal{G}(U \cap X) & \longleftarrow & \mathcal{G}(X)
\end{array}$$

inducing $\alpha(f)(U \cap Z) = \kappa^{-1}_{\mathcal{G},U} \circ \widetilde{f(U)}$, where $\mathcal{F}(\emptyset) = 0$ since $\mathcal{F}$ is a sheaf and $\kappa_{\mathcal{G},U}$ is invertible since $\mathcal{G}$ is a sheaf. On the other hand, $\beta(g)(U) = \widetilde{\kappa}_{\mathcal{G},U} \circ g(U \cap Z)$ defines the inverse β of α.

Exercise 5.4 (h).

Note that in general, if Z carries the initial topology with respect to a continuous map $Z \xrightarrow{i} Y$, the fully faithfulness of $i_\flat$ follows formally by an argument similar to the one given in (d).

Indeed, for open sets U in Y, the set $\{i^{-1}(V) : V$ is a neighborhood of $i(i^{-1}(U))$ in $Y\}$ ordered by inclusion has $i^{-1}(U)$ as its least element such that by the universal property of the colimit there exists a canonical isomorphism $i^\flat i_\flat \to 1_{\mathrm{PSh}(Z)}$, i.e. $i_\flat$ is fully faithful.

For the fully faithfulness of i_* note that, for open sets U in Y, there are canonical isomorphisms

$$i^! i_* \mathcal{F}(U \cap Z) \;=\; \mathrm{Ker}(\mathcal{F}(U \cap Z) \to \mathcal{F}(\emptyset) = 0) \xrightarrow{\cong} \mathcal{F}(U \cap Z)\,.$$

Evidently, $\mathrm{Im}(i_*) \subseteq \mathrm{Ker}(j^*)$ because of $j^* i_* \mathcal{G}(U) = \mathcal{G}(U \cap Z) = \mathcal{G}(\emptyset) = 0$ for $U \subseteq X$.

Conversely, $\mathcal{G} \cong i_* i^! \mathcal{G}$ for $\mathcal{G} \in \mathrm{Ker}(j^*)$ since $j^* \mathcal{G} = 0$ means that $\mathcal{G}(U \cap X) = j^* \mathcal{G}(U \cap X) = 0$ for all open U in Y such that the diagram from (g) yields canonical isomorphisms

$$i_* i^! \mathcal{G}(U) \;=\; \mathrm{Ker}\left(\mathcal{G}_X^{U \cup X}\right) \xrightarrow{\cong} \mathrm{Ker}(\mathcal{G}(U) \to \mathcal{G}(U \cap X) = 0) \xrightarrow{\cong} \mathcal{G}(U)\,.$$

Exercise 5.4 (i).

This is an immediate consequence of Exercise 5.1 and 5.4 (a)–(h).