REPRESENTATION THEORY
EXERCISES 7

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Let Λ be an artin algebra.

For a full additive subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ a morphism $X \xrightarrow{p} M$ in $\operatorname{mod}\Lambda$ is said to be a *right $\mathcal{C}$-approximation* of M if $X \in \mathcal{C}$ and any other morphism $X' \to M$ with $X' \in \mathcal{C}$ factors through p.

A *$\mathcal{C}$-resolution of M of length n* is by definition an exact sequence in $\operatorname{mod}\Lambda$

$$0 \to X_n \xrightarrow{p_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0 \xrightarrow{p_0} M \to 0$$

such that the induced maps $X_i \to \operatorname{Im}(p_i)$ are right $\mathcal{C}$-approximations. For $\mathcal{D} \subseteq \operatorname{mod}\Lambda$ define

$$\operatorname{res.dim}_{\mathcal{C}} \mathcal{D} \ := \ \sup_{M \in \mathcal{D}} \inf\{n \in \mathbb{N} : M \text{ admits a } \mathcal{C}\text{-resolution of length } n\} \, .$$

1. Prove the following:

(a) For all objects M and generators C of $\operatorname{mod}\Lambda$ we have
$$\operatorname{res.dim}_{\operatorname{add} C} \operatorname{add} M \ = \ \operatorname{proj.dim} \operatorname{Hom}_\Lambda(C, M)_{\operatorname{End}_\Lambda(C)} \, .$$

(b) If Λ is not semisimple and C is a generating-cogenerating object in $\operatorname{mod}\Lambda$, then
$$\operatorname{gldim} \operatorname{End}_\Lambda(C) \ = \ \operatorname{res.dim}_{\operatorname{add} C} \operatorname{mod}\Lambda + 2 \, .$$

(c) $\operatorname{rep.dim}\Lambda = 0$ iff Λ is semisimple.

(d) $\operatorname{rep.dim}\Lambda \neq 1$.

(e) $\operatorname{rep.dim}\Lambda = 2$ iff Λ is representation-finite and not semisimple.

Submodules of projective Λ-modules are called *torsionless*. The algebra Λ is *torsionless-finite* if there are only finitely many isomorphism classes of indecomposable torsionless modules in $\operatorname{mod}\Lambda$.

2. Show the following:

(a) If Λ is hereditary, then Λ is torsionless-finite.

(b) If $J(\Lambda)^2 = 0$, then Λ is torsionless-finite.

(c) If Λ is torsionless-finite, then $\operatorname{rep.dim}(\Lambda) \leq 3$.

3. Recall that $\mathfrak{q}A = \mathfrak{q}_A(A)$ where by definition for objects A, X in an abelian category $\mathcal{A}$ we have

$$\mathfrak{q}_A(X) \ = \ \operatorname{Coker}\Big(\operatorname{Ker}\Big(X \xrightarrow{\mu} A^{\operatorname{Rad}_{\mathcal{A}}(X,A)}\Big) \hookrightarrow X\Big) \quad \text{with } \mu_\phi = \phi \, .$$

Assume now that $\mathcal{A} = \operatorname{Mod} A$ for a semiprimary ring A. Show $\mathfrak{q}^n A_A = 0$ for sufficiently large n.

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4. Let k be a commutative ring and E the injective envelope of the k-module $\bigoplus_{\mathfrak{m}\in \mathrm{mSpec}(k)} k/\mathfrak{m}$, where $\mathrm{mSpec}(k)$ denotes the set of maximal ideals of k.

The contravariant endofunctor $D = \mathrm{Hom}_k(-, E)$ of $\mathrm{Mod}\, k$ is known as *Matlis duality*.

Verify the following facts:

(a) There are natural isomorphisms for all k-modules X and Y:

$$\mathrm{Hom}_k(X, DY) \;\cong\; D(X \otimes_k Y) \;\cong\; \mathrm{Hom}_k(Y, DX)$$

(b) The canonical map $X \to D^2 X$ given by evaluation is injective for each k-module X.

(c) $\mathrm{Ann}(X) = \mathrm{Ann}(DX)$, so in particular $X = 0 \Leftrightarrow DX = 0$, for all k-modules X

(d) D is faithful and exact.

(e) D induces a contravariant autoequivalence of the category $\mathrm{fl}\, k$ of finite-length modules.