GENERATION TIME IN DERIVED CATEGORIES

by

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Abstract

In the derived category of modules of a ring, a complex G is said to generate a complex X if the latter can be obtained from the former by taking finitely many cones and direct summands. The number of cones needed in this process is the generation time of X. This invariant captures some familiar invariants for modules, namely projective dimension and Loewy length, but has better homological properties for complexes. I present some local to global type results for computing this invariant, and also discuss some applications.

Contents

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Chapter 1

Introduction

There are various notions of generation in mathematics. It is a useful thing to consider if one wants to explore the structure of an object or a class of objects.

I am interested in complexes of modules over rings, more specifically in their homological properties. This leads me to study the derived category. In this category, every complex with zero homology is isomorphic to the zero complex, and a module, more generally any complex, is isomorphic to any of its resolutions.

A first thing to note about the derived category is that it is not abelian; there are no short exact sequences. However, one has exact triangles. When passing from the category of complexes to the derived category, every short exact sequence induces an exact triangle. Applying homology to such an exact triangle gives a long exact sequence in homology. Therefore, one can do homological algebra in this setting.

The derived category was the motivating example for Verdier [Ver96] to define the triangulated category. A triangulated category is a more abstract notion, and also appears in other areas. For example, the stable homotopy category in topology is a triangulated category. I consider generation in a triangulated category as introduced by [BvdB03] and further explored by [Rou08]. The generation time, called level, is the number of steps it takes to build an object X from an object G. This will be made more precise later. If level is finite, then G *generates* X. This invariant has many connections to other, more familiar, invariants.

Let R be a noetherian ring and M a finitely generated module over R. Then there exists a projective resolution of M

$$M \simeq P = (\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to 0)\,.$$

Now if this resolution is finite, that is $P_i = 0$ for $i \gg 0$, and the projective modules are finitely generated, then the projective modules 'generate' the module M. Moreover, every projective module is a direct summand of a direct sum of R, so R 'generates' M. It turns out that level of M with respect to R is precisely $\operatorname{pd}(M)+1$.

If R is a local ring with residue field k, then any module M has a filtration

$$\ldots \subseteq \mathfrak{m}^2 M \subseteq \mathfrak{m}M \subseteq M\,.$$

The subquotients in this filtration are direct sums of copies of k. If this filtration is finite and the subquotients are finite direct sums, then the residues field k 'generates' the module M. It turns out the level of M with respect to k is the Loewy length, which is the smallest integer n for which $\mathfrak{m}^n M = 0$.

These are just two examples of invariants connected to level. While both of these invariants are defined for modules, level extends these notions to complexes.

In Chapter 2, I recall the definitions of generation in the triangulated category as introduced by [BvdB03] and of level following [ABIM10], and discuss some properties. In Chapter 4, I discuss generation in the derived category, and show level encodes projective dimension and Loewy length on modules.

Despite their utility, there are few results on the behavior of level even under functors induced by a change of rings. I track the behavior of level under standard commutative algebra operations, notably localizations and completions.

There are two main theorems.

Theorem 1. *Let $\varphi\colon R \to S$ be a faithfully flat ring map with R a commutative noetherian ring and S a noetherian ring, so that R acts centrally on S. For complexes G and X of R-modules with bounded and degree-wise finitely generated homology, there is an equality*

$$\operatorname{level}_R^G(X) = \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X))$$

where $\varphi^* := S \otimes^{\mathbf{L}}_R -$.

This theorem notably applies when $(R,\mathfrak{m})$ is a local ring and $S = \widehat{R}$ its $\mathfrak{m}$-adic completion.

The second result considers the localizations.

Theorem 2. *Let R be a commutative noetherian ring. For G and X of R-modules with bounded and degree-wise finitely generated homology, there is an equality*

$$\operatorname{level}_R^G(X) = \sup\left\{\operatorname{level}_{R_\mathfrak{p}}^{G_\mathfrak{p}}(X_\mathfrak{p}) \,\middle|\, \mathfrak{p} \in \operatorname{Spec}(R)\right\}\,.$$

Moreover, if $\operatorname{level}^{G_{\mathfrak{p}}}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$ *for all prime ideals* $\mathfrak{p}$ *of* R*, then* $\operatorname{level}^{G}_{R}(X) < \infty$.

The two theorems are contained in Chapter 5. The principal tool in the proof is a converse coghost lemma by Oppermann and Šťovíček [OŠ12]. A map is G-coghost if it cannot be detected by post-composition with any suspension of G. The coghost index of X with respect to G is the minimal number n for which every n-fold composition of coghost maps that ends at X is zero. The coghost lemma by [Kel65] implies that the coghost index with respect to G is bounded above by level with respect to G. The converse coghost lemma yields equality in the bounded derived category of a Noether algebra.

I discuss coghost maps, as well as their dual in the opposite category, ghost maps, in a triangulated category in Chapter 3. A proof of the ghost lemma and a partial converse is given. For the proof of the converse coghost lemma in the derived category, see Chapter 5.

Chapter 6 has some applications for finite generation and Theorems 1 and 2. If a triangulated category is generated by one object, then, under some finiteness conditions, every homological functor is representable. That is, it is naturally isomorphic to the contravariant functor $\operatorname{Hom}(-, X)$.

From Hopkins' [Hop87, Theorem 11] and Neeman's [Nee92, Lemma 1.2] result about perfect complexes, I deduce that two complexes of finite injective dimension with finitely generated bounded homology who have the same support generate each other.

One can also track the behavior of proxy small, introduced in [DGI06]. A complex X is *proxy small* if $X \simeq 0$ or it generates a perfect complex $Y \not\simeq 0$ with the same support as X. I prove that X is proxy small if and only if it is proxy small locally, and X is proxy small precisely when it is proxy small under a faithfully flat base change.

By [Pol19], proxy small objects in $\mathrm{D}^{\mathrm{f}}(R)$ characterize whether a local ring R is a complete intersection. I conclude that proxy small objects also characterize whether a ring is *locally* a complete intersection.

Chapter 2

Generation in a Triangulated Category

Generation in a triangulated category was first introduced by [BvdB03] and further studied by [Rou08]. For convenience, I give the detailed construction of the filtration of the smallest thick subcategory containing an object G. A thick subcategory is a triangulated category closed under direct summands. The invariant level indicates when a complex appears the first time in this filtration. If the filtration stabilizes to the whole triangulated category, then G is a strong generator. The length of the shortest filtration for all strong generators is the dimension of the triangulated category. The end of this chapter discusses compact objects, and how these are the finite objects with respect to generation.

2.1 Triangulated Category

In this work, every category and every functor is additive. A *triangulated category* consists of

1. a category $\mathcal{T}$,

2. an auto-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ called the *suspension*, and

3. a collection of exact triangles (X, Y, Z, f, g, h) written as

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \text{or} \quad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y\,. \\ & {}_{h}\nwarrow^{+1} \quad \swarrow_{g} & \\ & Z & \end{array}$$

   The last notation for a triangle gives it its name, but to save space, it is common to use the first. The $+1$ indicates that the codomain of this morphism is the suspension of the indicated object.

This data has to satisfy some axioms; for details, see for example [Nee01, Chapter 1]. I record some properties.

The cone of a morphism $f\colon X \to Y$ in $\mathcal{T}$ is denoted by $\operatorname{cone}(f)$. This is an object that completes the morphism $X \to Y$ to an exact triangle:

$$X \xrightarrow{f} Y \to \operatorname{cone}(f) \to \Sigma X\,.$$

2.1.1. Given composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the octahedral axiom connects their cones with the cone of their composition: There exists an exact triangle

$$\operatorname{cone}(f) \to \operatorname{cone}(g \circ f) \to \operatorname{cone}(g) \to \Sigma\operatorname{cone}(f)$$

such that the morphisms are compatible. This property was introduced by Verdier [Ver96, Chapter II, 1.1].

I say a subcategory $\mathcal{C}$ is *closed under suspension* if the suspension Σ induces an auto-equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$. In particular, it is closed under Σ^n for all integers n.

Definition 2.1.2. Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$.

1. $\mathcal{C}$ is *strictly full* if it is closed under isomorphisms, and every morphism between objects in $\mathcal{C}$ viewed as objects in $\mathcal{T}$ lies in $\mathcal{C}$, that is for $X, Y \in \mathcal{C}$ one has

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathcal{T}}(X,Y)\,.$$

2. $\mathcal{C}$ is *triangulated* if it is strictly full, closed under suspension, and satisfies the two out of three property, that is when two objects of an exact triangle in $\mathcal{T}$ are in $\mathcal{C}$, then so is the third.

3. $\mathcal{C}$ is *thick* if it is triangulated and closed under direct summands.

The intersection of thick subcategories is again a thick subcategory. So there exists a smallest thick subcategory containing a subcategory $\mathcal{C}$, which I denote by $\operatorname{thick}(\mathcal{C})$. When $\mathcal{C}$ consists of a single object G, write $\operatorname{thick}(G)$. If the ambient category $\mathcal{T}$ is not evident from the context, write $\operatorname{thick}_{\mathcal{T}}(-)$.

2.2 Thickenings

In this section, I recall a filtration of $\operatorname{thick}(\mathcal{C})$ introduced in [BvdB03, 2.2].

Definition 2.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be strictly full subcategories of $\mathcal{T}$. Then $\mathcal{C} \star \mathcal{D}$ is the strictly full subcategory containing all objects $X \in \mathcal{T}$, for which an exact triangle

$$Y \to X \to Z \to \Sigma Y$$

exists with $Y \in \mathcal{C}$ and $Z \in \mathcal{D}$.

Note the operation $\star$ needs not be commutative.

Lemma 2.2.2. *The operation $\star$ is associative.*

Proof. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be strictly full subcategories of $\mathcal{T}$. Take X in $(\mathcal{C} \star \mathcal{D}) \star \mathcal{E}$. Then there exist exact triangles

$$Y \to X \to Z \xrightarrow{f} \Sigma Y \quad \text{with } Y \in \mathcal{C} \star \mathcal{D} \text{ and } Z \in \mathcal{E}\,,$$
$$V \to Y \xrightarrow{g} W \to \Sigma V \quad \text{with } V \in \mathcal{C} \text{ and } W \in \mathcal{D}\,.$$

Complete the map $Z \xrightarrow{f} \Sigma Y \xrightarrow{\Sigma g} \Sigma W$ to an exact triangle in $\mathcal{T}$ to get

$$W \to U \to Z \xrightarrow{(\Sigma g)\circ f} \Sigma W \quad \text{with } U \in \mathcal{D} \star \mathcal{E}\,.$$

By the octahedral axiom 2.1.1, there exists an exact triangle

$$V \to X \to U \to \Sigma V\,.$$

Thus X lies in $\mathcal{C} \star (\mathcal{D} \star \mathcal{E})$. The proof that the inclusion

$$\mathcal{C} \star (\mathcal{D} \star \mathcal{E}) \subseteq (\mathcal{C} \star \mathcal{D}) \star \mathcal{E}$$

holds is analogous. □

Let $\operatorname{add}(\mathcal{C})$ be the smallest strictly full subcategory closed under suspension and finite direct sums, and let $\operatorname{smd}(\mathcal{C})$ be the smallest strictly full subcategory closed under direct summands.

Definition 2.2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{T}$. Then

$$\mathcal{C} \diamond \mathcal{D} := \operatorname{smd}(\operatorname{add}(\mathcal{C}) \star \operatorname{add}(\mathcal{D}))\,.$$

By Lemma 2.2.2, the following notation is unambiguous.

Definition 2.2.4. Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$. Then

$$\mathcal{C}^{\star n} := \begin{cases} \{0\} & n = 0 \\ \mathcal{C} & n = 1 \\ \underbrace{\mathcal{C} \star \ldots \star \mathcal{C}}_{n \text{ copies}} & n \geq 2\,. \end{cases}$$

The following properties are stated in [BvdB03, 2.2]. For completeness, I give a proof.

Lemma 2.2.5. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be strictly full subcategories of $\mathcal{T}$. Then*

1. $\operatorname{smd}(\operatorname{smd}(\operatorname{add}(\mathcal{C})) \star \operatorname{add}(\mathcal{D})) = \operatorname{smd}(\operatorname{add}(\mathcal{C}) \star \operatorname{add}(\mathcal{D}))$*, and the same when* smd *is applied to the second factor,*
2. $\operatorname{add}(\mathcal{C} \diamond \mathcal{D}) = \mathcal{C} \diamond \mathcal{D}$*, and*
3. *the operation $\diamond$ is associative with*

$$\mathcal{C}_1 \diamond \ldots \diamond \mathcal{C}_n = \operatorname{smd}(\operatorname{add}(\mathcal{C}_1) \star \ldots \star \operatorname{add}(\mathcal{C}_n))\,.$$

Proof. In (1), the inclusion $\supseteq$ is clear. For the opposite inclusion, take X in the left-hand side. Then there exists an exact triangle

$$Y \to X \oplus X' \to Z \to \Sigma Y \quad \text{where } Y \in \operatorname{smd}(\operatorname{add}(\mathcal{C})) \text{ and } Z \in \operatorname{add}(\mathcal{D})\,.$$

In particular, $Y \oplus Y' \in \operatorname{add}(\mathcal{C})$ for some Y'. Since the triangle $Y' \to Y' \to 0 \to \Sigma Y'$ is exact, the triangle

$$Y \oplus Y' \to X \oplus X' \oplus Y' \to Z \to \Sigma(Y \oplus Y')$$

is exact and so $X \in \operatorname{smd}(\operatorname{add}(\mathcal{C}) \star \operatorname{add}(\mathcal{D}))$.

In (2), the inclusion $\supseteq$ is clear. For the opposite, I may assume that $\mathcal{C}$ and $\mathcal{D}$ are closed under finite direct sums and suspension. Let $X \in \operatorname{add}(\mathcal{C} \diamond \mathcal{D})$. Then $X = \bigoplus_{i=1}^{n} X^i$ with $X^i \oplus W^i \in \mathcal{C} \star \mathcal{D}$. Thus there are exact triangles

$$Y^i \to X^i \oplus W^i \to Z^i \to \Sigma Y^i \quad \text{with } Y^i \in \mathcal{C} \text{ and } Z^i \in \mathcal{D}\,.$$

Taking the direct sum of these exact triangles gives the exact triangle

$$\bigoplus_{i=1}^{n} Y^i \to X \oplus \bigoplus_{i=1}^{n} W^i \to \bigoplus_{i=1}^{n} Z^i \to \Sigma \bigoplus_{i=1}^{n} Y^i \quad \text{with } \bigoplus_{i=1}^{n} Y^i \in \mathcal{C} \text{ and } \bigoplus_{i=1}^{n} Z^i \in \mathcal{D}\,.$$

Then X lies in the right-hand side.

For (3), first prove that $\diamond$ is associative. By the previous parts

$$\begin{aligned}(\mathcal{C}_1 \diamond \mathcal{C}_2) \diamond \mathcal{C}_3 &= \operatorname{smd}(\operatorname{add}(\mathcal{C}_1 \diamond \mathcal{C}_2) \star \operatorname{add}(\mathcal{C}_3)) \\ &\overset{(2)}{=} \operatorname{smd}((\mathcal{C}_1 \diamond \mathcal{C}_2) \star \operatorname{add}(\mathcal{C}_3)) \\ &\overset{(1)}{=} \operatorname{smd}(\operatorname{add}(\mathcal{C}_1) \star \operatorname{add}(\mathcal{C}_2) \star \operatorname{add}(\mathcal{C}_3)) \\ &\overset{(1)}{=} \operatorname{smd}(\operatorname{add}(\mathcal{C}_1) \star (\mathcal{C}_2 \diamond \mathcal{C}_3)) \\ &\overset{(2)}{=} \mathcal{C}_1 \diamond (\mathcal{C}_2 \diamond \mathcal{C}_3)\,.\end{aligned}$$

These identifications use the associativity of $\star$ shown in 2.2.2. The third line gives the last statement for $n = 3$. By induction, it holds for arbitrary n. □

Definition 2.2.6. Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$. Then the *nth thickening* of $\mathcal{C}$ is

$$\operatorname{thick}^n(\mathcal{C}) := \begin{cases} \{0\} & n = 0 \\ \operatorname{smd}(\operatorname{add}(\mathcal{C})) & n = 1 \\ \operatorname{thick}^{n-1}(\mathcal{C}) \diamond \operatorname{thick}^1(\mathcal{C}) & n \geq 2\,. \end{cases}$$

For an object $G \in \mathcal{T}$, write

$$\operatorname{thick}^n(G) := \operatorname{thick}^n(\{G\})\,.$$

If the ambient category $\mathcal{T}$ is not evident from the context, I write $\operatorname{thick}_{\mathcal{T}}(-)$.

In [BvdB03] and [Rou08], the thickenings are denoted $\langle \mathcal{C} \rangle_n$. I follow the notation of [ABIM10].

Lemma 2.2.7. *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$. Then*

1. $\operatorname{thick}^n(\mathcal{C})$ *is closed under suspension, finite direct sums and direct summands, and*
2. *for $k + l = n$, one has*

$$\begin{aligned}\operatorname{thick}^n(\mathcal{C}) &= \operatorname{smd}(\operatorname{thick}^k(\mathcal{C}) \star \operatorname{thick}^l(\mathcal{C})) \\ &= \operatorname{smd}(\operatorname{add}(\mathcal{C})^{\star n})\,.\end{aligned}$$

Proof. By definition, $\operatorname{thick}^n(\mathcal{C})$ is closed under direct summands. To show it is closed under suspension and finite direct sums, it is enough to show that it is invariant when applying add. Use induction on n. For $n = 1$, one has

$$\begin{aligned}\operatorname{thick}^1(\mathcal{C}) &\subseteq \operatorname{add}(\operatorname{thick}^1(\mathcal{C})) = \operatorname{add}(\operatorname{smd}(\operatorname{add}(\mathcal{C}))) \\ &\subseteq \operatorname{smd}(\operatorname{add}(\operatorname{add}(\mathcal{C}))) = \operatorname{thick}^1(\mathcal{C})\,.\end{aligned}$$

When $n \geq 2$, the statement follows from Lemma 2.2.5 (2). This proves (1). For (2), the nth thickening decomposes by 2.2.5 (3) as

$$\operatorname{thick}^n(\mathcal{C}) = \underbrace{\operatorname{thick}^1(\mathcal{C}) \diamond \ldots \diamond \operatorname{thick}^1(\mathcal{C})}_{n\text{-times}} = \operatorname{smd}\big(\underbrace{\operatorname{thick}^1(\mathcal{C}) \star \ldots \star \operatorname{thick}^1(\mathcal{C})}_{n\text{-times}}\big).$$

Grouping the terms appropriately gives the first claim. The second equality follows from 2.2.5 (1) and the definition of $\operatorname{thick}^1(\mathcal{C})$. □

2.2.8. The thickenings give a filtration of $\operatorname{thick}(\mathcal{C})$:

$$\{0\} = \operatorname{thick}^0(\mathcal{C}) \subseteq \operatorname{thick}^1(\mathcal{C}) \subseteq \ldots \subseteq \bigcup_{n \geq 0} \operatorname{thick}^n(\mathcal{C}) = \operatorname{thick}(\mathcal{C}).$$

Observe the last is an equality, since the union of all thickenings is clearly thick, and $\operatorname{thick}(\mathcal{C})$ has to contain all the thickenings.

Definition 2.2.9. An object $G \in \mathcal{T}$ is a *classical generator* of $\mathcal{T}$, if $\mathcal{T} = \operatorname{thick}(G)$. If there exists an n such that $\mathcal{T} = \operatorname{thick}^n(G)$, then G is a *strong generator*.

Definition 2.2.10. Let $G, X \in \mathcal{T}$. I say G *finitely generates* X if X is in $\operatorname{thick}(G)$ and write

$$G \models X.$$

2.2.11. Note that generation is transitive:

$$X \models Y \quad \text{and} \quad Y \models Z \quad \text{then} \quad X \models Z.$$

By 2.2.8, if G finitely generates X, then there exists n such that $X \in \operatorname{thick}^n(G)$. In particular, X can be constructed from G by taking finitely many cones, suspensions, and direct summands.

2.3 Level

The following invariant was first introduced in [ABIM10, 2.3]. It encodes when an object first occurs in the filtration 2.2.8 for $\operatorname{thick}(\mathcal{C})$.

Definition 2.3.1. Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$ and X an object in $\mathcal{T}$. Then

$$\operatorname{level}^{\mathcal{C}}(X) := \inf\{n \geq 0 \mid X \in \operatorname{thick}^n(\mathcal{C})\}$$

is the *$\mathcal{C}$-level of* X. For G an object in $\mathcal{T}$, write

$$\operatorname{level}^G(X) := \operatorname{level}^{\{G\}}(X).$$

Note that $\mathrm{level}^G(X)$ is finite if and only if G finitely generates X.

Loosely speaking, level is the number of steps required to build X from G. In the building process, only the cones are counted.

Here a few properties of level are recorded; see [ABIM10, Lemma 2.4].

Lemma 2.3.2. *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$.*

1. *Level is invariant under suspension, that is*

$$\mathrm{level}^{\mathcal{C}}(\Sigma^d X) = \mathrm{level}^{\mathcal{C}}(X) \quad \text{for any } d \in \mathbb{Z}\,.$$

2. *Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. Then*

$$\mathrm{level}^{\mathcal{C}}(Y) \leq \mathrm{level}^{\mathcal{C}}(X) + \mathrm{level}^{\mathcal{C}}(Z)\,.$$

3. *Let X and Y be objects in $\mathcal{T}$, then*

$$\mathrm{level}^{\mathcal{C}}(X \oplus Y) = \max\{\mathrm{level}^{\mathcal{C}}(X), \mathrm{level}^{\mathcal{C}}(Y)\}\,.$$

Proof. (1) holds since by 2.2.7 (1), the nth thickening $\mathrm{thick}^n(\mathcal{C})$ is closed under suspension. For (2), assume $X \in \mathrm{thick}^k(\mathcal{C})$ and $Z \in \mathrm{thick}^l(\mathcal{C})$. Then by 2.2.7 (2), one has

$$Y \in \mathrm{thick}^k(\mathcal{C}) \star \mathrm{thick}^l(\mathcal{C}) \subseteq \mathrm{thick}^{k+l}(\mathcal{C})\,.$$

If $\mathrm{level}^{\mathcal{C}}(X) = \infty$ or $\mathrm{level}^{\mathcal{C}}(Z) = \infty$, then the statement is clear. (3) follows from the fact that $\mathrm{thick}^n(\mathcal{C})$ is closed under finite direct sums and direct summands: The direct sum $X \oplus Y$ lies in $\mathrm{thick}^n(\mathcal{C})$ if and only if X and Y lie in $\mathrm{thick}^n(\mathcal{C})$. □

Lemma 2.3.3 (Change of category for levels)**.** *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$.*

1. *If $\mathcal{C}$ is contained in a thick subcategory $\mathcal{S}$ of $\mathcal{T}$, then for every $X \in \mathcal{S}$, one has*

$$\mathrm{level}^{\mathcal{C}}_{\mathcal{T}}(X) = \mathrm{level}^{\mathcal{C}}_{\mathcal{S}}(X)\,.$$

2. *One has the equality $\mathrm{level}^{\mathcal{C}}_{\mathcal{T}}(X) = \mathrm{level}^{\mathcal{C}^{op}}_{\mathcal{T}^{op}}(X)$ for all $X \in \mathcal{T}$.*

3. *Let $\mathsf{f}\colon \mathcal{T} \to \mathcal{S}$ be an exact functor between triangulated categories and $X \in \mathcal{T}$. Then*

$$\mathrm{level}^{\mathsf{f}(\mathcal{C})}_{\mathcal{S}}(\mathsf{f}(X)) \leq \mathrm{level}^{\mathcal{C}}_{\mathcal{T}}(X)\,.$$

Proof. For (1): Since $\mathcal{S}$ is closed under finite direct sums, direct summands, suspension, and taking triangles, it contains $\operatorname{thick}_{\mathcal{T}}(\mathcal{C})$, and so

$$\operatorname{thick}^n_{\mathcal{T}}(\mathcal{C}) = \operatorname{thick}^n_{\mathcal{S}}(\mathcal{C}) \qquad \text{for all } n \geq 0\,.$$

For any strictly full subcategories $\mathcal{D}$ and $\mathcal{E}$ of $\mathcal{T}$, one has

$$\mathcal{E}^{op} \star \mathcal{D}^{op} = (\mathcal{D} \star \mathcal{E})^{op} \quad \text{and} \quad \operatorname{add}(\mathcal{D}^{op}) = \operatorname{add}(\mathcal{D})^{op} \quad \text{and} \quad \operatorname{smd}(\mathcal{D}^{op}) = \operatorname{smd}(\mathcal{D})^{op}\,.$$

Thus

$$(\operatorname{thick}^n_{\mathcal{T}}(\mathcal{C}))^{op} = \operatorname{thick}^n_{\mathcal{T}^{op}}(\mathcal{C}^{op})\,.$$

For (3): Since f is exact, it maps exact triangles to exact triangles, finite sums to finite sums, direct summands to direct summands, and it respects suspension. Thus if $X \in \operatorname{thick}^n_{\mathcal{T}}(\mathcal{C})$, then $\mathsf{f}(X) \in \operatorname{thick}^n_{\mathcal{S}}(\mathsf{f}(\mathcal{C}))$. □

Lemma 2.3.4. *Given G and H in $\mathcal{T}$. For any $X \in \mathcal{T}$, one has*

$$\operatorname{level}^G(X) \leq \operatorname{level}^G(H) \cdot \operatorname{level}^H(X)\,.$$

Proof. Set $n := \operatorname{level}^G(H)$ and $m := \operatorname{level}^H(X)$. I may assume these are finite. By 2.2.7 (2), one has

$$X \in \operatorname{smd}(\operatorname{thick}^1(H)^{\star m}) \subseteq \operatorname{smd}(\operatorname{thick}^n(G)^{\star m}) = \operatorname{thick}^{nm}(G)\,,$$

and the claim holds. □

This lemma implies the transitivity of generation mentioned in 2.2.11.

In the construction of the thickenings, one takes cones of maps. Here is a useful fact for the cone of a composition.

Lemma 2.3.5. *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$ and $X^0 \xrightarrow{f^1} \dots \xrightarrow{f^n} X^n$ be a collection of morphisms in $\mathcal{T}$ such that $\operatorname{cone}(f^i) \in \operatorname{thick}^1(\mathcal{C})$. Then*

$$\operatorname{cone}(f^n \circ \dots \circ f^1) \in \operatorname{thick}^n(\mathcal{C})\,.$$

Proof. Set $f(i,k) := f^{i+k-1} \circ \dots \circ f^i$ and $C(i,k) := \operatorname{cone}(f(i,k))$. By assumption, $C(i,1)$ lies in $\operatorname{thick}^1(\mathcal{C})$. I have to show $C(1,n)$ lies in $\operatorname{thick}^n(\mathcal{C})$. Show by induction on k that $C(i,k) \in \operatorname{thick}^k(\mathcal{C})$. This is true for $k = 1$. For $k > 1$, consider the exact triangles

$$\begin{aligned}
X^{i-1} &\xrightarrow{f(i,1)} X^i \longrightarrow C(i,1) \longrightarrow \Sigma X^{i-1}\,,\\
X^{i} &\xrightarrow{f(i+1,k-1)} X^{i+k-1} \longrightarrow C(i+1,k-1) \to \Sigma X^{i}\,,\\
X^{i-1} &\xrightarrow{f(i,k)} X^{i+k-1} \longrightarrow C(i,k) \longrightarrow \Sigma X^{i-1}\,.
\end{aligned}$$

By the octahedral axiom 2.1.1, there exists an exact triangle

$$C(i,1) \to C(i,k) \to C(i+1,k-1) \to \Sigma C(i,1)\,.$$

By assumption, $C(i,1)$ lies in $\operatorname{thick}^1(\mathcal{C})$ and by the induction hypothesis, $C(i+1,k-1)$ lies in $\operatorname{thick}^{k-1}(\mathcal{C})$. So $C(i,k)$ lies in $\operatorname{thick}^k(\mathcal{C})$. □

The following lemma is well-known.

Lemma 2.3.6. *Let $\eta\colon \mathsf{f} \to \mathsf{g}$ be a natural transformation of (co)homological functors $\mathsf{f}, \mathsf{g}\colon \mathcal{T} \to \mathcal{A}$ from a triangulated category to an abelian category. If $\eta(G)$ is an isomorphism for some $G \in \mathcal{T}$, then η is an isomorphism on* $\operatorname{thick}(G)$*. In particular, the category of all X for which $\eta(X)$ is an isomorphism is thick.* □

As in [BFK12, Definition 2.1], I define

Definition 2.3.7. Let G be a strong generator in $\mathcal{T}$. Then

$$\circleddash(G) := \min\left\{ n \geq 0 \,\middle|\, \operatorname{thick}^{n+1}(G) = \mathcal{T} \right\}$$

is the *generation time of* G. If G is not a strong generator, set $\circleddash(G) = \infty$.

The generation time of a strong generator G is zero if and only if every object is the direct summand of a direct sum of suspensions of G. That means no cones are required to build any object.

2.4 Dimension

As in [Rou08, Definition 3.2], I define

Definition 2.4.1. The *(Rouquier) dimension* of a triangulated category $\mathcal{T}$ is

$$\dim(\mathcal{T}) := \inf\left\{ \circleddash(G) \mid G \in \mathcal{T} \right\}\,.$$

The *ultimate dimension* of a triangulated category $\mathcal{T}$ is

$$\operatorname{udim}(\mathcal{T}) := \sup\left\{ \circleddash(G) \mid G \in \mathcal{T} \text{ with } \circleddash(G) < \infty \right\}\,,$$

if $\mathcal{T}$ has a strong generator. Otherwise $\operatorname{udim}(\mathcal{T}) := \infty$.

It is also possible to express the dimension using level or generation time ⊘:

$$\dim(\mathcal{T}) = \inf\left\{n \geq 0 \,\middle|\, \exists G \in \mathcal{T} \text{ with } \operatorname{level}^G(X) \leq n+1 \text{ for all } X \in \mathcal{T}\right\}$$

The dimension of $\mathcal{T}$ is finite precisely when $\mathcal{T}$ has a strong generator.

Lemma 2.4.2. *If* $\dim(\mathcal{T}) < \infty$, *then every classical generator is a strong generator.*

Proof. Let G be a classical generator of $\mathcal{T}$. Since $\mathcal{T}$ has finite dimension, there exists a strong generator $H \in \mathcal{T}$ with $\operatorname{thick}^n(H) = \mathcal{T}$ for some n. Since G is a classical generator, it generates H and so $H \in \operatorname{thick}^m(H)$ for some m. Thus by 2.3.4,

$$\operatorname{thick}^{mn}(G) \supseteq \operatorname{thick}^n(H) = \mathcal{T}$$

and G strongly generates $\mathcal{T}$ in at most mn steps, that is $⊘_{\mathcal{T}}(G) < mn$. □

The behavior of the dimension under an exact functor is not as easy as for level as recorded in 2.3.3. An exact functor $\mathsf{f}\colon \mathcal{T} \to \mathcal{S}$ has a *dense image*, if every object in $\mathcal{S}$ is the direct summand of an image of f.

Lemma 2.4.3. 1. *Let* $\mathsf{f}\colon \mathcal{T} \to \mathcal{S}$ *be an exact functor between triangulated categories with dense image. Then* $\dim(\mathcal{S}) \leq \dim(\mathcal{T})$.

2. $\dim(\mathcal{T}) = \dim(\mathcal{T}^{op})$.

Proof. For (1), one may assume $\mathcal{T}$ has a strong generator G. Given any $X \in \mathcal{S}$, there exists $Y \in \mathcal{T}$ such that $\mathsf{f}(Y) = X \oplus X'$. So by 2.3.3 (3) and 2.3.2 (3), one has

$$\operatorname{level}^G_{\mathcal{T}}(Y) \geq \operatorname{level}^{\mathsf{f}(G)}_{\mathcal{S}}(X \oplus X') \geq \operatorname{level}^{\mathsf{f}(G)}_{\mathcal{S}}(X)\,.$$

(2) holds since $\mathcal{T}$ and $\mathcal{T}^{op}$ are equivalent. □

It is worth noting that $\mathcal{S}$ could have a 'better' strong generator that does not come from $\mathcal{T}$. That is its pre-image is not a generator in $\mathcal{T}$ and its generation time is less than the generation time of any strong generator coming from $\mathcal{T}$.

2.5 Compact and Cocompact Objects

Let $\mathcal{X}$ be a set of objects in a category $\mathcal{C}$. Suppose the coproduct of the objects in $\mathcal{X}$ exists in $\mathcal{C}$. Then the inclusion maps

$$X \to \coprod_{X \in \mathcal{X}} X$$

induce an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\coprod_{X\in\mathcal{X}} X, Y) \xrightarrow{\cong} \prod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{C}}(X, Y)\,. \tag{2.5.1}$$

Analogously suppose the product exists, then the projection maps

$$\prod_{X\in\mathcal{X}} X \to X$$

induce an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Y, \prod_{X\in\mathcal{X}} X) \xrightarrow{\cong} \prod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{C}}(Y, X)\,. \tag{2.5.2}$$

The inclusion maps, respectively the projection maps, also induce maps

$$\begin{aligned}&\coprod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, \coprod_{X\in\mathcal{X}} X)\,, \quad \text{respectively}\\ &\coprod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(\prod_{X\in\mathcal{X}} X, Y)\,.\end{aligned}$$

These maps need not be isomorphisms. This motivates the definition:

Definition 2.5.3. An object $C \in \mathcal{C}$ is *compact* if for any set of objects $\mathcal{X}$ of $\mathcal{C}$, for which the coproduct $\coprod_{X\in\mathcal{X}} X$ exists in $\mathcal{C}$, the canonical map

$$\coprod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{C}}(C, \coprod_{X\in\mathcal{X}} X)$$

is an isomorphism. Write $\mathcal{C}^c$ for the strictly full subcategory of compact objects of $\mathcal{C}$.

The object C is *cocompact* if for any set of objects $\mathcal{X}$ of $\mathcal{C}$, for which the product $\prod_{X\in\mathcal{X}} X$ exists in $\mathcal{C}$, the canonical map

$$\coprod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{C}}(X, C) \to \mathrm{Hom}_{\mathcal{C}}(\prod_{X\in\mathcal{X}} X, C)$$

is an isomorphism. Write $\mathcal{C}^{cc}$ for the strictly full subcategory of cocompact objects of $\mathcal{C}$.

The definitions for compact and cocompact are dual to each other in that an object is compact in $\mathcal{T}$ if and only if it is cocompact in the opposite category $\mathcal{T}^{op}$. In particular, one has $(\mathcal{T}^c)^{op} = (\mathcal{T}^{op})^{cc}$.

Lemma 2.5.4. *The categories $\mathcal{T}^c$ and $\mathcal{T}^{cc}$ are thick subcategories of the triangulated category $\mathcal{T}$.*

Proof. I only show $\mathcal{T}^c$ is thick. The proof for $\mathcal{T}^{cc}$ works analogously. Clearly $\mathcal{T}^c$ is closed under suspension. It remains to show it satisfies the two out of three property and it is closed under direct summands. Let $\mathcal{X}$ be a set of objects in $\mathcal{T}$, such that the coproduct $\coprod_{X\in\mathcal{X}} X$ exists in $\mathcal{T}$. Set

$$\mathsf{f} := \coprod_{X\in\mathcal{X}} \mathrm{Hom}_{\mathcal{T}}(-, X) \xrightarrow{\eta} \mathrm{Hom}_{\mathcal{T}}(-, \coprod_{X\in\mathcal{X}} X) =: \mathsf{g}\,.$$

I first show $\mathcal{T}^c$ is closed under direct summands. If $C = C' \oplus C''$, then $\eta(C)$ decomposes as

$$\mathsf{f}(C) = \mathsf{f}(C') \oplus \mathsf{f}(C'') \xrightarrow{\begin{pmatrix} \eta(C') & 0 \\ 0 & \eta(C'') \end{pmatrix}} \mathsf{g}(C') \oplus \mathsf{g}(C'') = \mathsf{g}(C)\,.$$

So C is compact if and only if C' and C'' are.

Let $C' \to C \to C'' \to \Sigma C'$ be an exact triangle with C' and C'' compact. Then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathsf{f}(C') & \longrightarrow & \mathsf{f}(C) & \longrightarrow & \mathsf{f}(C'') & \longrightarrow & \mathsf{f}(\Sigma C') & \longrightarrow & \mathsf{f}(\Sigma C') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{g}(C'') & \longrightarrow & \mathsf{g}(C) & \longrightarrow & \mathsf{g}(C'') & \longrightarrow & \mathsf{g}(\Sigma C') & \longrightarrow & \mathsf{g}(\Sigma C'')\,.
\end{array}$$

Since the first two and last two vertical maps are isomorphisms, by the 5-lemma, the middle one is also an isomorphism and thus C is compact. □

The (co)compact objects can be considered the finite objects in the following sense: An object is compact if any map to a coproduct factors through a *finite* coproduct of a subset of the initial set of objects. Similarly, for cocompact objects, the map from any product factors through a *finite* product of a subset of the initial set of objects.

Let $\mathrm{Add}(G)$ be the smallest strictly full subcategory of $\mathcal{T}$ that contains G and is closed under all coproducts that exist in $\mathcal{T}$ and suspension. Similarly, let $\mathrm{Prod}(G)$ be the smallest strictly full subcategory of $\mathcal{T}$, which contains G and is closed under all products that exist in $\mathcal{T}$ and suspension.

The next two lemmata express another way that the (co)compact objects are the finite objects in $\mathcal{T}$. The first is due to [BvdB03, Proposition 2.2.4], and I state it without proof.

Lemma 2.5.5. *Let G be a compact object in $\mathcal{T}$. Then*

$$\mathrm{thick}^n(G) = \mathrm{thick}^n(\mathrm{Add}(G)) \cap \mathcal{T}^c\,,$$

and $\mathrm{level}^G(X) = \mathrm{level}^{\mathrm{Add}(G)}(X)$ *for* $X \in \mathcal{T}^c$. □

Lemma 2.5.6. *Let G be a cocompact object of $\mathcal{T}$. Then*

$$\operatorname{thick}^n(G) = \operatorname{thick}^n(\operatorname{Prod}(G)) \cap \mathcal{T}^{cc},$$

and $\operatorname{level}^G(X) = \operatorname{level}^{\operatorname{Prod}(G)}(X)$ *for* $X \in \mathcal{T}^{cc}$.

Proof. Since $G \in \operatorname{Prod}(G)$, one has $\operatorname{thick}^n(G) \subseteq \operatorname{thick}^n(\operatorname{Prod}(G))$, and by 2.5.4, every object generated by G is cocompact.

For the reverse inclusion, take $X \in \operatorname{thick}^n(\operatorname{Prod}(G)) \cap \mathcal{T}^{cc}$. Set $X^n = X$. Then by 2.2.7 (2), the object X^n is a direct summand of some $Y^n \in \operatorname{Prod}(G)^{\star n}$. So there exist $Y^i \in \operatorname{Prod}(G)^{\star i}$ and exact triangles

$$Z^i \to Y^i \to Y^{i-1} \to \Sigma Z^i \quad \text{with } Z^i \in \operatorname{Prod}(G) \quad \text{for all } 1 \le i \le n\,.$$

Note that $Y^0 = 0$. Now construct corresponding cocompact objects $X^i \in \operatorname{thick}^i(\operatorname{Prod}(G))$: Since X^n is cocompact, the composition of $Z^n \to Y^n$ with the natural projection map $Y^n \to X^n$ factors through a finite product of objects in G. That is there exists $W^n \in \operatorname{add}(G)$ such that the following diagram commutes

$$\begin{array}{ccccccc} Z^n & \longrightarrow & Y^n & \longrightarrow & Y^{n-1} & \longrightarrow & \Sigma Z^n\,. \\ \downarrow & & \downarrow & & & & \\ W^n & \longrightarrow & X^n & & & & \end{array}$$

Completing the bottom row to an exact triangle gives

$$W^n \to X^n \to X^{n-1} \to \Sigma W^n\,,$$

and a map $Y^{n-1} \to X^{n-1}$. Since W^n and X^n are cocompact, so is X^{n-1}. Repeating this construction gives the commutative diagram

$$\begin{array}{ccccccccc} Y^n & \longrightarrow & Y^{n-1} & \longrightarrow & \cdots & \longrightarrow & Y^1 & \longrightarrow & Y^0 = 0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X = X^n & \longrightarrow & X^{n-1} & \longrightarrow & \cdots & \longrightarrow & X^1 & \longrightarrow & X^0\,. \end{array}$$

Since X^n is a direct summand of Y^n and the top row is zero, so is the bottom row. By construction $\operatorname{cone}(X^i \to X^{i-1}) \in \operatorname{add}(G)$. So by 2.3.5, one has

$$\Sigma X \oplus X^0 = \operatorname{cone}(0) = \operatorname{cone}(X \to X^0) \in \operatorname{thick}^n(G)\,.$$

In particular, $X \in \operatorname{thick}^n(G)$. □

Chapter 3

Ghost and Coghost Index

To compute the level of an object, one has to find a sequence of exact triangles that construct the object up to a direct summand using 2.3.2. However, it is hard to be sure that this is the construction with the least number of exact triangles. Therefore, such a sequence only gives an upper bound. In particular, it is rather tricky to compute the level of an object, and even more so the dimension of a triangulated category.

To find lower bounds for level, one uses its connection to two other invariants, the ghost and coghost index. The ghost index in the opposite category $\mathcal{T}^{op}$ is the coghost index in $\mathcal{T}$. In this chapter, I will introduce these invariants and explain their relationship to level.

3.1 (Co)ghost Lemma

Let $\mathcal{A}b$ be the category of abelian groups.

Definition 3.1.1. A covariant functor $\mathsf{h}\colon \mathcal{T} \to \mathcal{A}b$ is *homological*, if for every exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

the associated sequence in $\mathcal{A}b$

$$\cdots \to \mathsf{h}(X) \xrightarrow{\mathsf{h}(f)} \mathsf{h}(Y) \xrightarrow{\mathsf{h}(g)} \mathsf{h}(Z) \xrightarrow{\mathsf{h}(h)} \mathsf{h}(\Sigma X) \to \cdots$$

is exact.

A contravariant functor $\mathsf{h}\colon \mathcal{T}^{op} \to \mathcal{A}b$ is *cohomological*, if for every exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

the associated sequence in $\mathcal{A}b$

$$\cdots \to \mathsf{h}(\Sigma X) \xrightarrow{\mathsf{h}(h)} \mathsf{h}(Z) \xrightarrow{\mathsf{h}(g)} \mathsf{h}(Y) \xrightarrow{\mathsf{h}(f)} \mathsf{h}(X) \to \cdots$$

is exact.

Example 3.1.2. For every $X \in \mathcal{T}$, the covariant functor $\mathrm{Hom}_{\mathcal{T}}(X, -)$ is homological, and the contravariant functor $\mathrm{Hom}_{\mathcal{T}}(-, X)$ is cohomological.

Any map $f\colon X \to Y$ induces natural transformations

$$f^*\colon \mathrm{Hom}_{\mathcal{T}}(Y, -) \to \mathrm{Hom}_{\mathcal{T}}(X, -) \quad \text{and} \quad f_*\colon \mathrm{Hom}_{\mathcal{T}}(-, X) \to \mathrm{Hom}_{\mathcal{T}}(-, Y)\,.$$

Kelly [Kel65] was the first to use the vanishing of maps to give a lower bound for level. The following is a generalized version by [Rou08, Lemma 4.11].

Lemma 3.1.3 (Generalized (co-)ghost lemma)**.** *Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$ closed under suspension, and $\mathsf{h}_0, \ldots, \mathsf{h}_n$ (co)homological functors on $\mathcal{T}$ with natural transformations $\eta_i\colon \mathsf{h}_{i-1} \to \mathsf{h}_i$ for $1 \leq i \leq n$, which vanish on $\mathcal{C}$. Then $\eta := \eta_1 \circ \ldots \circ \eta_n$ vanishes on* $\mathrm{thick}^n(\mathcal{C})$.

Proof. I prove the claim for cohomological functors. The statement for homological functors holds analogously. For $n = 1$, realize that the functors h_i and natural transformations η_i respect direct sums. So since η_1 vanishes on $\mathcal{C}$, it vanishes on $\mathrm{thick}^1(\mathcal{C})$.

For $n > 1$, take $X \in \mathrm{thick}^n(\mathcal{C})$ and $\eta' := \eta_2 \circ \ldots \circ \eta_n$. Then there exists an exact triangle

$$X^1 \to X \oplus X' \to X^{n-1} \to \Sigma X^1 \quad \text{with } X^1 \in \mathrm{thick}^1(\mathcal{C}) \text{ and } X^{n-1} \in \mathrm{thick}^{n-1}(\mathcal{C})\,.$$

By induction, η' vanishes on X^{n-1}. Now consider the following commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathsf{h}_0(X^{n-1}) & \longrightarrow & \mathsf{h}_0(X \oplus X') & \longrightarrow & \mathsf{h}_0(X^1) \\
{\scriptstyle \eta_1(X^{n-1})}\downarrow & \swarrow & \downarrow{\scriptstyle \eta_1(X \oplus X')} & & \downarrow{\scriptstyle \eta_1(X^1)=0} \\
\mathsf{h}_1(X^{n-1}) & \longrightarrow & \mathsf{h}_1(X \oplus X') & \longrightarrow & \mathsf{h}_1(X^1) \\
{\scriptstyle \eta'(X^{n-1})=0}\downarrow & & \downarrow{\scriptstyle \eta'(X \oplus X')} & & \downarrow{\scriptstyle \eta'(X^1)=0} \\
\mathsf{h}_n(X^{n-1}) & \longrightarrow & \mathsf{h}_n(X \oplus X') & \longrightarrow & \mathsf{h}_n(X^1)\,.
\end{array}$$

Then $\eta_1(X \oplus X')$ factors through $\mathsf{h}_1(X^{n-1})$, and so $\eta(X \oplus X') = 0$ and $\eta(X) = 0$. □

The lemma can be used to find a lower bound for level: Suppose X is an object in $\mathcal{T}$. If $(\eta_1 \circ \ldots \circ \eta_n)(X) \neq 0$, then X does not lie in $\mathrm{thick}^n(\mathcal{C})$; that is to say $\mathrm{level}^{\mathcal{C}}(X) \geq n + 1$.

The rest of this section considers the special case when the (co)homological functors are as in Example 3.1.2 and the natural transformations are induced by morphisms. For convenience, write

$$\mathrm{Ext}_{\mathcal{T}}(X, Y) := \coprod_{d \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^d Y)\,. \tag{3.1.4}$$

In the case that $\mathcal{T} = \mathrm{D}(R)$ is the derived category of a ring R and X, Y are modules over R, this definition coincides with the usual definition of a graded Ext-module. For details, see 4.3.2.

Definition 3.1.5. Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$. A morphism $f\colon X \to Y$ is *$\mathcal{C}$-ghost*, if the natural transformation

$$f_*\colon \operatorname{Ext}_{\mathcal{T}}(-, X) \to \operatorname{Ext}_{\mathcal{T}}(-, Y)$$

vanishes on $\mathcal{C}$.

A map $f\colon X \to Y$ is *n-fold $\mathcal{C}$-ghost*, if it is a composition of n $\mathcal{C}$-ghost maps. The *ghost index with respect to $\mathcal{C}$* is

$$\operatorname{gin}^{\mathcal{C}}(X) := \inf\{n \geq 1 \mid \text{all } n\text{-fold } \mathcal{C}\text{-ghost maps } X \to Y \text{ are zero}\}$$

when the object X is non-zero, and $\operatorname{gin}^{\mathcal{C}}(0) := 0$.

A morphism $f\colon X \to Y$ is *$\mathcal{C}$-coghost*, if the natural transformation

$$f^*\colon \operatorname{Ext}_{\mathcal{T}}(Y, -) \to \operatorname{Ext}_{\mathcal{T}}(X, -)$$

vanishes on $\mathcal{C}$.

A map $f\colon X \to Y$ is *n-fold $\mathcal{C}$-coghost*, if it is a composition of n $\mathcal{C}$-coghost maps. The *coghost index with respect to $\mathcal{C}$* is

$$\operatorname{cogin}^{\mathcal{C}}(Y) := \inf\{n \geq 1 \mid \text{all } n\text{-fold } \mathcal{C}\text{-coghost maps } X \to Y \text{ are zero}\}$$

when the object Y is non-zero, and $\operatorname{cogin}^{\mathcal{C}}(0) := 0$.

As before, when the ambient category $\mathcal{T}$ is not evident, write $\operatorname{gin}^{\mathcal{C}}_{\mathcal{T}}(-)$ and $\operatorname{cogin}^{\mathcal{C}}_{\mathcal{T}}(-)$.

The following corollary of 3.1.3 is called the (co)ghost lemma.

Corollary 3.1.6. *Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$ and $X \in \mathcal{T}$. Then*

$$\operatorname{gin}^{\mathcal{C}}(X) \leq \operatorname{level}^{\mathcal{C}}(X) \quad \textit{and} \quad \operatorname{cogin}^{\mathcal{C}}(X) \leq \operatorname{level}^{\mathcal{C}}(X)\,.$$

Proof. I will give a proof for the first inequality, and the second can be proved along the same lines. If X is the zero object, both invariants are zero. When X is non-zero, set $n := \operatorname{gin}^{\mathcal{C}}(X) - 1$. If $n = 0$, there is nothing to prove. For $n > 0$, there exists a sequence of $\mathcal{C}$-ghost maps

$$X = X^0 \xrightarrow{f^1} X^1 \xrightarrow{f^2} \cdots \xrightarrow{f^n} X^n$$

such that the composition $f := f^n \circ \ldots \circ f^1$ is non-zero. To apply Lemma 3.1.3, consider the cohomological functors $\mathrm{Hom}_{\mathcal{T}}(-, X^i)$ and natural transformations $(f^i)^*$. Since the maps f^i are $\mathcal{C}$-ghost, the natural transformations $(f^i)^*$ vanish on $\mathrm{thick}^1(\mathcal{C})$. Then their composition f^* vanishes on $\mathrm{thick}^n(\mathcal{C})$. But $f^*(X)(\mathrm{id}_X) = f \neq 0$. So

$$X \notin \mathrm{thick}^n(\mathcal{C}) \quad \text{and} \quad \mathrm{level}^{\mathcal{C}}(X) \geq n+1 = \mathrm{gin}^{\mathcal{C}}(X)\,. \qquad \square$$

3.2 Properties

To further compare ghost and coghost indices to level, I discuss properties of the former analogous to those of the latter from Section 2.3.

Lemma 3.2.1. *Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$.*

1. *Ghost and coghost index are invariant under suspension, that is for $X \in \mathcal{T}$, one has*

$$\mathrm{gin}^{\mathcal{C}}(\Sigma^d X) = \mathrm{gin}^{\mathcal{C}}(X) \quad \textit{and} \quad \mathrm{cogin}^{\mathcal{C}}(\Sigma^d X) = \mathrm{cogin}^{\mathcal{C}}(X) \quad \textit{for any } d \in \mathbb{Z}\,.$$

2. *Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. Then*

$$\mathrm{gin}^{\mathcal{C}}(Y) \leq \mathrm{gin}^{\mathcal{C}}(X) + \mathrm{gin}^{\mathcal{C}}(Z) \quad \textit{and} \quad \mathrm{cogin}^{\mathcal{C}}(Y) \leq \mathrm{cogin}^{\mathcal{C}}(X) + \mathrm{cogin}^{\mathcal{C}}(Z)\,.$$

3. *Let X and Y be objects in $\mathcal{T}$, then*

$$\begin{aligned} \mathrm{gin}^{\mathcal{C}}(X \oplus Y) &= \max\{\mathrm{gin}^{\mathcal{C}}(X), \mathrm{gin}^{\mathcal{C}}(X)\} \qquad \textit{and} \\ \mathrm{cogin}^{\mathcal{C}}(X \oplus Y) &= \max\{\mathrm{cogin}^{\mathcal{C}}(X), \mathrm{cogin}^{\mathcal{C}}(X)\}\,. \end{aligned}$$

4. *One has the equality $\mathrm{gin}^{\mathcal{C}}_{\mathcal{T}}(X) = \mathrm{cogin}^{\mathcal{C}^{op}}_{\mathcal{T}^{op}}(X)$ for all $X \in \mathcal{T}$.*

5. *For any object X in $\mathcal{T}$, one has*

$$\mathrm{gin}^{\mathcal{C}}(X) = \mathrm{gin}^{\mathrm{Add}(\mathcal{C})}(X) \quad \textit{and} \quad \mathrm{cogin}^{\mathcal{C}}(X) = \mathrm{cogin}^{\mathrm{Prod}(\mathcal{C})}(X)\,.$$

Proof. The first statement follows from the fact that Σ is an auto-equivalence. In particular, it preserves $\mathcal{C}$-ghost and non-zero maps.

For (2), set $m := \mathrm{gin}^{\mathcal{C}}(X)$ and $n := \mathrm{gin}^{\mathcal{C}}(Z)$. It is enough to show any $(n+m)$-fold $\mathcal{C}$-ghost map starting at Y is zero. Given an $(n+m)$-fold $\mathcal{C}$-ghost map $Y \to Y'$ and decompose it such that $Y \to Y''$ is an m-fold $\mathcal{C}$-ghost map and $Y'' \to Y'$ is an n-fold $\mathcal{C}$-ghost map. Then

$X \to Y \to Y''$ is an m-fold $\mathcal{C}$-ghost map and thus zero. So there exists a map $Z \to Y''$ such that the diagram commutes

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ 0 & \longrightarrow & Y'' & = & Y'' & \longrightarrow & 0\,. \end{array}$$

Then $Z \to Y'' \to Y'$ is an n-fold $\mathcal{C}$-ghost map and thus zero. In particular, $Y \to Y'$ is zero and

$$\operatorname{gin}^{\mathcal{C}}(Y) \leq m + n = \operatorname{gin}^{\mathcal{C}}(X) + \operatorname{gin}^{\mathcal{C}}(Z)\,.$$

An analog argument shows the inequality for cogin.

For (3), observe that $X \oplus Y \to Z$ is non-zero if and only if the pre-composition with one of the inclusion maps $X \to Z$ or $Y \to Z$ is non-zero. If $X \oplus Y \to Z$ is $\mathcal{C}$-ghost, then

$$X \to X \oplus Y \to Z \quad \text{and} \quad Y \to X \oplus Y \to Z$$

are $\mathcal{C}$-ghost. On the other hand, if $X \to Z$ is a $\mathcal{C}$-ghost map, then

$$X \oplus Y \to X \to Z$$

is $\mathcal{C}$-ghost. This shows the equality for the ghost index. For the coghost index, the proof works the same.

For (4), realize that pre-composition in $\mathcal{T}$ becomes post-composition in $\mathcal{T}^{op}$ and in reverse. Thus ghost become coghost maps. Also the domain and codomain switch places, so a composition starting at X in $\mathcal{T}$ is a composition ending at X in $\mathcal{T}^{op}$.

The last claim follows from the isomorphisms (2.5.1) and (2.5.2) induced by the coproduct and product. □

Statements (1)–(4) are analogous to the properties of level. The last property (5) does not hold in full generality for level, since thick is not closed under coproducts or products. In the special case that X is compact, respectively cocompact, property (5) holds for level; see 2.5.5, respectively 2.5.6.

For gin and cogin, there is no analog of Lemma 2.3.3 (3). An exact functor need not map a (co)ghost map to a (co)ghost map. Additionally, a functor need not be faithful.

I can make the following statements when enlarging the generators $\mathcal{C}$ or the ambient category $\mathcal{T}$:

3.2.2. Let $\mathcal{C} \subseteq \mathcal{D}$ be strictly full subcategories of $\mathcal{T}$ and X an object in $\mathcal{T}$. Then

$$\operatorname{gin}^{\mathcal{C}}(X) \leq \operatorname{gin}^{\mathcal{D}}(X) \quad \text{and} \quad \operatorname{cogin}^{\mathcal{C}}(X) \leq \operatorname{cogin}^{\mathcal{D}}(X)\,.$$

Let $\mathcal{T}$ be a triangulated subcategory of $\mathcal{U}$. Then for any object $X \in \mathcal{T}$, one has

$$\operatorname{gin}^{\mathcal{C}}_{\mathcal{T}}(X) \leq \operatorname{gin}^{\mathcal{C}}_{\mathcal{U}}(X) \quad \text{and} \quad \operatorname{cogin}^{\mathcal{C}}_{\mathcal{T}}(X) \leq \operatorname{cogin}^{\mathcal{C}}_{\mathcal{U}}(X)\,.$$

These last inequalities show another difference to level: The level with respect to $\mathcal{C}$ only depends on $\operatorname{thick}(\mathcal{C})$ and not on its ambient category. On the other hand, the (co)ghost index might change when enlarging the ambient category. The reason for this is that there are more morphisms in a larger category.

Lemma 3.2.3. *Fix $G \in \mathcal{T}$ and an n-fold G-ghost (resp. G-coghost) map $f\colon X \to Y$. If H is an object of $\mathcal{T}$ with* $\operatorname{level}^G(H) \leq n$*, then f is H-ghost (resp. G-coghost).*

Proof. It is enough to show the claim for $\operatorname{level}^G(H) = n$. Use induction on n. For $n = 1$, there is nothing to prove. For $n > 1$, there exists an exact triangle

$$H^1 \to H \oplus H' \to H^{n-1} \to \Sigma H^1 \quad \text{with } H^1 \in \operatorname{thick}^1(\mathcal{C}) \text{ and } H^{n-1} \in \operatorname{thick}^{n-1}(\mathcal{C})\,.$$

Write f as a composition $X \xrightarrow{g} Z \xrightarrow{h} Y$ where g is a G-ghost and h an $(n-1)$-fold G-ghost map. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc}
\operatorname{Hom}_{\mathcal{T}}(H^{n-1}, X) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(H \oplus H', X) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(H^1, X) \\
\downarrow{\scriptstyle g_*(H^{n-1})} & \swarrow & \downarrow{\scriptstyle g_*(H \oplus H')} & & \downarrow{\scriptstyle g_*(H^1)=0} \\
\operatorname{Hom}_{\mathcal{T}}(H^{n-1}, Z) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(H \oplus H', Z) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(H^1, Z) \\
\downarrow{\scriptstyle h_*(H^{n-1})=0} & & \downarrow{\scriptstyle h_*(H \oplus H')} & & \downarrow{\scriptstyle h_*(H^1)=0} \\
\operatorname{Hom}_{\mathcal{T}}(H^{n-1}, Y) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(H \oplus H', Y) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(H^1, Y)\,.
\end{array}$$

It follows $f_* = 0$ on $H \oplus H'$. So $f_* = 0$ on H and f is H-ghost. Similarly, the claim for G-coghost holds. □

This statement is similar to Lemma 2.3.4 for level. The version for the (co)ghost index does not give such a nice inequality as the one for level. Lemma 3.2.3 gives the inequality

$$\left\lfloor \frac{\operatorname{gin}^G(X)}{\operatorname{level}^G(H)} \right\rfloor \leq \operatorname{gin}^H(X)$$

where the left-hand side is the biggest integer below the fraction.

3.3 Converse (Co)ghost Lemma

In some situations, level coincides with the (co)ghost index.

Definition 3.3.1. Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$ closed under suspension. A morphism $n(X)\colon C \to X$ is a *right $\mathcal{C}$-approximation of* X if $C \in \mathcal{C}$ and for any morphism $f\colon C' \to X$ with $C' \in \mathcal{C}$ there exists $g\colon C' \to C$ with $f = n(X) \circ g$. If every $X \in \mathcal{T}$ admits a right $\mathcal{C}$-approximation, then $\mathcal{C}$ is a *contravariantly finite* subcategory.

A morphism $m(X)\colon X \to C$ is a *left $\mathcal{C}$-approximation of* X if $C \in \mathcal{C}$ and for any morphism $f\colon X \to C'$ with $C' \in \mathcal{C}$ there exists $g\colon C \to C'$ with $f = g \circ m(X)$. If every $X \in \mathcal{T}$ admits a left $\mathcal{C}$-approximation, then $\mathcal{C}$ is a *covariantly finite* subcategory.

To get a better intuition for these definitions, it is useful to regard the corresponding diagrams

$$\begin{array}{ccc} C' & \xrightarrow{f} & X \\ {\scriptstyle \exists g}\downarrow & \nearrow_{n(X)} & \\ C & & \end{array} \quad \text{and respectively} \quad \begin{array}{ccc} X & \xrightarrow{f} & C' \\ & {}_{m(X)}\searrow & \uparrow{\scriptstyle \exists g} \\ & & C\,. \end{array}$$

Note that the left/right approximation need not be unique.

3.3.2. A left/right $\mathcal{C}$-approximation is also a left/right approximation with respect to any direct summands of direct sums of $\mathcal{C}$. So a left/right $\mathcal{C}$-approximation in a triangulated category is a left/right $\operatorname{thick}^1(\mathcal{C})$-approximation.

The terms left and right to differentiate between the two dual notions of approximations are connected to the left/right adjoint of the inclusion functor:

3.3.3. Suppose C is an object in $\mathcal{C}$ and X an object in $\mathcal{T}$. Let $n(X)\colon C \to X$ be a morphism in $\mathcal{T}$. Then $n(X)$ is a right $\mathcal{C}$-approximation of X if and only if the natural transformation

$$n(X)_*\colon \operatorname{Hom}_{\mathcal{C}}(-, C) \to \operatorname{Hom}_{\mathcal{T}}(-, X)$$

is surjective on $\mathcal{C}$.

If the inclusion functor $\mathcal{C} \to \mathcal{T}$ has a right adjoint $\mathsf{r}\colon \mathcal{T} \to \mathcal{C}$, then for any $X \in \mathcal{T}$ there exists a natural map

$$\varepsilon(X) \in \operatorname{Hom}_{\mathcal{T}}(\mathsf{r}(X), X) \cong \operatorname{Hom}_{\mathcal{C}}(\mathsf{r}(X), \mathsf{r}(X))\,.$$

The induced natural transformation $\varepsilon(X)_*$ is an isomorphism. So $\mathcal{C}$ is contravariantly finite. The converse is not true in general, for the induced natural transformation $n(X)_*$ need not be injective.

Analogously, a morphism $m(X)\colon X \to C$ is a left $\mathcal{C}$-approximation of X if and only if the natural transformation

$$m(X)^*\colon \operatorname{Hom}_{\mathcal{C}}(C,-) \to \operatorname{Hom}_{\mathcal{T}}(X,-)$$

is surjective on $\mathcal{C}$. If the inclusion functor $\mathcal{C} \to \mathcal{T}$ has a left adjoint, then $\mathcal{C}$ is covariantly finite. The converse is not true in general, for the induced natural transformation $n(X)^*$ need not be injective.

3.3.4. The left/right approximations help with the construction of (co)ghost maps. Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$ closed under suspension, and $n(X)\colon C \to X$ a right $\mathcal{C}$-approximation of some $X \in \mathcal{T}$. Complete $n(X)$ to an exact triangle

$$C \xrightarrow{n(X)} X \xrightarrow{f} Y \to \Sigma C\,.$$

Given a map $C' \to X$ with $C' \in \operatorname{thick}^1(\mathcal{C})$, there exists an commuting diagram

$$\begin{array}{ccccc} C' & \longrightarrow & X & & \\ \downarrow & & \| & & \\ C & \xrightarrow{n(X)} & X & \xrightarrow{f} & Y \end{array}$$

and $C' \to X \to Y$ is zero, thus f is a $\mathcal{C}$-ghost map.

Conversely, a $\mathcal{C}$-ghost map need not induce a right $\mathcal{C}$-approximation. For a special case in which it does hold, see the proof of Lemma 3.4.7.

There is a similar connection between left approximations and coghost maps.

3.3.5. The $\mathcal{C}$-ghost map $f\colon X \to Y$ constructed from a right $\mathcal{C}$-approximation $n(X)\colon C \to X$ is universal: Let $g\colon X \to Z$ be a $\mathcal{C}$-ghost map. Then the following diagram commutes

$$\begin{array}{ccccccc} C & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \Sigma C \\ \downarrow & & \downarrow{\scriptstyle g} & & & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{\mathrm{id}} & Z & \longrightarrow & 0\,. \end{array}$$

So there exists a map $Y \to Z$ such that g factors through f.

The same holds for the $\mathcal{C}$-coghost map constructed from the left $\mathcal{C}$-approximation.

Lemma 3.3.6. *Let X and Y be objects in $\mathcal{T}$.*

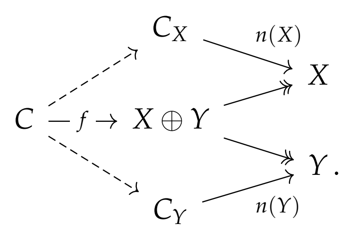
1. *If X has a right (resp. left) $\mathcal{C}$-approximation $n(X)$ (resp. $m(X)$), then the suspension ΣX also has a right (resp. left) $\mathcal{C}$-approximation given by $n(\Sigma X) = \Sigma n(X)$ (resp. $m(\Sigma X) = \Sigma m(X)$).*

2. *The direct sum $X \oplus Y$ has a right (resp. left) $\mathcal{C}$-approximation if and only if X and Y do. The right (resp. left) $\mathcal{C}$-approximation is given by*

$$n(X) \oplus n(Y) = n(X \oplus Y) \quad \textit{(resp. } m(X) \oplus m(Y) = m(X \oplus Y)\textit{)}.$$

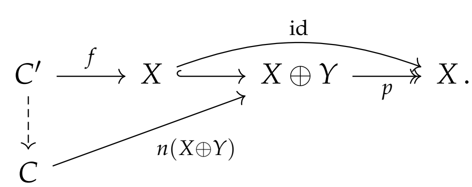
Proof. For (1), let $f\colon C' \to \Sigma X$ be a map with $C' \in \mathcal{C}$. Then $\Sigma^{-1} f\colon \Sigma^{-1} C' \to X$ factors through $n(X)$. In particular, f factors through $\Sigma n(X)$ and $n(\Sigma X) = \Sigma n(X)$ is a right $\mathcal{C}$-approximation of ΣX.

For (2), first assume X and Y have right $\mathcal{C}$-approximations. Given a map $f\colon C' \to X \oplus Y$ with $C' \in \mathcal{C}$. Then f composed with the projection onto X factors through $n(X)$, and similarly for Y. This gives the following commutative diagram



Then f factors through $n(X) \oplus n(Y)$.

For the reverse direction, assume $X \oplus Y$ has a right $\mathcal{C}$-approximation. Let $f\colon C' \to X$ be a map with $C' \in \mathcal{C}$. Using the natural inclusion and projection maps for X as a direct summand gives the commutative diagram



In particular, f factors through $p \circ n(X \oplus Y)$.

The claims hold $\mathcal{C}$-approximations similarly. □

For right $\mathcal{C}$-approximations, (2) does not hold for infinite coproducts, but it does hold for infinite products. Dually, for left $\mathcal{C}$-approximations, (2) holds for infinite coproducts but not for infinite products.

Lemma 3.3.7. *Let $\mathsf{f}\colon \mathcal{S} \to \mathcal{T}$ be a full exact functor between triangulated categories, and $\mathcal{C}$ a strictly full subcategory of $\mathcal{C}$ closed under suspension. If $n(X)$ (resp. $m(X)$) is a right (resp. left) $\mathcal{C}$-approximation of X in $\mathcal{S}$, then $\mathsf{f}(n(X))$ (resp. $\mathsf{f}(m(X))$) is a right $\mathsf{f}(\mathcal{C})$-approximation of $\mathsf{f}(X)$ in $\mathcal{T}$.*

Proof. Let $g \colon D \to \mathsf{f}(X)$ be a map in $\mathcal{T}$ with $D \in \mathsf{f}(\mathcal{C})$. Then $D = \mathsf{f}(C')$ for some $C' \in \mathcal{C}$ and there exists a map $f \colon C' \to X$ in $\mathcal{S}$ that is mapped to g by the functor f. Since $n(X)$ is a right $\mathcal{C}$-approximation, f factors through $n(X)$. So g factors through $\mathsf{f}(n(X))$. The claim for the left $\mathcal{C}$-approximation holds analogously. □

Lemma 3.3.8 (Converse (co-)ghost lemma for (contra-)covariantly finite)**.** *Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$ closed under suspension.*

1. *If $\mathcal{C}$ is contravariantly finite and $X \in \mathcal{T}$, then*
$$\operatorname{gin}^{\mathcal{C}}(X) = \operatorname{level}^{\mathcal{C}}(X)\,.$$
2. *If $\mathcal{C}$ is covariantly finite and $X \in \mathcal{T}$, then*
$$\operatorname{cogin}^{\mathcal{C}}(X) = \operatorname{level}^{\mathcal{C}}(X)\,.$$

Proof. I will verify the equality in (1), and a similar argument yields (2). By Lemma 3.1.6, it is enough to show $\operatorname{gin}^{\mathcal{C}}(X) \geq \operatorname{level}^{\mathcal{C}}(X)$. Set $n := \operatorname{gin}^{\mathcal{C}}(X)$ and $X^0 := X$. Define X^i and $f^i \colon X^{i-1} \to X^i$ for $1 \leq i \leq n$ inductively by completing the right $\mathcal{C}$-approximation of X^{i-1} to an exact triangle
$$C^i \xrightarrow{n(X^{i-1})} X^{i-1} \xrightarrow{f^i} X^i \to \Sigma C^i\,.$$
Then $C^i = \Sigma^{-1}\operatorname{cone}(f^i) \in \operatorname{thick}^1(\mathcal{C})$ and the maps f^i are $\mathcal{C}$-ghost by 3.3.4. By Lemma 2.3.5, the cone of $f^n \circ \ldots \circ f^1$ lies in $\operatorname{thick}^n(\mathcal{C})$. Now $f^n \circ \ldots \circ f^1 = 0$, so one has
$$\begin{aligned} &\Sigma X^0 \oplus X^n = \operatorname{cone}(0) = \operatorname{cone}(f^n \circ \ldots \circ f^1) \in \operatorname{thick}^n(\mathcal{C})\,, \\ &X = X^0 \in \operatorname{thick}^n(\mathcal{C}) \quad \text{and} \quad \operatorname{level}^{\mathcal{C}}(X) \leq n = \operatorname{gin}^{\mathcal{C}}(X)\,. \end{aligned}$$
□

This lemma is called the converse (co)ghost lemma, because in [Kel65], the original (co)ghost lemma was not stated as an inequality as in 3.1.6, but as: If $n \geq \operatorname{level}^{\mathcal{C}}(X)$, then any n-fold (co)ghost map is zero.

3.3.9. The proof of the converse ghost lemma 3.3.8 (2) constructs a sequence of exact triangles

$$\begin{array}{ccccccccc} X = X^0 & \xrightarrow{f^1} & & X^1 & \xrightarrow{f^2} & & X^2 & \longrightarrow & \ldots \\ & \nwarrow_{n(X^0)} & \swarrow_{+1} & & \nwarrow_{n(X^1)} & \swarrow_{+1} & & \nwarrow_{n(X^2)} \ \swarrow_{+1} & \\ & & C^1 & & & C^2 & & & C^3 \end{array}$$

where the horizontal maps on top are $\mathcal{C}$-ghost. The composition $f^n \circ \ldots \circ f^1$ are a universal n-fold $\mathcal{C}$-ghost map in the sense of 3.3.5.

3.4 Adams Resolution

This section gives another approach to a converse (co)ghost lemma. This is due to [Chr98] and I will show it is equivalent to 3.3.8.

Definition 3.4.1. Given a class of object $\mathcal{P}$ and a class of morphisms $\mathcal{M}$, let X be an object in $\mathcal{T}$. An *Adams resolution of X with respect to* $(\mathcal{P}, \mathcal{M})$ is a diagram of the form

$$\begin{array}{ccccccccc} X = X^0 & & \longrightarrow & & X^1 & & \longrightarrow & & X^2 & & \longrightarrow & \dots \\ & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} \\ & & P^1 & & & & P^2 & & & & P^3 & \end{array} \tag{3.4.2}$$

where

1. the P^i's are objects in $\mathcal{P}$ and the morphisms $X^i \to X^{i+1}$ lie in $\mathcal{M}$, and
2. the triangles are exact triangles in $\mathcal{T}$.

These resolutions become interesting if the maps in $\mathcal{M}$ are $\mathcal{P}$-ghost.

Lemma 3.4.3. *Let $\mathcal{P}$ be a family of objects $\mathcal{P}$ and $\mathcal{M}$ a family of $\mathcal{P}$-ghost maps. If an object $X \in \mathcal{T}$ has an Adams resolution of the form* (3.4.2), *then*

$$\operatorname{level}^{\mathcal{P}}(X) = \operatorname{gin}^{\mathcal{P}}(X) = \inf\{n \geq 0 \mid X \to X^n \textit{ is the zero map}\} .$$

Proof. By the definition of the Adams resolution

$$\operatorname{cone}(X^{i-1} \to X^i) = P^i \in \operatorname{thick}^1(\mathcal{P}) .$$

Then by Lemma 2.3.5, the cone of $X \to X^n$ lies in $\operatorname{thick}^n(\mathcal{P})$. If $X \to X^n$ is the zero map, then

$$\operatorname{cone}(X \to X^n) = \Sigma X \oplus X^n \in \operatorname{thick}^n(\mathcal{P}) \implies X \in \operatorname{thick}^n(\mathcal{P}) .$$

Thus

$$\operatorname{level}^{\mathcal{P}}(X) \leq \inf\{n \geq 0 \mid X \to X^n \text{ is the zero map}\} \leq \operatorname{gin}^{\mathcal{P}}(X) \leq \operatorname{level}^{\mathcal{P}}(X) ,$$

and they are all equal. □

The following definition is due to [EM65, 2].

Definition 3.4.4. Let $\mathcal{P}$ be a class of objects in $\mathcal{T}$ closed under suspension and $\mathcal{M}$ a class of morphisms in $\mathcal{T}$. The pair $(\mathcal{P}, \mathcal{M})$ is called a *projective class*, if

1. $\mathcal{M}$ is the class of all $\mathcal{P}$-ghost maps in $\mathcal{T}$,

2. $\mathcal{P}$ is the class of all objects P in $\mathcal{T}$ such that all maps in $\mathcal{M}$ are P-ghost, and

3. for any objects X in $\mathcal{T}$, there exists an exact triangle

$$P \to X \to Y \to \Sigma P$$

   such that P is in $\mathcal{P}$ and $X \to Y$ in $\mathcal{M}$.

Lemma 3.4.5. *Let $\mathcal{P}$ be a class of objects in $\mathcal{T}$ closed under suspension and $\mathcal{M}$ a class of morphisms in $\mathcal{T}$. If*

1. *$\mathcal{P}$ and $\mathcal{M}$ satisfy conditions* (1) *and* (3) *of Definition 3.4.4, and*

2. *$\mathcal{P}$ is closed under direct summands and isomorphisms,*

then $(\mathcal{P}, \mathcal{M})$ is a projective class.

Proof. It is enough to show (2) of definition of a projective class. Given $P \in \mathcal{T}$, assume every map in $\mathcal{M}$ is P-ghost. It is enough to show $P \in \mathcal{P}$. By (3) of 3.4.4, there exists an exact triangle

$$P' \to P \to Y \to \Sigma P'$$

with $P' \in \mathcal{P}$ and $P \to Y$ a morphism in $\mathcal{M}$. In particular, $P \to Y$ is P-ghost and so it is the zero map. Thus P is a direct summand of P' and by (2) the object P lies in $\mathcal{P}$. □

In particular, this guarantees the existence of an Adams resolution.

Lemma 3.4.6. *Let $(\mathcal{P}, \mathcal{M})$ be a projective class. Then for every object X in $\mathcal{T}$, there exists an Adams resolution of X.* □

Let $\mathcal{C}\text{-}\operatorname{gh}$ be the family of all $\mathcal{C}$-ghost maps.

Lemma 3.4.7. *Let $\mathcal{C}$ be a strictly full subcategory of $\mathcal{T}$ closed under suspension and direct summands. Then $\mathcal{C}$ is contravariantly finite in $\mathcal{T}$ if and only if $(\operatorname{thick}^1(\mathcal{C}), \mathcal{C}\text{-}\operatorname{gh})$ is a projective class.*

Proof. Assume $\mathcal{C}$ is contravariantly finite. By Lemma 3.4.5, it is enough to check that for every X, there exists an exact triangle

$$C \to X \to Y \to \Sigma C$$

with $C \in \mathcal{C}$ and $X \to Y$ a $\mathcal{C}$-ghost map. Let X be an object in $\mathcal{T}$. Since $\mathcal{C}$ is contravariantly finite, there exists a right $\mathcal{C}$-approximation $C \to X$. Complete this to an exact triangle

$$C \to X \xrightarrow{f} Y \to \Sigma C\,.$$

By 3.3.4, the map f is $\mathcal{C}$-ghost.

For the opposite direction, assume $(\mathrm{thick}^1(\mathcal{C}), \mathcal{C}\text{-}\mathrm{gh})$ is a projective class. Then for any $X \in \mathcal{T}$, there exists an exact triangle

$$C \xrightarrow{n} X \xrightarrow{f} Y \to \Sigma C$$

with $C \in \mathcal{C}$ and f a $\mathcal{C}$-ghost map. Given a map $C' \to X$ with $C' \in \mathcal{C}$. Then the composition with f is zero. Consider the following diagram

$$\begin{array}{ccccccc} C' & \xrightarrow{\mathrm{id}} & C' & \longrightarrow & 0 & \longrightarrow & \Sigma C' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{n} & X & \xrightarrow{f} & Y & \longrightarrow & \Sigma C\,. \end{array}$$

Since the rows are both exact triangles, there exists a map $C' \to C$ so that the diagram commutes. That is n is a right $\mathcal{C}$-approximation. $\square$

This approach also works in the opposite category: Instead of an Adams resolution, one gets an *Adams coresolution*

$$\begin{array}{ccccccccc} X = X^0 & & \longleftarrow & & X^1 & & \longleftarrow & X^2 & \longleftarrow \cdots \\ & \searrow & & \nearrow_{+1} & & \searrow & \nearrow_{+1} & & \searrow \quad \nearrow_{+1} \\ & & P^1 & & & P^2 & & & P^3 \end{array}$$

and instead of a projective class, one gets an injective class. The analogous of the preceding statements hold in this setting.

Chapter 4

Dimension and Level in the Derived Category

In this chapter, I review what is known about dimension and level in the derived category; see [ABIM10, KK06, Rou08]. First, I present the connection of level to some homological dimensions: projective dimension, flat dimension, and injective dimension. Then I state and prove the results for Rouquier dimension.

4.1 Generation in the Derived Category

Let R be a noetherian ring, not necessarily commutative. An R-module will mean a left R-module. The category of R-modules is denoted by $R\text{-}\mathrm{Mod}$, and its subcategory of finitely generated R-modules by $R\text{-}\mathrm{mod}$.

I will index complexes homologically, that is

$$X = \cdots \to X_{i+1} \xrightarrow{\partial_{i+1}} X_i \to \cdots .$$

In particular, bounded below means $X_i = 0$ for $i \ll 0$ and bounded above means $X_i = 0$ for $i \gg 0$.

The derived category of R is $\mathrm{D}(R)$. This is a triangulated category with suspension given by the shift

$$(\Sigma X)_i = X_{i-1}\,, \quad \partial_i^{\Sigma X} = (-1)\partial_{i-1}^X \quad \text{and} \quad (\Sigma f)_i = f_{i-1}$$

for a complex X and a map of complexes f.

4.1.1. By Lemma 2.3.3 (1), level in $\mathrm{D}(R)$ and in a thick subcategory of $\mathrm{D}(R)$ are the same. To simplify notation, I write

$$\mathrm{thick}_R^n(\mathcal{C}) := \mathrm{thick}_{\mathrm{D}(R)}^n(\mathcal{C}) \quad \text{and} \quad \mathrm{level}_R^{\mathcal{C}}(X) := \mathrm{level}_{\mathrm{D}(R)}^{\mathcal{C}}(X)\,.$$

The derived category of complexes with finitely generated homology is $\mathrm{D}^{\mathrm{f}}(R)$. It contains all complexes for which the *total homology*

$$\mathrm{H}(X) := \coprod_{i \in \mathbb{Z}} \Sigma^i \mathrm{H}_i(X)$$

is finitely generated. In particular, $\mathrm{H}_i(X) = 0$ for $|i| \gg 0$ and $\mathrm{H}_i(X)$ is finitely generated for all i.

The derived category of complexes with bounded below and degree-wise finitely generated homology is $\mathrm{D}_+(R\text{-}\mathrm{mod})$. It contains the complexes X with $\mathrm{H}_i(X) = 0$ for $i \ll 0$ and $\mathrm{H}_i(X)$ finitely generated for all i.

4.2 Filtration

The following lemma due to [ABIM10] shows that a filtration gives a bound on level.

Lemma 4.2.1. *Let $\mathcal{C}$ be a strictly full subcategory of $\mathrm{D}(R)$ and X a complex of R-modules.*

1. *If X has a filtration by subcomplexes $0 = X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n = X$, then*

$$\mathrm{level}^{\mathcal{C}}(X) \leq \sum_{i=1}^{n} \mathrm{level}^{\mathcal{C}}(X^i / X^{i-1}) \,.$$

2. *One has*

$$\mathrm{level}^{\mathcal{C}}(X) \leq \sum_{i \in \mathbb{Z}} \mathrm{level}^{\mathcal{C}}(\mathrm{H}_i(X)) \,.$$

3. *If X has bounded cohomology, then*

$$\mathrm{level}^{\mathcal{C}}(X) \leq \inf \left\{ \sum_{i \in \mathbb{Z}} \mathrm{level}^{\mathcal{C}}(Y_i) \;\middle|\; Y \simeq X \textit{ with } Y \in \mathrm{K}_b(R\text{-}\mathrm{Mod}) \right\} \,.$$

Proof. Given a filtration as in (1), one has exact triangles

$$X^{i-1} \to X^i \to X^i / X^{i-1} \to \Sigma X^{i-1} \,.$$

Thus by 2.3.2 (2)

$$\mathrm{level}^{\mathcal{C}}(X^i) \leq \mathrm{level}^{\mathcal{C}}(X^{i-1}) + \mathrm{level}^{\mathcal{C}}(X^i / X^{i-1})$$

and the claim holds.

For (2), set $w(X) = |\{d \in \mathbb{Z} \mid \mathrm{H}_d(X) \neq 0\}|$. If $w(X) = \infty$, then the right side of the inequality is ∞, thus there is nothing to prove. So I may assume $w(X) < \infty$. Then X has a filtration by soft truncations. A soft truncation of X is

$$\sigma_{\geqslant i}X\colon \; \cdots \to X_{i+1} \to \ker(\partial_i) \to 0\,.$$

Then X has a filtration

$$0 \simeq \sigma_{\geqslant b+1}X \subseteq \sigma_{\geqslant b}X \subseteq \ldots \subseteq \sigma_{\geqslant a+1}X \simeq X$$

where

$$a = \min\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \quad \text{and} \quad b = \max\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}\,.$$

Now

$$\sigma_{\geqslant i}X/\sigma_{\geqslant i+1}X = (0 \to X_{i+1}/\ker(\partial_{i+1}) \to \ker(\partial_i) \to 0) \simeq \Sigma^i\,\mathrm{H}_i(X)$$

and by (1), it is

$$\mathrm{level}^{\mathcal{C}}(X) \leq \sum_{i \in \mathbb{Z}} \mathrm{level}^{\mathcal{C}}(\mathrm{H}_i(X))\,.$$

For (3), one has $\mathrm{level}^{\mathcal{C}}(X) = \mathrm{level}^{\mathcal{C}}(Y)$ for $X \simeq Y$. Thus it is enough to prove

$$\mathrm{level}^{\mathcal{C}}(Y) \leq \sum_{i \in \mathbb{Z}} \mathrm{level}^{\mathcal{C}}(Y_i)$$

for a bounded complex Y. This time I consider a filtration by brutal truncations. A brutal truncation of X is

$$\tau_{\leqslant i}X\colon 0 \to X_i \to X_{i-1} \to \cdots\,.$$

Then Y has a filtration

$$0 = \tau_{\leqslant a-1}Y \subseteq \tau_{\leqslant a}Y \subseteq \ldots \subseteq \tau_{\leqslant b}Y = Y$$

where

$$a = \min\{i \in \mathbb{Z} \mid Y_i \neq 0\} \quad \text{and} \quad b = \max\{i \in \mathbb{Z} \mid Y_i \neq 0\}\,.$$

Now $\tau_{\leqslant i}Y/\tau_{\leqslant i-1}Y \simeq \Sigma^i Y_i$ and so the claim follows from (1). □

Corollary 4.2.2. *Let G be a complex in $\mathrm{D}^{\mathrm{f}}(R)$, such that for any finitely generated R-module M*

$$\mathrm{level}^{G}(M) \leq n\,.$$

Then $\mathrm{level}^{G}(X) \leq 2n$ *for any* $X \in \mathrm{D}^{\mathrm{f}}(R)$.

Proof. Since any module has level less or equal than n, any bounded complex with 0-differential also has level less or equal than n. In particular, given a complex $X \in \mathrm{D}^{\mathrm{f}}(R)$, the complex of cycles $\mathrm{Z}(X)$ and the complex of boundaries $\mathrm{B}(X)$ are such complexes. Now there exists an exact triangle in $\mathrm{D}(R)$

$$\mathrm{Z}(X) \to X \to \Sigma\,\mathrm{B}(X) \to \Sigma\,\mathrm{Z}(X)$$

induced from the short exact sequence $0 \to \mathrm{Z}(X) \to X \to \Sigma\,\mathrm{B}(X) \to 0$. Thus by 2.3.2 (2)

$$\mathrm{level}^G(X) \leq \mathrm{level}^G(\mathrm{Z}(X)) + \mathrm{level}^G(\mathrm{B}(X)) \leq 2n\,. \qquad \square$$

4.3 Level and Projective Dimension

4.3.1. A *perfect complex* is a complex quasi-isomorphic to a bounded complex of finitely generated projective modules. The category of perfect complexes is $\mathrm{Perf}(R)$ and one has

$$\mathrm{thick}(R) = \mathrm{Perf}(R)\,,$$

since $\mathrm{thick}^1(R)$ contains all finitely generated projective modules and $\mathrm{Perf}(R)$ is thick; for a proof, see for example [DGI06, 3.7]. It is well-known that the perfect complexes are also precisely the compact objects in $\mathrm{D}(R)$.

For an R-module M, the projective dimension $\mathrm{pd}(M)$ is the minimal length of a projective resolution of M. This is connected to the vanishing of Ext:

$$\mathrm{pd}(M) = \sup\left\{n \geq 0 \,|\, \mathrm{Ext}^n_R(M,-) \neq 0\right\}\,.$$

First I will discuss the Ext-groups for an abelian category $\mathcal{A}$. By [Wei94, 10.7], the Ext-groups are connected to the morphisms in the derived category $\mathrm{D}(A)$:

Lemma 4.3.2. *If $\mathcal{A}$ has enough projectives or enough injectives, then for $X, Y \in \mathcal{A}$ one has*

$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X, \Sigma^n Y)\,.$$

In particular, this gives $\mathrm{Ext}_{\mathrm{D}(\mathcal{A})}(-,-) = \mathrm{Ext}_{\mathcal{A}}(-,-)$ *for objects in $\mathcal{A}$ viewed as a complex in degree zero in* $\mathrm{D}(\mathcal{A})$.

Proof. I prove the statement when $\mathcal{A}$ has enough projectives. If $\mathcal{A}$ has enough injectives, the claim holds similarly. Let $P \xrightarrow{\sim} X$ be a projective resolution. Then

$$\mathrm{Z}^n(\mathrm{Hom}_{\mathcal{A}}(P,Y)) = \left\{f\colon P_n \to Y \,\middle|\, f \circ \partial^P = 0\right\}\,, \qquad \text{and}$$

$$\begin{aligned}\mathrm{B}^n(\mathrm{Hom}_{\mathcal{A}}(P,Y)) &= \left\{f\colon P_n \to Y \,\middle|\, \exists g\colon P_{n-1} \to Y \text{ such that } f = g \circ \partial^P\right\} \\ &= \left\{f\colon P_n \to Y \,|\, f \text{ is null-homotopic}\right\}\,.\end{aligned}$$

Taking the quotient gives

$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) = \mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}}(P,Y)) = \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P,\Sigma^n Y) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(X,\Sigma^n Y)\,,$$

where $\mathrm{K}(\mathcal{A})$ is the homotopy category. □

Following [ML95, III.5–6], I discuss another description of the Ext-groups.

Definition 4.3.3. Let X and Y be objects in $\mathcal{A}$. A *degree d Yoneda extension of X by Y* is an exact sequence

$$E : 0 \to X \to E_d \to \ldots \to E_1 \to Y \to 0$$

in $\mathcal{A}$. Two Yoneda extensions E and E' of the same degree are *equivalent*, if there exists a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & X & \longrightarrow & E_d & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & Y & \longrightarrow & 0 \\ \uparrow & & \| & & \uparrow & & & & \uparrow & & \| & & \uparrow \\ 0 & \longrightarrow & X & \longrightarrow & E''_d & \longrightarrow & \cdots & \longrightarrow & E''_1 & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & E'_d & \longrightarrow & \cdots & \longrightarrow & E'_1 & \longrightarrow & Y & \longrightarrow & 0 \end{array} \tag{4.3.4}$$

where the middle row is a degree d Yoneda extension as well.

Lemma 4.3.5. *The map*

$$\delta\colon \left\{\begin{matrix}\textit{equivalence classes of degree } d\\ \textit{Yoneda extensions of } X \textit{ by } Y\end{matrix}\right\} \to \mathrm{Ext}^d_{\mathcal{A}}(Y,X)$$

given by

$$\delta(E) = g_E \circ (f_E)^{-1} \quad \textit{with } f_E\colon E_Y \xrightarrow{\sim} Y \textit{ and } g_E\colon E_Y \to \Sigma^d X$$
$$\textit{where } E_Y = (0 \to X \to E_d \to \cdots \to E_1 \to 0)$$

is a bijection.

Proof. (well-defined) In the diagram (4.3.4), the maps are quasi-isomorphisms. The definition of δ is invariant under quasi-isomorphisms, thus all representatives of an equivalence class are mapped to the same element in $\mathrm{Ext}^d_{\mathcal{A}}(Y,X)$.

(surjective) Take $h \in \mathrm{Ext}^d_{\mathcal{A}}(Y,X)$. By 4.3.2, this can be viewed as a morphism $h\colon Y \to \Sigma^d X$ in $\mathrm{D}(\mathcal{A})$. Then there exists a quasi-isomorphism $f\colon \widetilde{Y} \xrightarrow{\sim} Y$, such that $g = h \circ f$ is a chain map. Set

$$E_i := \widetilde{Y}_{i-1} \quad \text{for } 1 \le i \le d-1 \quad \text{and}$$
$$E_d := (\widetilde{Y}_{d-1} \oplus X) / \operatorname{im}(\widetilde{Y}_d \xrightarrow{\left(\begin{smallmatrix} \widetilde{Y}_d \to \widetilde{Y}_{d-1} \\ -(\widetilde{Y}_d \to X) \end{smallmatrix}\right)} \widetilde{Y}_{d-1} \oplus X)\,.$$

Then the sequence

$$E : 0 \to X \to E_d \to E_{d-1} \to \cdots \to E_1 \to Y \to 0$$

is a degree d Yoneda extension of X by Y and $\delta(E) = h$.

(injective) Let E and E' be degree d Yoneda extensions with $\delta(E) = \delta(E')$. That is $g_E \circ (f_E)^{-1} = g_{E'} \circ (f_{E'})^{-1}$. In particular, there are quasi-isomorphisms

$$\begin{array}{ccc} Z & \overset{\sim}{\dashrightarrow} & E'_Y \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ E_Y & \xrightarrow{\sim} & Y \end{array}$$

for some $Z \in \mathrm{D}(\mathcal{A})$. Then

$$h = (Z \to E_Y \to \Sigma^d X) \quad \text{and} \quad (Z \to E'_Y \to \Sigma^d X) = h'$$

are homotopic via a homotopy $s\colon Z_{d-1} \to X$. Now consider the following commutative diagram with exact rows

$$\begin{array}{cccccccccccccc}
E: & 0 & \to & X & \longrightarrow & E_d & \longrightarrow & E_{d-1} & \to & \cdots & \longrightarrow & E_1 & \longrightarrow & Y \to 0 \\
 & & & \| & & \uparrow{\scriptstyle (u\ v)} & & \uparrow & & & & \uparrow & & \| \\
 & 0 & \to & X & \xrightarrow{\binom{0}{1}} & \dfrac{Z_{d-1} \oplus X}{\operatorname{im}\binom{\partial}{-h}} & \xrightarrow{(\partial\ 0)} & Z_{d-2} & \to & \cdots & \to & \ker(Z_0 \to Z_{-1}) & \to & Y \to 0 \\
 & & & \| & & \downarrow{\scriptstyle (u' + v' \circ s\ \ v')} & & \downarrow & & & & \downarrow & & \| \\
E': & 0 & \to & X & \longrightarrow & E'_d & \longrightarrow & E'_{d-1} & \to & \cdots & \longrightarrow & E'_1 & \longrightarrow & Y \to 0\,,
\end{array}$$

where $u^{(\prime)}\colon Z_{d-1} \to E_d^{(\prime)}$ is the map in degree $d-1$ of $Z \to E_Y^{(\prime)}$ and $v^{(\prime)}\colon X \to E_d^{(\prime)}$ is the map in E. This defines an equivalence of the Yoneda extensions E and E'. □

Using this description of the Ext-groups by Yoneda extensions gives a lower bound for level of objects in the abelian category; see [KK06, 2.4].

Lemma 4.3.6. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{P} \subseteq \mathcal{A}$ the strictly full subcategory of projective objects. If $X \in \mathcal{A}$ with $\operatorname{Ext}^n_{\mathcal{A}}(X, -) \neq 0$, then $X \notin \operatorname{thick}^n_{\mathrm{D}(\mathcal{A})}(\mathcal{P})$.*

Proof. Set $X^0 := X$. Then there exists $X^n \in \mathcal{A}$, such that $\operatorname{Ext}^n_{\mathcal{A}}(X^0, X^n) \neq 0$. Let f be a non-zero element in $\operatorname{Ext}^n_{\mathcal{A}}(X^0, X^n)$ and identify it with the corresponding Yoneda extension

$$f = (0 \to X^n \to E_n \to \cdots \to E_1 \to X^0 \to 0)\,.$$

This sequence is exact with $E_i \in \mathcal{A}$. Define inductively

$$f^i = (0 \to X^i \to E_i \to X^{i-1} \to 0) \in \operatorname{Ext}^1_{\mathcal{A}}(X^{i-1}, X^i)\,.$$

I can rewrite f^i as

$$f^i = (X^{i-1} \xleftarrow{\sim} (0 \to X^i \to E_i \to 0) \to \Sigma X^i) \in \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(X^{i-1}, \Sigma X^i)\,.$$

This is the connecting morphism in $\mathrm{D}(\mathcal{A})$ that completes the short exact sequence to an exact triangle. These maps induce natural transformations

$$f^i_*\colon\ \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(-, X^{i-1}) \to \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(-, \Sigma X^i) = \operatorname{Ext}^1_{\mathcal{A}}(-, X^i)$$

which vanish on $\operatorname{thick}^1_{\mathrm{D}(\mathcal{A})}(\mathcal{P})$, that is f^i is $\mathcal{P}$-ghost for all i. So

$$0 \neq f = (\Sigma^{n-1} f^n) \circ \ldots \circ f^1$$

is n-fold $\mathcal{P}$-ghost and by the ghost lemma 3.1.6, one has $X = X^0 \notin \operatorname{thick}^n_{\mathrm{D}(\mathcal{A})}(\mathcal{P})$. □

Now I discuss R as a generator in the derived category $\mathrm{D}(R)$.

4.3.7. Note that $\mathrm{H}_0(-) = \operatorname{Hom}_{\mathrm{D}(R)}(R, -)$. In particular, a map $f\colon X \to Y$ is R-ghost if and only if $\mathrm{H}(f) = 0$.

Given $G \in \mathrm{D}(R)$. Then a map $f\colon X \to Y$ is G-ghost if and only if

$$\mathbf{R}\operatorname{Hom}_R(G, f)\colon\ \mathbf{R}\operatorname{Hom}_R(G, X) \to \mathbf{R}\operatorname{Hom}_R(G, Y)$$

is R-ghost. Similarly, the map f is G-coghost if and only if $\mathbf{R}\operatorname{Hom}_R(f, G)$ is R-ghost.

4.3.8. Lemma 4.3.6 can be applied in two ways: If $\mathcal{A} = R\text{-}\operatorname{Mod}$ is the category of all R-modules, then $\mathcal{P}$ is the category of all projective modules. If $\mathcal{A} = R\text{-}\operatorname{mod}$ is the category of all finitely generated R-modules, then $\mathcal{P}$ consists of all finitely generated R-modules. In the latter case, it is possible to replace $\mathcal{P}$ by R.

Notice that $R\text{-}\operatorname{mod}$ is only abelian if R is noetherian. In this case, level with respect to all projective modules and level with respect to R are the same for objects in $\mathrm{D}^{\mathrm{f}}(R)$; see 2.5.5.

Lemma 4.3.9. *Let R be a noetherian ring and M a finitely generated R-module. Then*

$$\operatorname{level}^R(M) = \operatorname{pd}(M) + 1\,.$$

Proof. By 4.3.1, the ring R generates all perfect complexes. In particular, it generates M if and only if M has finite projective dimension. So it remains to show the equality when both sides are finite. Let $P \xrightarrow{\sim} M$ be a minimal projective resolution of M. Then by Lemma 4.2.1 (3)

$$\operatorname{level}^R(M) = \operatorname{level}^R(P) \leq \sum_{d\in\mathbb{Z}} \operatorname{level}^R(P_d) \leq \operatorname{pd}(M) + 1$$

since $P_d \neq 0$ precisely when $0 \leq d \leq \operatorname{pd}(M)$ and then $\operatorname{level}^R(P_d) = 1$.

For the opposite inequality, set $n := \operatorname{pd}(M)$. Then $\operatorname{Ext}^n_R(M, -) \neq 0$. By Lemma 4.3.6

$$M \notin \operatorname{thick}^n(R)\,, \quad \text{and thus} \quad \operatorname{level}^R(M) \geq n + 1 = \operatorname{pd}(M) + 1\,.$$

□

Lemma 4.3.10. *Let R be a noetherian ring. Then any $X \in \mathrm{D}(R)$ has a right R-approximation. Moreover, if $n(X)\colon P \to X$ is any right R-approximation of X, then*

$$\operatorname{pd}(\mathrm{H}(X)) = \operatorname{pd}(\mathrm{H}(\operatorname{cone}(n(X)))) + 1\,.$$

Proof. For every $d \in \mathbb{Z}$, take a projective module P^d such that $P^d \twoheadrightarrow \mathrm{H}_d(X)$ is surjective. Set $P := \coprod_{d\in\mathbb{Z}} \Sigma^d P^d$. Consider the diagram

$$\begin{array}{ccc} & \mathrm{Z}(X) & \hookrightarrow X\,. \\ \nearrow & \downarrow\!\!\!\downarrow & \\ P & \twoheadrightarrow \mathrm{H}(X) & \end{array}$$

Since P is projective, there exists a map $P \to \mathrm{Z}(X)$. I claim the induced map $P \to X$ is a right R-approximation. Let $f\colon \Sigma^n R \to X$ be a map in $\mathrm{D}(R)$. Then f is an element in $\mathrm{H}_n(X) = \operatorname{Hom}_{\mathrm{D}(R)}(\Sigma^n R, X)$. Since R is projective, it factors through $P \to X$ by construction of the latter.

For the second claim, consider the exact triangle

$$Y \to P \to X \to \Sigma Y\,.$$

The map $X \to \Sigma Y$ is R-ghost by 3.3.4, so it vanishes in homology. Applying homology $\mathrm{H}(-)$ to the exact triangle gives the short exact sequence

$$0 \to \mathrm{H}(Y) \to P \to \mathrm{H}(X) \to 0\,.$$

In particular, $\mathrm{H}(Y)$ is the first syzygy of $\mathrm{H}(X)$. This proves the claim. □

With this lemma and 3.4.7, I recover a result from [Chr98, 8.2].

Corollary 4.3.11. *The pair* $(\operatorname{thick}^1(R), R\text{-}\operatorname{gh})$ *is a projective class.* □

4.3.12. Using the right R-approximation, one can construct an Adams resolution. For an R-module M, the Adams resolution is of the form

$$\begin{array}{ccccccccc} M = \Omega^0 M & & \longrightarrow & & \Sigma\Omega^1 M & & \longrightarrow & & \Sigma^2\Omega^2 M & \longrightarrow & \cdots \\ & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} \\ & & P_0 & & & & \Sigma P_1 & & & & \Sigma^2 P_2 \end{array}$$

where $\Omega^n M$ is the nth syzygy of M and the P_n's are projective R-modules.

This gives a projective resolution of M

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

where the differentials are the composition of the diagonal maps of the Adams resolution.

The horizontal maps are constructed by 'cutting off' the part of the projective resolution that has homology. That is

$$\begin{array}{ccccccccc} M & \simeq & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\ \downarrow & & & & \| & & \downarrow & & \\ \Sigma\Omega^1 M & \simeq & \cdots & \longrightarrow & P_1 & \longrightarrow & 0\,. & & \end{array}$$

Corollary 4.3.13. *Let R be a noetherian ring. Then for any $X \in \mathrm{D}(R)$, one has*

$$\operatorname{level}^{R\text{-}\operatorname{Proj}}(X) \le \operatorname{pd}(\mathrm{H}(X)) + 1\,.$$

If X has bounded finitely generated homology, then the left side coincides with the level with respect to R. In particular

$$\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(R) = \operatorname{gldim}(R)\,.$$

Proof. The inequality holds by 4.3.10. Then $\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(R) \le \operatorname{gldim}(R)$. If $\operatorname{gldim}(R) = n < \infty$, then there exists a finitely generated R-module M with $\operatorname{pd}(M) = n$. So by 4.3.9, equality holds. If $\operatorname{gldim}(R) = \infty$, then for every integer n, there exists a finitely generated R-module M such that $\operatorname{level}^R(M) = \operatorname{pd}(M) + 1 \ge n$. So $\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(R) = \infty$. □

4.3.14. There are various approaches on how to extend projective dimension to complexes. Lemma 4.3.9 motivates another one. Given $X \in \mathrm{D}^{\mathrm{f}}(R)$, set

$$\operatorname{pd}_l(X) := \operatorname{level}^R(X) - 1\,.$$

This differs from the following two definitions for projective dimension for complexes:

$$\mathrm{pd}_a(X) := \inf\{b - a \mid X \simeq P \in \mathrm{Perf}(R) \text{ where } P_i \neq 0 \text{ for } a \leq i \leq b\} \quad \text{and}$$
$$\mathrm{pd}_s(X) := \inf\{b \mid X \simeq P \in \mathrm{Perf}(R) \text{ where } P_i = 0 \text{ for } i > b\}\ .$$

These definitions for the projective dimension both measure the 'size' of a minimal projective resolution. While $\mathrm{pd}_a(X)$ gives the amplitude of the resolution, the second invariant $\mathrm{pd}_s(X)$ is the highest degree in which a projective resolution has a non-zero module.

The definition via level, $\mathrm{pd}_l(-)$, behaves better with respect to suspension and direct sums. By 2.3.2 (1), level is invariant under suspension. Measuring the amplitude of a complex is also invariant under suspension, but the highest non-zero degree shifts:

$$\mathrm{pd}_l(\Sigma X) = \mathrm{pd}_l(X)\,, \quad \mathrm{pd}_a(\Sigma X) = \mathrm{pd}_a(X) \quad \text{and} \quad \mathrm{pd}_s(\Sigma X) = \mathrm{pd}_s(X) + 1\,.$$

The amplitude of a direct sum needs not be related to the amplitude of the summands. For example

$$\mathrm{pd}_a(R \oplus \Sigma^n R) = n \quad \text{while} \quad \mathrm{pd}_a(R) = \mathrm{pd}_a(\Sigma^n R) = 0\,.$$

For the other two invariants, one has

$$\mathrm{pd}_{l/s}(X \oplus Y) = \max\{\mathrm{pd}_{l/s}(X), \mathrm{pd}_{l/s}(Y)\}\,.$$

Using the same ideas as [Nee01, Lemma C.4.2], I get

Lemma 4.3.15. *Let R be a noetherian ring and $X \in \mathrm{D^f}(R)$ with $\mathrm{level}^R(\mathrm{H}(X)) \leq 2$. Then $X \simeq \mathrm{H}(X)$.*

Proof. If $\mathrm{level}^R(\mathrm{H}(X)) = 1$, then $\mathrm{H}(X)$ is a finite direct sum of suspensions of projective modules, that is it is projective. Thus there exists a map

$$\mathrm{H}(X) \underset{\twoheadleftarrow}{\dashrightarrow} \mathrm{Z}(X) \hookrightarrow X$$

that is a quasi-isomorphism. Thus $\mathrm{H}(X) \simeq X$.

If $\mathrm{level}^R(\mathrm{H}(X)) = 2$, then $\mathrm{pd}(\mathrm{H}(X)) = 1$ and there exists a short exact sequence

$$0 \to Q \to P \to \mathrm{H}(X) \to 0$$

where P and Q are bounded complexes of finitely generated projective modules with 0-differential. By the proof of Lemma 4.3.10, the map $P \twoheadrightarrow \mathrm{H}(X)$ induces a morphism $P \to X$ in $\mathrm{D}(R)$. Complete this to an exact triangle

$$P \to X \to Y \to \Sigma P\,.$$

Taking homology and comparing the resulting short exact sequence to the one above gives $\mathrm{H}(\Sigma^{-1}Y) \simeq Q$. Then by the first case, one has $Y \simeq \mathrm{H}(Y) \in \operatorname{thick}^1(R)$. This gives the commuting diagram

$$\begin{array}{ccccccc} Q & \longrightarrow & P & \longrightarrow & \mathrm{H}(X) & \longrightarrow & \Sigma Q \\ \downarrow{\scriptstyle\sim} & & \| & & & & \downarrow{\scriptstyle\sim} \\ \Sigma^{-1}Y & \longrightarrow & P & \longrightarrow & X & \longrightarrow & Y \end{array}$$

where the rows are exact triangles. Thus $X \simeq \mathrm{H}(X)$. □

4.4 Flat and Injective Dimension

Flat and injective dimension can, similarly as projective dimension, be described by the vanishing of a functor:

$$\operatorname{fd}(M) = \sup\left\{n \geq 0 \,\middle|\, \operatorname{Tor}_n^R(M,-) \neq 0\right\} \quad \text{and}$$
$$\operatorname{injdim}(M) = \sup\left\{n \geq 0 \mid \operatorname{Ext}_R^n(-,M) \neq 0\right\}.$$

As for projective dimension for modules, these are the same as the level with respect to all flat respectively injective modules. If R is noetherian, the finitely generated projective modules lie in $\operatorname{thick}^1(R)$. So the generation by finitely generated projective modules is the same as generation by R. Something similar is not possible for flat or injective modules. Further, it is not possible to restrict to only finitely generated flat/injective modules. Let R-Flat be the category of all flat R-modules and R-Inj the category of all injective R-modules.

The following results are well-known.

Lemma 4.4.1. *Let R be a ring and M an R-module. Then*

$$\operatorname{level}^{R\text{-}\mathrm{Flat}}(M) = \operatorname{fd}(M) + 1\,.$$

Proof. Let $F \xrightarrow{\sim} M$ be a minimal flat resolution of M. Then by Lemma 4.2.1 (3), one has

$$\operatorname{level}^{R\text{-}\mathrm{Flat}}(M) = \operatorname{level}^{R\text{-}\mathrm{Flat}}(F) \leq \sum_{d\in\mathbb{Z}} \operatorname{level}^{R\text{-}\mathrm{Flat}}(F_d) \leq \operatorname{fd}(M) + 1$$

since $\operatorname{level}^{R\text{-}\mathrm{Flat}}(F_d) = 1$, if $F_d \neq 0$.

For the opposite inequality, realize that the category of all complexes quasi-isomorphic to a bounded complex of flat modules is thick. In particular, it contains $\operatorname{thick}(R\text{-}\mathrm{Flat})$. So if the flat dimension of M is infinite, then the level with respect to R-Flat is also infinite.

Now I may assume M has finite flat dimension. Set $n := \operatorname{fd}(M)$. Then there exists a module $N \in R^{op}$- Flat, such that $\operatorname{Tor}_n^R(N,M) \neq 0$. Let $F \xrightarrow{\sim} N$ be a flat resolution of N in R^{op}- Flat. Set

$$N_0 := N \quad \text{and} \quad N_i := \operatorname{coker}(F_i \to F_{i-1}) \, .$$

Then there exist short exact sequences $0 \to N_i \to F_{i-1} \to N_{i-1} \to 0$. These give natural transformations

$$\operatorname{Tor}_n^R(N_0,-) \xrightarrow{\eta_1} \operatorname{Tor}_{n-1}^R(N_1,-) \to \cdots \to \operatorname{Tor}_1^R(N_{n-1},-) \xrightarrow{\eta_n} \operatorname{Tor}_0^R(N_n,-)$$

where η_i is an isomorphism on R- Mod for $i < n$ and η_n is injective on R- Mod. The functors $\operatorname{Tor}_i^R(N_{n-i},-)$ vanish on R- Flat for $i > 0$. So the η_i's vanish on R- Flat and by the generalized (co-)ghost lemma 3.1.3, the composition $\eta := \eta_1 \circ \ldots \circ \eta_n$ vanishes on $\operatorname{thick}^n(R\text{- Flat})$. Since

$$0 \neq \operatorname{Tor}_n^R(N,M) \xhookrightarrow{\eta(M)} \operatorname{Tor}_0^R(N_n,M)$$

is injective, $\eta(M) \neq 0$ and so $M \notin \operatorname{thick}^n(R\text{- Flat})$. This gives

$$\operatorname{level}^{R\text{- Flat}}(M) \geq n+1 = \operatorname{fd}(M)+1 \, . \qquad \square$$

Lemma 4.4.2. *Let R be a ring and M an R-module. Then*

$$\operatorname{level}^{R\text{- Inj}}(M) = \operatorname{injdim}(M)+1 \, .$$

Proof. This proof works similar to the proof of 4.4.1 for flat dimension. The only difference occurs for the proof of $\geq$ when M has finite injective dimension. Set $n := \operatorname{injdim}(M)$. Then $\operatorname{Ext}_R^n(-,M) \neq 0$. Let $\mathcal{A} = (R\text{- Mod})^{op}$ be the opposite category. Then

$$\operatorname{Ext}_R^n(-,M) = \operatorname{Ext}_{\mathcal{A}}^n(M,-) \quad \text{and} \quad R\text{- Inj} = \mathcal{P} \, ,$$

where $\mathcal{P}$ is the class of projective objects of $\mathcal{A}$. By 4.3.6, one has

$$M \notin \operatorname{thick}_{\mathrm{D}(\mathcal{A})}^n(\mathcal{P}) = \operatorname{thick}_R^n(R\text{- Inj})$$

where the last identification holds by 2.3.3 (2). This shows the claim. $\square$

4.5 Level and Loewy Length

Definition 4.5.1. Let $(R,\mathfrak{m},k)$ be a local ring and M an R-module. Then

$$\operatorname{ll}(M) := \inf \{ n \geq 0 \mid \mathfrak{m}^n M = 0 \}$$

is the *Loewy length of M*.

By [ABIM10, Theorem 6.2], the Loewy length is connected to the level with respect to the residue field.

Lemma 4.5.2. *Let $(R, \mathfrak{m}, k)$ be a local ring, $X \in \mathrm{D}^{\mathrm{f}}(R)$ and M a finitely generated R-module. Then*

1. $\operatorname{level}^k(M) = \operatorname{ll}(M)$, *and*

2. $$loewy(\mathrm{H}(X)) \leq \operatorname{level}^k(X) \leq \operatorname{ll}(R).$$

Proof. I first show $\operatorname{level}^k(M) \leq \operatorname{ll}(M)$. If the Loewy length is infinite, there is nothing to prove. If $n := \operatorname{ll}(M)$ is finite, consider the filtration

$$0 = \mathfrak{m}^n M \subseteq \mathfrak{m}^{n-1} M \subseteq \ldots \subseteq \mathfrak{m} M \subseteq M\,.$$

The quotients $\mathfrak{m}^{i-1}M/\mathfrak{m}^i M$ are finite direct sums of k. Then by Lemma 4.2.1 (1), one has

$$\operatorname{level}^k(M) \leq \sum_{i=1}^{n} \operatorname{level}^k(\mathfrak{m}^{i-1}M/\mathfrak{m}^i M) = n = \operatorname{ll}(M)\,.$$

The opposite inequality is a special case of the first inequality of (2). For this, it is enough to show $\mathfrak{m}^i$ annihilates every object in $\operatorname{thick}^i(k)$. A complex X is *annihilated* by $\mathfrak{m}^i$ if $\mathfrak{m}^i \mathrm{H}(X) = 0$. Since $\mathfrak{m}$ annihilates k, it annihilates $\operatorname{thick}^1(k)$. Now if $X \in \operatorname{thick}^i(k)$, there exists an exact triangle

$$Y \to X \to Z \to \Sigma Y \quad \text{with } Y \in \operatorname{thick}^1(k) \text{ and } Z \in \operatorname{thick}^{i-1}(k)$$

and by induction, $\mathfrak{m}$ annihilates Y and $\mathfrak{m}^{i-1}$ annihilates Z. The triangle induces a long exact sequence

$$\cdots \to \mathrm{H}(Y) \to \mathrm{H}(X) \to \mathrm{H}(Z) \to \mathrm{H}(\Sigma Y) \to \cdots$$

and so there is a short exact sequence

$$0 \to K \to \mathrm{H}(X) \to C \to 0$$

where $\mathfrak{m}$ annihilates K and $\mathfrak{m}^{i-1}$ annihilates C. Thus $\mathfrak{m}^i$ annihilates $\mathrm{H}(X)$.

It remains to show the second inequality of (2). If the Loewy length of R is infinite, there is nothing to show. If $n := \operatorname{ll}(R)$ is finite, then $\mathfrak{m}^n \mathrm{H}(X) \cong 0$ and the claim holds by the same argument as above. □

4.5.3. This lemma inspires the definition of Lowey length for complexes

$$\mathrm{ll}_l(X) := \mathrm{level}^k(X)$$

that extends the definition for modules. There are two other approaches for such an invariant

$$\mathrm{ll}_c(X) := \inf\{n \geq 0 \mid \mathfrak{m}^n X = 0\} \quad \text{and} \quad \mathrm{ll}_h(X) := \inf\{\mathrm{ll}_c(Y) \mid Y \simeq X\}\ .$$

The first of these invariants is not well-defined on the derived category. That is quasi-isomorpic complexes need not have the same Loewy length when using $\mathrm{ll}_c(-)$.

The second invariant, $\mathrm{ll}_h(-)$, is the *homotopical Loewy length*; for details, see [AIM06, 6.2]. The homotopical Loewy length does not respect exact triangles; see the following example.

Example 4.5.4. Consider the ring $R = k[x]/(x^n)$ for $n \geq 4$. Then there exists an exact triangle

$$\Sigma k \xrightarrow{x^{n-1}} X \to k \to \Sigma^2 k$$

where $X = 0 \to R \xrightarrow{x} R \to 0$. The homotopical Loewy length for these complexes are

$$\mathrm{ll}_h(k) = \mathrm{ll}_h(\Sigma^2 k) = 1 \quad \text{and} \quad \mathrm{ll}_h(X) \geq \mathrm{ll}(\mathrm{H}(X)) \geq \mathrm{ll}(xR) = n-1 \geq 3\,.$$

The bound for the Rouquier dimension of an artinian ring is due to [Rou08, Proposition 7.37].

Proposition 4.5.5. *Let $(R, \mathfrak{m}, k)$ be an artinian local ring. Then k is a strong generator of* $\mathrm{D}^{\mathrm{f}}(R)$ *with*

$$\dim(\mathrm{D}^{\mathrm{f}}(R)) \leq \circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(k) = \mathrm{ll}(R) - 1\,.$$

Proof. For any $X \in \mathrm{D}^{\mathrm{f}}(R)$, one has $\mathrm{level}^k(X) \leq \mathrm{ll}(R)$ by Lemma 4.5.2 (2). Thus

$$\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(k) \leq \mathrm{ll}(R) - 1\,.$$

Since R is artinian, the ring has finite Loewy length and k is a strong generator. Since $\mathrm{level}^k(R) = \mathrm{ll}(R)$, the generation time of k is precisely $\mathrm{ll}(R) - 1$. □

4.6 Rouquier Dimension for Regular Rings

The regular local rings of Krull dimension one are precisely the discrete valuation rings (DVR). A generator of the maximal ideal of such a ring is called a *uniformizer* and denoted by ϖ.

Proposition 4.6.1. *Let R be a DVR. Then the Rouquier dimension is*

$$\dim(\mathrm{D}^{\mathrm{f}}(R)) = 1\,.$$

Proof. By Proposition 4.3.13, the ring R is a strong generator with

$$\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(R) \leq \operatorname{gldim}(R) = 1\,.$$

Assume there exists a generator G with $\operatorname{thick}^1(G) = \mathrm{D}^{\mathrm{f}}(R)$. Since $\operatorname{level}^R(G) \leq 2$, one has $G \simeq \mathrm{H}(G)$ by Lemma 4.3.15. Now R is a principal ideal domain (PID) and by the structure theorem for PID's, one has

$$\mathrm{H}(G) \cong R^e \oplus \bigoplus_{j=1}^{N} R/(\varpi^{i_j})$$

where ϖ is a uniformizer of R. Set $n := \max\left\{ i_j \,\middle|\, 1 \leq j \leq N \right\}$. Then

$$R/(\varpi^{n+1}) \notin \operatorname{thick}^1(\mathrm{H}(G)) = \operatorname{thick}^1(G)\,.$$

This is a contradiction. □

4.6.2. Let R be a commutative finitely generated k-algebra and a domain. Then by [Eis95, Chapter 13, Theorem A], one has

$$\dim(R) = \dim(R_{\mathfrak{m}}) \qquad \text{for all } \mathfrak{m} \in \operatorname{Max}(R)\,.$$

Also, any non-empty open set in $\operatorname{Spec}(R)$ contains a maximal ideal.

Proposition 4.6.3. *Let R be a finitely generated commutative k-algebra and a domain. If R is regular, then*

$$\dim(\mathrm{D}^{\mathrm{f}}(R)) = \dim(R)$$

and R is a strong generator with minimal generation time.

Proof. Let G be a strong generator of $\mathrm{D}^{\mathrm{f}}(R)$. Set

$$\mathcal{V} := \left\{ \mathfrak{p} \in \operatorname{Spec}(R) \,\middle|\, G_{\mathfrak{p}} \in \operatorname{thick}^1_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \right\}\,.$$

In 5.4.5, it will be shown this set is open. Since $(0) \in \mathcal{V}$, the free locus is non-empty. So by 4.6.2, it contains a maximal ideal $\mathfrak{m} \in \mathcal{V}$. Then $\operatorname{thick}^1_{R_\mathfrak{m}}(R_\mathfrak{m}) = \operatorname{thick}^1_{R_\mathfrak{m}}(G_\mathfrak{m})$ and, by 4.6.2 and 4.3.13, one has

$$\begin{aligned}\dim(R) = \dim(R_\mathfrak{m}) = \operatorname{gldim}(R_\mathfrak{m}) = \circleddash_{\mathrm{D^f}(R_\mathfrak{m})}(R_\mathfrak{m}) \\ = \circleddash_{\mathrm{D^f}(R_\mathfrak{m})}(G_\mathfrak{m}) \leq \circleddash_{\mathrm{D^f}(R)}(G)\,.\end{aligned}$$

So one has $\dim(R) \leq \dim(\mathrm{D^f}(R))$. However, since $\circleddash_{\mathrm{D^f}(R)}(R) = \dim(R)$ by 4.3.13, equality holds. □

4.7 Support

Instead of computing the exact value of level, often it is sufficient to know whether it is finite.

Definition 4.7.1. Let X be a complex of R-modules. Then the *support* of X is

$$\operatorname{Supp}_R(X) := \operatorname{Supp}_R(\mathrm{H}(X)) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathrm{H}(X)_\mathfrak{p} \neq 0\}\ .$$

Since the localization $R_\mathfrak{p}$ is flat over R, one has

$$\mathrm{H}(X)_\mathfrak{p} = 0 \iff X_\mathfrak{p} = R_\mathfrak{p} \otimes^{\mathbf{L}}_R X \simeq 0\,.$$

The support of a finitely generated modules is closed, so for complexes in $\mathrm{D^f}(R)$, the support is also closed.

4.7.2. Let $X \to Y \to Z \to \Sigma X$ be an exact triangle of objects in $\mathrm{D}(R)$. Applying the functor $- \otimes^{\mathbf{L}}_R R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ to the exact triangle yields

$$X_\mathfrak{p} \to Y_\mathfrak{p} \to Z_\mathfrak{p} \to \Sigma X_\mathfrak{p}\,.$$

Now if $\mathfrak{p}$ is neither in the support of X nor in the support of Z, then their localizations at $\mathfrak{p}$ are zero. Then $Y_\mathfrak{p} \simeq 0$ and thus

$$\operatorname{Supp}_R(Y) \subseteq \operatorname{Supp}_R(X) \cup \operatorname{Supp}_R(Z)\,.$$

The support gives a necessary condition when the level is finite.

Lemma 4.7.3. *Let X and Y be objects in $\mathrm{D^f}(R)$. Then*

$$X \models Y \implies \operatorname{Supp}_R(X) \supseteq \operatorname{Supp}_R(Y)\,.$$

Proof. This follows immediately from 4.7.2. □

The reverse does not hold in general.

Example 4.7.4. Let R be a local ring that is not regular. That is the residue field k does not have finite projective dimension. Let $\operatorname{Kos}(R)$ be the Koszul complex on the maximal ideal. Since the Koszul complex has finite length homology, its support consists only of $\mathfrak{m}$. Now by 4.3.1 one has

$$\operatorname{Kos}(R) \in \operatorname{thick}(R) \quad \text{and} \quad k \notin \operatorname{thick}(R)\,.$$

Since generation is transitive, by 2.2.11, the Koszul complex does not generate the residue field.

For perfect complexes, Hopkins [Hop87, Theorem 11] and Neeman [Nee92, Lemma 1.2] proved a converse of 4.7.3.

Theorem 4.7.5. *Let R be a commutative noetherian ring, and let X, Y be perfect complexes over R. If $\operatorname{Supp}_R(X) \subseteq \operatorname{Supp}_R(Y)$, then $\operatorname{level}^Y(X) < \infty$.* □

Chapter 5

Local to Global Principle

The goal of this chapter is to give a relationship between generation in $\mathrm{D}(R)$ and $\mathrm{D}(R_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p}$. As mentioned in Chapter 3, the ghost and coghost index give lower bounds for level. The converse (co)ghost lemma, as proved in Section 3.3, does not hold for an arbitrary generator in $\mathrm{D}^{\mathrm{f}}(R)$. However, a converse coghost lemma for $\mathrm{D}^{\mathrm{f}}(R)$ was established by [OŠ12]. I will give a detailed proof of this. It is not possible to prove a converse ghost lemma using the same idea, but it is possible to deduce a converse ghost lemma when the ring has a dualizing complex.

Using the converse coghost lemma, I will give a local to global principle for generation and level. This chapter presents the content of my paper [Let19].

5.1 Converse Coghost Lemma

To prove a converse coghost lemma, one first establishes that $\mathrm{D}^{\mathrm{f}}(R)$ are precisely the cocompact objects in $\mathrm{D}_{+}(R\text{-}\mathrm{mod})$. To classify the cocompact objects in $\mathrm{D}_{+}(R\text{-}\mathrm{mod})$, one needs to know which products exist.

Definition 5.1.1. A set $\{X^i \mid i \in I\}$ of bounded below complexes is *descending*, if

1. there exists an integer N, such that $X^i_n = 0$ for all $n \leq N$, and
2. for each $n \in \mathbb{Z}$, the set $\{i \in I \mid X^i_n \neq 0\}$ is finite.

Then [OŠ12, Proposition 13] show that products of descending sets are the only products that exist in $\mathrm{D}_{+}(R\text{-}\mathrm{mod})$.

Lemma 5.1.2. *Let $\{X^i \mid i \in I\}$ be a set of complexes in $\mathrm{D}_{+}(R\text{-}\mathrm{mod})$. Then $\prod_{i \in I} X^i$ exists in $\mathrm{D}_{+}(R\text{-}\mathrm{mod})$ if and only if $\exists Y^i \simeq X^i$ such that the set $\{Y^i \mid i \in I\}$ is descending.*

Proof. The product $X := \prod_{i\in I} X^i$ exists in $\mathrm{D}(R)$. Now $X \in \mathrm{D}_+(R\text{-}\mathrm{mod})$ if and only if $\mathrm{H}_n(X) = 0$ for $n \ll 0$ and $\mathrm{H}_n(X)$ is finitely generated for all n. Because homology commutes with the product, one has

$$\mathrm{H}_n(X) = 0 \text{ for } n \ll 0 \iff \exists N : \mathrm{H}_n(X^i) = 0\, \forall n \le N, i \in I\,, \quad \text{and}$$
$$\mathrm{H}_n(X) \text{ f.g. } \forall n \iff \left\{ i \in I \,\middle|\, \mathrm{H}_n(X^i) \neq 0 \right\} \text{ finite } \forall n \text{ and } \mathrm{H}_n(X^i) \text{ f.g. } \forall i \in I, \forall n\,.$$

The two conditions on the right are equivalent to $\exists Y^i \simeq X^i$ such that the set $\left\{ Y^i \,\middle|\, i \in I \right\}$ is descending. □

Following [OŠ12, Theorem 18], this gives a classification of the cocompact objects.

Proposition 5.1.3. *Let R be a noetherian ring. Then*

$$\mathrm{D}^{\mathrm{f}}(R) = \mathrm{D}_+(R\text{-}\mathrm{mod})^{cc}\,.$$

Proof. I first verify that any object in $\mathrm{D}^{\mathrm{f}}(R)$ is cocompact in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. Let $C \in \mathrm{D}^{\mathrm{f}}(R)$. One may assume $C_n = 0$ for $n \gg 0$. Set $N := \max\{n \mid C_n \neq 0\}$. Let $\left\{ X^i \,\middle|\, i \in I \right\}$ be a set of complexes in $\mathrm{D}_+(R\text{-}\mathrm{mod})$, such that their product exists in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. By Lemma 5.1.2, I may assume this set is descending. Then the set

$$J := \left\{ i \in I \,\middle|\, X^i_n \neq 0 \text{ for some } n \le N \right\}$$

is finite. For any complex Y with $Y_n = 0$ for $n \le N$, one has $\mathrm{Hom}_{\mathrm{D}(R)}(Y, C) = 0$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(R)}(\prod_{i\in I} X^i, C) &= \mathrm{Hom}_{\mathrm{D}(R)}(\prod_{i\in J} X^i \oplus \prod_{i\notin J} X^i, C) \\
&= \mathrm{Hom}_{\mathrm{D}(R)}(\prod_{i\in J} X^i, C) \oplus \mathrm{Hom}_{\mathrm{D}(R)}(\prod_{i\notin J} X^i, C) \\
&= \mathrm{Hom}_{\mathrm{D}(R)}(\prod_{i\in J} X^i, C) \\
&= \coprod_{i\in J} \mathrm{Hom}_{\mathrm{D}(R)}(X^i, C) \\
&= \coprod_{i\in I} \mathrm{Hom}_{\mathrm{D}(R)}(X^i, C)\,.
\end{aligned}$$

For the opposite inclusion, let $C \in \mathrm{D}_+(R\text{-}\mathrm{mod})$ be a cocompact object. One may assume $C_n = 0$ for $n \ll 0$ and C_n is finitely generated for all n. Set $N := \min\{n \mid C_n \neq 0\}$. The set $\{\Sigma^i \mathrm{Z}_i(C)\}_{i \ge N}$ is descending, where $\mathrm{Z}_i(C) = \ker(\partial_i)$ are the cycles in degree i.

So the product exists in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. Now there are canonical maps $f_i\colon \Sigma^i \mathrm{Z}_i(C) \to C$. Because all $\mathrm{Z}_i(C)$ are concentrated in different degrees, one has

$$\prod_{i\geq N} \Sigma^i \mathrm{Z}_i(C) = \coprod_{i\geq N} \Sigma^i \mathrm{Z}_i(C)$$

and by the universal property for the coproduct, there exists an induced map

$$f\colon \prod_{i\geq N} \Sigma^i \mathrm{Z}_i(C) \to C\,.$$

Since C is cocompact, all but finitely many of the maps f_i vanish in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. That is they are null-homotopic and f_i factors through ∂_{i-1} for almost all i. Then $\mathrm{H}_i(C) = 0$ for almost all i. So $C \in \mathrm{D}^{\mathrm{f}}(R)$. □

5.1.4. Let R be a noetherian ring. By 2.5.6 and 5.1.3, for $G, X \in \mathrm{D}^{\mathrm{f}}(R)$, the equality

$$\mathrm{level}^G(X) = \mathrm{level}^{\mathrm{Prod}_+(G)}(X)$$

holds, where $\mathrm{Prod}_+(G)$ is the smallest subcategory of $\mathrm{D}_+(R\text{-}\mathrm{mod})$, which contains G and is closed under all products that exist in $\mathrm{D}_+(R\text{-}\mathrm{mod})$.

Lemma 5.1.5. *Let R be a noetherian ring and $G, X \in \mathrm{D}^{\mathrm{f}}(R)$. Then*

$$\mathrm{cogin}^G_{\mathrm{D}^{\mathrm{f}}(R)}(X) = \mathrm{cogin}^{\mathrm{Prod}_+(G)}_{\mathrm{D}_+(R\text{-}\mathrm{mod})}(X)\,.$$

Proof. By Lemma 3.2.1 (5), a map is G-coghost if and only if it is $\mathrm{Prod}_+(G)$-coghost. So clearly, the left-hand side is no more than the right-hand side. To show the opposite inequality, I will construct from a non-zero composition in $\mathrm{D}_+(R\text{-}\mathrm{mod})$ of G-coghost maps ending in X a non-zero composition in $\mathrm{D}^{\mathrm{f}}(R)$ of G-ghost maps ending in X of the same length.

Given a non-zero composition of G-coghost maps in $\mathrm{D}_+(R\text{-}\mathrm{mod})$

$$X^n \to X^{n-1} \to \cdots \to X^1 \to X^0 = X$$

I replace the X^i's by their projective resolutions. Then the X^i_d's are finitely generated projective modules and for every i, one has $X^i_d = 0$ for $d \ll 0$. One has

$$\mathrm{Hom}_{\mathrm{D}(R)}(X^n, X) = \mathrm{Hom}_{\mathrm{D}(R)}((X^n)_{\leq i_n}, X) \quad \text{for some } i_n \gg 0\,.$$

In particular, the composition $(X^n)_{\leq i_n} \to X^n \to \cdots \to X$ is non-zero. The goal is to split this map into a composition of n G-coghost maps in $\mathrm{D}^{\mathrm{f}}(R)$. The object $(X^n)_{\leq i_n}$ is a perfect complex, so the set $J := \left\{ j \in \mathbb{Z} \,\middle|\, \operatorname{Hom}_{\mathrm{D}(R)}((X^n)_{\leq i_n}, \Sigma^j G) \neq 0 \right\}$ is finite and there exists $i_{n-1} \geq i_n$, such that

$$\operatorname{Hom}_{\mathrm{D}(R)}((X^{n-1})_{\leq i_{n-1}}, \Sigma^j G) = \operatorname{Hom}_{\mathrm{D}(R)}(X^{n-1}, \Sigma^j G) \quad \text{for all } j \in J\,.$$

Then the induced map $(X^n)_{\leq i_n} \to (X^{n-1})_{\leq i_{n-1}}$ is G-coghost. Repeating this process gives a commutative diagram

$$\begin{array}{ccccccccc} X^n & \longrightarrow & X^{n-1} & \longrightarrow & \cdots & \longrightarrow & X^1 & \longrightarrow & X \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ (X^n)_{\leq i_n} & \longrightarrow & (X^{n-1})_{\leq i_{n-1}} & \longrightarrow & \cdots & \longrightarrow & (X^1)_{\leq i_1} & \longrightarrow & (X)_{\leq i_0} \end{array}$$

where the horizontal maps are G-coghost and the composition $(X^n)_{\leq i_n} \to \cdots \to X$ is non-zero. Moreover, the complexes $(X^l)_{\leq i_l}$ are perfect and in particular lie in $\mathrm{D}^{\mathrm{f}}(R)$. Thus

$$\operatorname{cogin}^G_{\mathrm{D}_+(R\text{-}\mathrm{mod})}(X) \leq \operatorname{cogin}^G_{\mathrm{D}^{\mathrm{f}}(R)}(X)$$

and by 3.2.2, equality holds. □

It is left to establish a converse coghost lemma in $\mathrm{D}_+(R\text{-}\mathrm{mod})$.

A *Noether algebra* is a noetherian ring R, which is finitely generated as a module over its center $\mathrm{Z}(R)$.

Lemma 5.1.6. *Let R be a Noether algebra and $G \in \mathrm{D}^{\mathrm{f}}(R)$. Then* $\operatorname{Prod}_+(G)$ *is covariantly finite in* $\mathrm{D}_+(R\text{-}\mathrm{mod})$.

Proof. I may assume G is bounded, that is $G_n = 0$ for $|n| \gg 0$. Let Y be an object in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. Then for any i, the abelian group $\operatorname{Hom}_{\mathrm{D}(R)}(Y, \Sigma^i G)$ is a finitely generated module over the center $\mathrm{Z}(R)$. Let $f_{i,1}, \dots, f_{i,n_i}$ be a set of generators. Since

$$\operatorname{Hom}_{\mathrm{D}(R)}(Y, \Sigma^i G) = 0 \quad \text{for } i \ll 0$$

one can choose $n_i = 0$ for $i \ll 0$. Then the set $\left\{ \Sigma^i G^{n_i} \,\middle|\, i \in \mathbb{Z} \right\}$ is descending and the product $G_Y := \prod_{i \in \mathbb{Z}} \Sigma^i G^{n_i}$ exists in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. By the universal property of the product, there exists a map $n(Y) \colon Y \to G_Y$ such that post-composition with the projection onto the component (i, l) is the map $f_{i,l}$. This map $n(Y)$ is a left $\operatorname{Prod}_+(G)$-approximation of Y: Let $g \colon Y \to \Sigma^j G$ be some morphism. It can be written as $g = \sum_l a_{j,l} f_{j,l}$ for $a_{j,l} \in \mathrm{Z}(R)$, and it factors as

$$Y \xrightarrow{n(Y)} G_Y \to \Sigma^j G\,.$$

The second map is induced by the zero map when $i \neq j$, and multiplication by $a_{j,l}$ on the (j,l)th component. This map exists since the coproduct and the product are the same in $\mathrm{D}_+(R)$. □

Corollary 5.1.7. *Let R be a Noether algebra. Then for $G \in \mathrm{D}^{\mathrm{f}}(R)$ and $X \in \mathrm{D}_+(R\text{-}\mathrm{mod})$, one has the equality*

$$\mathrm{level}^{\mathrm{Prod}_+(G)}(X) = \mathrm{cogin}^{\mathrm{Prod}_+(G)}_{\mathrm{D}_+(R\text{-}\mathrm{mod})}(X)\,.$$

Proof. This follows from 3.3.8 and 5.1.6. □

Theorem 5.1.8. *Let R be a Noether algebra. Then for any $G, X \in \mathrm{D}^{\mathrm{f}}(R)$, one has the equality*

$$\mathrm{level}^{G}(X) = \mathrm{cogin}^{G}_{\mathrm{D}^{\mathrm{f}}(R)}(X)\,.$$

Proof. By 5.1.4, and 5.1.7, and 5.1.5, one has

$$\mathrm{level}^{G}(X) = \mathrm{level}^{\mathrm{Prod}_+(G)}(X) = \mathrm{cogin}^{\mathrm{Prod}_+(G)}_{\mathrm{D}_+(R\text{-}\mathrm{mod})}(X) = \mathrm{cogin}^{G}_{\mathrm{D}^{\mathrm{f}}(R)}(X)\,.$$ □

5.2 Converse Ghost Lemma

All the steps but Lemma 5.1.5 in the previous section can be adjusted by replacing coghost with ghost maps and $\mathrm{D}_+(R\text{-}\mathrm{mod})$ by $\mathrm{D}_-(R\text{-}\mathrm{mod})$. In 5.1.5, the projective resolution needs to be replaced by an injective resolution. However, there need not exist enough injective modules in the category of finitely generated modules. So it is not possible to truncate the injective resolutions and stay in $\mathrm{D}^{\mathrm{f}}(R)$. It is still possible to establish a converse ghost lemma if the ring has a dualizing complex.

5.2.1. Let $\mathsf{d}\colon \mathcal{S}^{op} \to \mathcal{T}$ and $\mathsf{d}'\colon \mathcal{S}^{op} \to \mathcal{T}$ be a duality of triangulated categories in the sense that d and d' are contravariant functors, and $\mathsf{d}\mathsf{d}' \cong \mathrm{id}_{\mathcal{S}}$ and $\mathsf{d}'\mathsf{d} \cong \mathrm{id}_{\mathcal{T}}$. The duality interchanges ghost and coghost maps, so that

$$\mathrm{gin}^{G}_{\mathcal{S}}(X) = \mathrm{cogin}^{\mathsf{d}(G)}_{\mathcal{T}}(\mathsf{d}(X)) \quad \text{and} \quad \mathrm{cogin}^{G}_{\mathcal{S}}(X) = \mathrm{gin}^{\mathsf{d}(G)}_{\mathcal{T}}(\mathsf{d}(X))\,; \qquad (5.2.2)$$

compare to 3.2.1 (4). Thus the converse coghost lemma holds for $G \in \mathcal{S}$ in $\mathcal{S}$ if and only if the converse ghost lemma holds for $\mathsf{d}(G) \in \mathcal{T}$ in $\mathcal{T}$.

The dualizing complex gives a class of dualities on the derived categories. The following definition was introduced by [CFH06, Definition 1.1].

Definition 5.2.3. Let S be a left noetherian ring and R a right noetherian ring. A *dualizing complex* of the ordered pair $\langle S, R\rangle$ is a complex ω of S-R-bimodules, such that

1. ω is a bounded complex of injective modules over S and R^{op},
2. $\mathrm{H}(\omega)$ is finitely generated over S and R^{op},
3. there exists a quasi-isomorphism $P \xrightarrow{\sim} \omega$ where P is a bounded below complex of projective modules over S and R^{op}, and
4. the canonical morphisms
$$S \to \mathbf{R}\mathrm{Hom}_{R^{op}}(\omega,\omega) \quad \text{and} \quad R \to \mathbf{R}\mathrm{Hom}_S(\omega,\omega)$$
are quasi-isomorphisms.

If R is additionally left noetherian, there exists a contravariant auto-equivalence

$$\mathrm{D^f}(S) \underset{\mathbf{R}\mathrm{Hom}_{R^{op}}(-,\omega)}{\overset{\mathbf{R}\mathrm{Hom}_S(-,\omega)}{\rightleftarrows}} \mathrm{D^f}(R^{op})\,;$$

see [IK06, 3.4]. These functors send ghost maps to coghost maps and reverse.

Theorem 5.2.4 (Converse ghost lemma)**.** *Let S be a left noetherian ring and R a Noether algebra with ω a dualizing complex of $\langle S, R\rangle$. Fix $G \in \mathrm{D^f}(S)$. Then for any $X \in \mathrm{D^f}(S)$, one has*

$$\mathrm{gin}^G_{\mathrm{D^f}(S)}(X) = \mathrm{level}^G_S(X)\,.$$

Proof. Set $(-)^\dagger := \mathbf{R}\mathrm{Hom}_S(-,\omega)$ and $(-)^{\dagger'} := \mathbf{R}\mathrm{Hom}_{R^{op}}(-,\omega)$. These functors are a duality in the sense of 5.2.1. Then

$$\mathrm{level}^G_S(X) = \mathrm{level}^{G^\dagger}_{R^{op}}(X^\dagger) = \mathrm{cogin}^{G^\dagger}_{\mathrm{D^f}(R^{op})}(X^\dagger) = \mathrm{gin}^G_{\mathrm{D^f}(S)}(X)$$

where the converse coghost Lemma 5.1.8 gives the equality in the middle. □

If R is a commutative noetherian ring, the definition of a dualizing complex of $\langle R, R\rangle$ coincides with Grothendieck's definition of a dualizing complex [Har66, V §2]. Then R has a dualizing complex if and only if it is the homomorphic image of a Gorenstein ring of finite Krull dimension (see [Kaw02, Corollary 1.4]). So for any such ring, the converse ghost lemma also holds.

5.3 Finite Flat Dimension

In this section, I look at cases when level is unchanged by the functor $W \otimes^{\mathbf{L}}_R -$ for a complex of S-R-bimodules W.

Lemma 5.3.1. *Let R and S be noetherian rings, $X \in \mathrm{D}^{\mathrm{f}}(S)$ and Y a complex of S-R-bimodules and $W \in \mathrm{D}(R)$. Assume one of the following conditions is satisfied*

1. *X is perfect, or*
2. *Y is bounded above, that is $Y_i = 0$ for $i \gg 0$, and $W \in \operatorname{thick}_R(R\text{-}\mathrm{Flat})$.*

Then the natural morphism of complexes of abelian groups

$$\mathbf{R}\mathrm{Hom}_S(X, Y) \otimes^{\mathbf{L}}_R W \to \mathbf{R}\mathrm{Hom}_S(X, Y \otimes^{\mathbf{L}}_R W)$$

is a quasi-isomorphism.

Proof. For (1), the claim holds for $X = R$ and thus by 2.3.6 for any perfect complex. For (2), one first proves the claim for flat modules. Then the claim holds again by 2.3.6. □

5.3.2. For the rest of the section, suppose R is a commutative noetherian ring and S a noetherian ring. Let W be a complex of S-R-bimodules, such that W is a bounded complex of finitely generated projective S-modules and has finite flat dimension over R. Additionally, let the left and right action of R on $\mathrm{Hom}_S(W, W)$ be the same, that is the canonical map $R \to \mathrm{Hom}_S(W, W)$ is central. Note, the complex W has finite flat dimension over R if and only if it lies in $\operatorname{thick}_R(R\text{-}\mathrm{Flat})$; for details, see Section 4.4.

This gives adjoint functors

$$\mathrm{D}(R) \underset{\mathsf{h} := \mathrm{Hom}_S(W, -)}{\overset{\mathsf{t} := W \otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathrm{D}(S)\,, \tag{5.3.3}$$

and t restricts to a functor from $\mathrm{D}^{\mathrm{f}}(R)$ to $\mathrm{D}^{\mathrm{f}}(S)$. I track how coghost maps behave under the functor t, when restricted to $\mathrm{D}^{\mathrm{f}}(R) \to \mathrm{D}^{\mathrm{f}}(S)$.

Lemma 5.3.4. *As an R-complex, $\mathrm{Hom}_S(W, W)$ lies in $\operatorname{thick}_R(R\text{-}\mathrm{Flat})$.*

Proof. Since $W = \mathrm{Hom}_S(S, W)$ lies in $\operatorname{thick}_R(R\text{-}\mathrm{Flat})$, the complex $\mathrm{Hom}_S(P, W)$ lies in $\operatorname{thick}_R(R\text{-}\mathrm{Flat})$ for any perfect complex P over S. In particular, the complex $\mathrm{Hom}_S(W, W)$ lies in $\operatorname{thick}_R(R\text{-}\mathrm{Flat})$. □

Lemma 5.3.5. *For any $X, Y \in \mathrm{D^f}(R)$, there is a quasi-isomorphism*

$$\mathbf{R}\mathrm{Hom}_R(X,Y) \otimes^{\mathbf{L}}_R \mathrm{Hom}_S(W,W) \simeq \mathbf{R}\mathrm{Hom}_S(\mathsf{t}(X),\mathsf{t}(Y)).$$

Proof. Since R is commutative, the left R-action on Y induces a right R-action on Y. Also, the left and right R-action on $\mathrm{Hom}_S(W,W)$ are the same, so that there is a natural isomorphism

$$Y \otimes^{\mathbf{L}}_R \mathrm{Hom}_S(W,W) \cong \mathrm{Hom}_S(W,W) \otimes^{\mathbf{L}}_R Y.$$

One has the following equivalences

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_R(X,Y) \otimes^{\mathbf{L}}_R \mathrm{Hom}_S(W,W) &\overset{5.3.1\,(2)}{\simeq} \mathbf{R}\mathrm{Hom}_R(X, \mathrm{Hom}_S(W,W) \otimes^{\mathbf{L}}_R Y) \\ &\overset{5.3.1\,(1)}{\simeq} \mathbf{R}\mathrm{Hom}_R(X, \mathsf{h}(\mathsf{t}(Y))) \\ &\cong \mathbf{R}\mathrm{Hom}_S(\mathsf{t}(X),\mathsf{t}(Y)). \end{aligned}$$

The last step holds by the adjunction in (5.3.3). □

The next lemma shows how coghost maps act under the functor t. The statement is similar to [AIN18, Lemma 2.6].

Lemma 5.3.6. *Let V be an R-complex with $\mathrm{level}^{R\text{-}\mathrm{Flat}}_R(V) \leq l$ and G in $\mathrm{D^f}(R)$. Then for any l-fold R-ghost map $f\colon X \to Y$ in $\mathrm{D^f}(R)$, the map $f \otimes^{\mathbf{L}}_R V$ is R-ghost.*

Proof. It is enough to show $\mathrm{H}(f \otimes^{\mathbf{L}}_R V) = 0$. Use induction on l. If $l = 1$, I may assume V is a flat module concentrated in degree 0. Then it commutes with taking homology, and

$$\mathrm{H}(f \otimes^{\mathbf{L}}_R V) \cong \mathrm{H}(f) \otimes_R V = 0$$

since f is R-ghost. For $l > 1$, there exists an exact triangle $V' \to V \to V''$ with $\mathrm{level}^{\mathcal{F}}_R(V') = 1$ and $\mathrm{level}^{\mathcal{F}}_R(V'') = l-1$. Since f is an l-fold R-ghost map, it can be written as $f = h \circ g$ with $h\colon X \to Z$ an $(l-1)$-fold R-ghost map and $h\colon Z \to Y$ an R-ghost map. Applying $\mathrm{H}(- \otimes^{\mathbf{L}}_R -)$ to the composition in the first component and the exact triangle in the second component gives the commuting diagram

$$\begin{array}{ccccc} \mathrm{H}(X \otimes^{\mathbf{L}}_R V') & \longrightarrow & \mathrm{H}(X \otimes^{\mathbf{L}}_R V) & \longrightarrow & \mathrm{H}(X \otimes^{\mathbf{L}}_R V'') \\ \downarrow{\scriptstyle \mathrm{H}(g \otimes^{\mathbf{L}}_R V')=0} & & \downarrow{\scriptstyle \mathrm{H}(g \otimes^{\mathbf{L}}_R V)} & & \downarrow{\scriptstyle \mathrm{H}(g \otimes^{\mathbf{L}}_R V'')=0} \\ \mathrm{H}(Z \otimes^{\mathbf{L}}_R V') & \longrightarrow & \mathrm{H}(Z \otimes^{\mathbf{L}}_R V) & \longrightarrow & \mathrm{H}(Z \otimes^{\mathbf{L}}_R V'') \\ \downarrow{\scriptstyle \mathrm{H}(h \otimes^{\mathbf{L}}_R V')=0} & & \downarrow{\scriptstyle \mathrm{H}(h \otimes^{\mathbf{L}}_R V)} & & \downarrow{\scriptstyle \mathrm{H}(h \otimes^{\mathbf{L}}_R V'')} \\ \mathrm{H}(Y \otimes^{\mathbf{L}}_R V') & \longrightarrow & \mathrm{H}(Y \otimes^{\mathbf{L}}_R V) & \longrightarrow & \mathrm{H}(Y \otimes^{\mathbf{L}}_R V''). \end{array}$$

In the diagram, all rows are exact and the maps on the left and the vertical map from the top right corner are zero by induction. Then the composition of the middle column is also zero and thus $\mathrm{H}(f \otimes^{\mathbf{L}}_R V) = 0$. So $f \otimes^{\mathbf{L}}_R V$ is R-ghost. □

Corollary 5.3.7. *If* $\operatorname{level}^{R\text{-}\mathrm{Flat}}_R(\operatorname{Hom}_S(W,W)) \leq l$ *and* f *is an* l*-fold* G*-coghost map in* $\mathrm{D}^{\mathrm{f}}(R)$*, then* $\mathsf{t}(f)$ *is* $\mathsf{t}(G)$*-coghost.*

Proof. By 4.3.7, the map $\mathbf{R}\operatorname{Hom}_R(f,G)$ is an l-fold R-ghost map. By Lemma 5.3.6, I have $\mathrm{H}(\mathbf{R}\operatorname{Hom}_R(f,G) \otimes^{\mathbf{L}}_R \operatorname{Hom}_S(W,W)) = 0$, and Lemma 5.3.5 gives the identification

$$\mathrm{H}(\mathbf{R}\operatorname{Hom}_R(f,G) \otimes^{\mathbf{L}}_R \operatorname{Hom}_S(W,W)) \cong \operatorname{Ext}_{\mathrm{D}(S)}(\mathsf{t}(f),\mathsf{t}(G)) .$$

Thus $\mathsf{t}(f)$ is $\mathsf{t}(G)$-coghost. □

From the corollary, it follows that if $\operatorname{Hom}_S(W,W)$ is isomorphic in $\mathrm{D}(R)$ to a finite direct sum of suspensions of flat modules, the functor t preserves coghost maps. This does not imply that it also preserves the coghost index. For that, the functor t needs to be faithful.

Lemma 5.3.8. *If* $\operatorname{Hom}_S(W,W) \in \operatorname{add}(R\text{-}\mathrm{Flat})$ *and* t *is faithful, then for* X *and* G *in* $\mathrm{D}^{\mathrm{f}}(R)$

$$\operatorname{cogin}^{G}_{\mathrm{D}^{\mathrm{f}}(R)}(X) \leq \operatorname{cogin}^{\mathsf{t}(G)}_{\mathrm{D}^{\mathrm{f}}(S)}(\mathsf{t}(X)) .$$

Proof. Given a non-zero n-fold G-coghost map f. By 5.3.7, the map $\mathsf{t}(f)$ is n-fold $\mathsf{t}(G)$-coghost, and it is non-zero, because t is faithful. □

Theorem 5.3.9. *Suppose* R *is a commutative noetherian ring and* S *a noetherian ring. Let* W *be a complex of* S*-*R*-bimodules and set*

$$\mathsf{t} := W \otimes^{\mathbf{L}}_R - : \mathrm{D}^{\mathrm{f}}(R) \to \mathrm{D}^{\mathrm{f}}(S) .$$

Assume

- W *is a bounded complex of finitely generated projective* S*-modules,*
- W *has finite flat dimension over* R*,*
- *the natural map* $R \to \operatorname{Hom}_S(W,W)$ *is central,*
- $\operatorname{Hom}_S(W,W) \in \operatorname{add}(R\text{-}\mathrm{Flat})$*, and*
- t *is faithful.*

Then for any $G, X \in \mathrm{D^f}(R)$, *one has* $\operatorname{level}_R^G(X) = \operatorname{level}_S^{\mathsf{t}(G)}(\mathsf{t}(X))$.

Proof. I have the (in)equalities

$$\operatorname{level}_R^G(X) = \operatorname{cogin}_{\mathrm{D^f}(R)}^G(X) \le \operatorname{cogin}_{\mathrm{D^f}(S)}^{\mathsf{t}(G)}(\mathsf{t}(X)) \le \operatorname{level}_S^{\mathsf{t}(G)}(\mathsf{t}(X))$$

where the equality holds by the converse coghost lemma 5.1.8. The first inequality holds by 5.3.8, and the second by 3.1.6. The opposite inequality holds by 2.3.3 (3). □

Note in this proof that the converse coghost lemma does not need to hold in $\mathrm{D^f}(S)$. I only require it to hold in $\mathrm{D^f}(R)$.

An important class of examples for which Theorem 5.3.9 applies comes from faithfully flat ring maps $\varphi\colon R \to S$ with R a commutative noetherian ring and S a noetherian ring. This induces the functor

$$\varphi^* := S \otimes_R - \colon \mathrm{D^f}(R) \to \mathrm{D^f}(S)\,.$$

Lemma 5.3.10. *If S is faithfully flat as an R-module, then the functor φ^* is faithful.*

Proof. Since φ is faithful the map of abelian groups

$$\operatorname{Hom}_{\mathrm{D}(R)}(X, Y) \hookrightarrow S \otimes_R \operatorname{Hom}_{\mathrm{D}(R)}(X, Y)$$

is injective. Because S is flat, one has

$$\begin{aligned} S \otimes_R \operatorname{Hom}_{\mathrm{D}(R)}(X, Y) &\cong \mathrm{H}_0(S \otimes_R \mathbf{R}\mathrm{Hom}_R(X, Y)) \\ &\overset{5.3.1\,(2)}{\cong} \operatorname{Hom}_{\mathrm{D}(R)}(X, \varphi^*(Y)) \\ &\cong \operatorname{Hom}_{\mathrm{D}(S)}(\varphi^*(X), \varphi^*(Y))\,. \end{aligned}$$

The last equivalence holds by adjunction. □

If R acts centrally on S, then the functor φ^* with $W = S$ satisfies all the conditions of Theorem 5.3.9. The following answers a question posed in [DGI06, Remark 9.6].

Corollary 5.3.11. *Let $\varphi\colon R \to S$ be a faithfully flat ring map with R a commutative ring and S a noetherian ring, so that R acts centrally on S. For $X, G \in \mathrm{D^f}(R)$, one has* $\operatorname{level}_R^G(X) = \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X))$. □

In particular, level remains unchanged after completion.

Corollary 5.3.12. *Let $(R, \mathfrak{m}, k)$ be a local ring and let $\widehat{(-)}$ be the completion with respect to $\mathfrak{m}$. Then for any $X, G \in \mathrm{D^f}(R)$, one has*

$$\operatorname{level}_R^G(X) = \operatorname{level}_{\widehat{R}}^{\widehat{G}}(\widehat{X})\,.$$ □

5.4 A Local to Global Principle

In this section, I investigate the behavior of level and finite generation in the derived category of a Noether algebra under the localization at prime ideals of the center.

Let R be a Noether algebra with center $\mathrm{Z}(R)$ and $\mathfrak{p}$ a prime ideal of $\mathrm{Z}(R)$. Then write $(-)_\mathfrak{p} := R_\mathfrak{p} \otimes_R -$ for the localization functor. For any left R-module M, one has

$$M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M \cong (\mathrm{Z}(R)_\mathfrak{p} \otimes_{\mathrm{Z}(R)} R) \otimes_R M \cong \mathrm{Z}(R)_\mathfrak{p} \otimes_{\mathrm{Z}(R)} M$$

as a left module over $\mathrm{Z}(R)_\mathfrak{p} \otimes_{\mathrm{Z}(R)} R \cong R_\mathfrak{p}$. Since $\mathrm{Z}(R) \to \mathrm{Z}(R)_\mathfrak{p}$ is flat, so is the ring map $R \to R_\mathfrak{p}$. These maps need not be faithful.

An R-module M is zero if and only if $M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$ of $\mathrm{Z}(R)$. Thus a map of R-modules f is zero if and only if $f_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$. The same holds for maps in the derived category:

Lemma 5.4.1. *Let $f\colon X \to Y$ be a morphism in $\mathrm{D^f}(R)$. Then the following conditions are equivalent*

1. *$f = 0$ in $\mathrm{D}(R)$,*
2. *$f_\mathfrak{p} = 0$ in $\mathrm{D}(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))$, and*
3. *$f_\mathfrak{m} = 0$ in $\mathrm{D}(R_\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max}(\mathrm{Z}(R))$.*

Proof. (1) $\implies$ (2) and (2) $\implies$ (3) are obvious. For (3) $\implies$ (1): Since X and Y lie in $\mathrm{D^f}(R)$, I have

$$\operatorname{Hom}_{\mathrm{D}(R)}(X,Y)_\mathfrak{m} = \operatorname{Hom}_{\mathrm{D}(R_\mathfrak{m})}(X_\mathfrak{m}, Y_\mathfrak{m})\,.$$

Then the submodule $\mathrm{Z}(R) \cdot f$ of the left side is locally zero, since

$$(\mathrm{Z}(R) \cdot f)_\mathfrak{m} = \mathrm{Z}(R_\mathfrak{m}) \cdot f_\mathfrak{m} = 0 \quad \forall \mathfrak{m} \in \operatorname{Max}(\mathrm{Z}(R))\,.$$

Then $\mathrm{Z}(R) \cdot f = 0$ and $f = 0$. □

Lemma 5.4.2. *Given the map $f\colon X \to Y$ in $\mathrm{D^f}(R)$. The subset*

$$\{\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R)) \mid f_\mathfrak{p} = 0 \text{ in } \mathrm{D}(R)\}$$

of $\operatorname{Spec}(\mathrm{Z}(R))$ is open in the Zariski topology.

Proof. The map f fits in an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X\,.$$

Applying $\mathrm{Hom}_{\mathrm{D}(R)}(X,-)$ to this exact triangle gives the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathrm{D}(R)}(X,X) \xrightarrow{f_*(X)} \mathrm{Hom}_{\mathrm{D}(R)}(X,Y) \xrightarrow{g_*(X)} \mathrm{Hom}_{\mathrm{D}(R)}(X,Z) \to \cdots\,.$$

Then $f = 0$ if and only if $\ker(g_*(X)) = 0$. Since X and Y are in $\mathrm{D}^{\mathrm{f}}(R)$, one has

$$(\ker(g_*(X)))_{\mathfrak{p}} = \ker((g_*(X))_{\mathfrak{p}}) = \ker((g_{\mathfrak{p}})_*(X))\,.$$

A module being zero is a open property, so the set

$$\{\mathfrak{p} \in \mathrm{Spec}(\mathrm{Z}(R)) \mid f_{\mathfrak{p}} = 0\} = \{\mathfrak{p} \in \mathrm{Spec}(\mathrm{Z}(R)) \mid (\ker(g_*(X)))_{\mathfrak{p}} = 0\}$$

is open in the Zariski topology. □

Lemma 5.4.3. *Fix $G \in \mathrm{D}^{\mathrm{f}}(R)$. Then for any $X \in \mathrm{D}^{\mathrm{f}}(R)$, one has*

$$\begin{aligned}\mathrm{cogin}^{G}_{\mathrm{D}^{\mathrm{f}}(R)}(X) &\le \sup\left\{\mathrm{cogin}^{G_{\mathfrak{m}}}_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{m}})}(X_{\mathfrak{m}}) \,\middle|\, \mathfrak{m} \in \mathrm{Max}(\mathrm{Z}(R))\right\}\\ &\le \sup\left\{\mathrm{cogin}^{G_{\mathfrak{p}}}_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{p}})}(X_{\mathfrak{p}}) \,\middle|\, \mathfrak{p} \in \mathrm{Spec}(\mathrm{Z}(R))\right\}\,.\end{aligned}$$

Proof. Given a n-fold G-coghost map f. Then $f_{\mathfrak{m}}$ is a n-fold $G_{\mathfrak{m}}$-coghost map by 5.3.7. If $f_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$, then $f = 0$ by Lemma 5.4.1. This proves the first inequality. The second is obvious. □

The next theorem allows calculating level locally.

Theorem 5.4.4. *Let R be a Noether algebra. Fix G and X in $\mathrm{D}^{\mathrm{f}}(R)$. Then*

$$\begin{aligned}\mathrm{level}^{G}_{R}(X) &= \sup\left\{\mathrm{level}^{G_{\mathfrak{p}}}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \,\middle|\, \mathfrak{p} \in \mathrm{Spec}(\mathrm{Z}(R))\right\}\\ &= \sup\left\{\mathrm{level}^{G_{\mathfrak{m}}}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \,\middle|\, \mathfrak{m} \in \mathrm{Max}(\mathrm{Z}(R))\right\}\,.\end{aligned}$$

Proof. Given a prime ideal $\mathfrak{p}$, there exists a maximal ideal $\mathfrak{m} \supseteq \mathfrak{p}$ and by 2.3.3 (3), one has

$$\mathrm{level}^{G}_{R}(X) \ge \mathrm{level}^{G_{\mathfrak{m}}}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \ge \mathrm{level}^{G_{\mathfrak{p}}}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}})\,.$$

So it is enough to show the claim for all prime ideals. By the converse coghost lemma 5.1.8 and Lemma 5.4.3, one has

$$\begin{aligned}\mathrm{level}^{G}_{R}(X) &= \mathrm{cogin}^{G}_{\mathrm{D}^{\mathrm{f}}(R)}(X)\\ &\le \sup\left\{\mathrm{cogin}^{G_{\mathfrak{p}}}_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{p}})}(X_{\mathfrak{p}}) \,\middle|\, \mathfrak{p} \in \mathrm{Spec}(\mathrm{Z}(R))\right\}\\ &= \sup\left\{\mathrm{level}^{G_{\mathfrak{p}}}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \,\middle|\, \mathfrak{p} \in \mathrm{Spec}(\mathrm{Z}(R))\right\}\,.\end{aligned}$$

The opposite inequality follows from

$$\operatorname{level}_R^G(X) \geq \operatorname{level}_{R_\mathfrak{p}}^{G_\mathfrak{m}}(X_\mathfrak{p})$$

which holds by 2.3.3 (3) for all prime ideals $\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))$. □

In [BM67, Lemma 4.5], it is proved that a module M has finite projective dimension if and only if $M_\mathfrak{p}$ has finite projective dimension for all prime ideals $\mathfrak{p}$. This was extended to perfect complexes by [AIL10, Theorem 4.1]. The following result generalizes this to level with respect to any generator G. It complements Theorem 5.4.4, in that it is not only possible to compute level locally, but also to check finiteness of level locally.

Theorem 5.4.5. *Let R be a Noether algebra. Suppose G and X are objects in $\mathrm{D}^{\mathrm{f}}(R)$. Then for any integer n, the set*

$$\left\{\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R)) \,\middle|\, \operatorname{level}_{R_\mathfrak{p}}^{G_\mathfrak{p}}(X_\mathfrak{p}) \leq n\right\} \subseteq \operatorname{Spec}(\mathrm{Z}(R))$$

is Zariski open. Moreover, the following conditions are equivalent

1. $\operatorname{level}_R^G(X) < \infty$,
2. $\operatorname{level}_{R_\mathfrak{p}}^{G_\mathfrak{p}}(X_\mathfrak{p}) < \infty$ *for all* $\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))$, *and*
3. $\operatorname{level}_{R_\mathfrak{m}}^{G_\mathfrak{m}}(X_\mathfrak{m}) < \infty$ *for all* $\mathfrak{m} \in \operatorname{Max}(\mathrm{Z}(R))$.

Proof. By Lemma 5.1.6, the subcategory $\operatorname{Prod}_+(G)$ is covariantly finite in $\mathrm{D}_+(R\text{-}\mathrm{mod})$. Then by the analogs of 3.4.6 and 3.4.7 in the opposite category, there exists an Adams coresolution

$$\begin{array}{ccccccc} X = X^0 & \xleftarrow{f^1} & X^1 & \xleftarrow{f^2} & X^2 & \xleftarrow{f^3} & \cdots . \\ & \searrow \quad \nearrow_{+1} & & \searrow \quad \nearrow_{+1} & & \searrow \quad \nearrow_{+1} & \\ & H^0 & & H^1 & & H^2 & \end{array}$$

where the f^i's are G-coghost and $H^i \in \operatorname{thick}^1(\operatorname{Prod}_+(G))$ and $X^i \to H^i$ are left $\operatorname{Prod}_+(G)$-approximations.

By the analogous statement of 3.4.3 in the opposite category, I have

$$\operatorname{level}_R^G(X) = \operatorname{cogin}_{\mathrm{D}_+(R)}^G(X) = \inf\left\{n \geq 0 \,\middle|\, f^1 \circ \ldots \circ f^n = 0 \text{ in } \mathrm{D}(R)\right\} .$$

Then by 5.1.5, there exist G-coghost maps $g^i \colon Y^i \to Y^{i-1}$ with $Y^0 = X$ and Y^i perfect, such that a composition $g^1 \circ \ldots \circ g^n$ is zero if and only if $f^1 \circ \ldots \circ f^n$ is zero. That gives

$$\operatorname{level}^G(X) = \inf\left\{n \geq 0 \,\middle|\, g^1 \circ \ldots \circ g^n = 0\right\} .$$

While products need not localize in general, the products in $\mathrm{D}_+(R\text{-}\operatorname{mod})$ localize. The reason is that if a product exists, it is the componentwise product, and one may assume in each component the product is finite. Thus $\operatorname{Prod}_+(G_\mathfrak{p}) = \operatorname{Prod}_+(G)_\mathfrak{p}$.

The functor $\mathrm{D}^\mathrm{f}(R) \to \mathrm{D}^\mathrm{f}(R_\mathfrak{p})$ is full and by 3.3.7, the localization of a left $\operatorname{Prod}_+(G)$-approximation in $\mathrm{D}_+(R\text{-}\operatorname{mod})$ is a left $\operatorname{Prod}_+(G_\mathfrak{p})$-approximation in $\mathrm{D}_+(R_\mathfrak{p}\text{-}\operatorname{mod})$. So the Adams coresolution of X localizes to an Adams coresolution of $X_\mathfrak{p}$ in $\mathrm{D}_+(R_\mathfrak{p}\text{-}\operatorname{mod})$. The truncations used in 5.1.5 descend to the localization, so that $(f^1 \circ \cdots \circ f^n)_\mathfrak{p}$ is zero if and only if $(g^1 \circ \cdots \circ g^n)_\mathfrak{p}$ is zero. This gives

$$\mathcal{V}_n := \left\{\mathfrak{p} \in \operatorname{Spec}(R) \,\middle|\, \operatorname{level}^{G_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p}) \leq n\right\} = \left\{\mathfrak{p} \in \operatorname{Spec}(R) \,\middle|\, (g^1 \circ \ldots \circ g^n)_\mathfrak{p} = 0\right\}$$

is open by Lemma 5.4.2.

For the second part, (2) $\iff$ (3) and (1) $\implies$ (2) are clear. For (2) $\implies$ (1), assume $\operatorname{level}^{G_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p})$ is finite for all prime ideals $\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))$. That is the union of all $\mathcal{V}_n$ is $\operatorname{Spec}(\mathrm{Z}(R))$. The sets $\mathcal{V}_n$ form an ascending chain of open sets. Since $\mathrm{Z}(R)$ is noetherian, the space $\operatorname{Spec}(\mathrm{Z}(R))$ is noetherian and the chain stabilizes. So there exists an N, such that $\mathcal{V}_N = \mathcal{V}_n$ for $n \geq N$. Thus $\operatorname{Spec}(\mathrm{Z}(R)) = \mathcal{V}_N$, and $\operatorname{level}^{G_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p}) \leq N$ for all prime ideals $\mathfrak{p}$. By Theorem 5.4.4, then $\operatorname{level}^{G}_{R}(X) \leq N < \infty$. □

To detect whether an object is a strong generator locally, one has to be able to lift objects from the localizations. A functor is called *essentially surjective* if it is surjective on objects.

Lemma 5.4.6. *For any prime ideal* $\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))$*, the functor* $\mathrm{D}^\mathrm{f}(R) \to \mathrm{D}^\mathrm{f}(R_\mathfrak{p})$ *is essentially surjective.*

Proof. Every finitely generated module over $R_\mathfrak{p}$ can be lifted to a finitely generated module over R. Also any $R_\mathfrak{p}$-linear map can be lifted to a R-linear map. Given a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of finitely generated modules over $R_\mathfrak{p}$ with $g \circ f = 0$. This lifts to a sequence

$$\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{\tilde{g}} \tilde{Z}$$

of finitely generated modules over R. The composition $\tilde{g} \circ \tilde{f}$ need not be zero, but one has $((\tilde{g} \circ \tilde{f})(\tilde{X}))_\mathfrak{p} = 0$. Since X is finitely generated, there exists $r \in R \setminus \mathfrak{p}$, such that $r \cdot (\tilde{g} \circ \tilde{f})(X) = 0$. Replacing $\tilde{g}$ by $r\tilde{g}$ gives a sequence whose composition is zero. Since r is a unit in $R_\mathfrak{p}$, this sequence localizes, up to isomorphism, to the original sequence. Thus inductively, any bounded complex of finitely generated $R_{\mathfrak{p}}$-modules lifts to a complex of finitely generated R-modules. □

It is possible to detect a strong generator locally, though the generation time has to have an upper bound for all localizations.

Theorem 5.4.7. *Let R be a Noether algebra. Fix $G \in \mathrm{D}^{\mathrm{f}}(R)$ and a positive integer N. Then the following are equivalent:*

1. *G is a strong generator of $\mathrm{D}^{\mathrm{f}}(R)$ with $\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(G) \leq N$,*
2. *$G_{\mathfrak{p}}$ is a strong generator of $\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{p}})$ with $\circleddash_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{p}})}(G_{\mathfrak{p}}) \leq N$ for all prime ideals $\mathfrak{p}$ of $\mathrm{Z}(R)$, and*
3. *$G_{\mathfrak{m}}$ is a strong generator of $\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{m}})$ with $\circleddash_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{m}})}(G_{\mathfrak{m}}) \leq N$ for all maximal ideals $\mathfrak{m}$ of $\mathrm{Z}(R)$.*

Proof. (2) $\Longrightarrow$ (3) is obvious. For (1) $\Longrightarrow$ (2), let X be an object in $\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{p}})$. By 5.4.6, there exists $Y \in \mathrm{D}^{\mathrm{f}}(R)$ with $Y_{\mathfrak{p}} = X$. One has

$$\mathrm{level}_{R_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(X) \leq \mathrm{level}_{R}^{G}(Y) \leq \circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(G) + 1 \leq N + 1 \,.$$

So $G_{\mathfrak{p}}$ is a strong generator of $\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{p}})$ with generation time $\leq N$.

It remains to show (3) $\Longrightarrow$ (1). For any $X \in \mathrm{D}^{\mathrm{f}}(R)$, I have, by 5.4.4,

$$\begin{aligned} \mathrm{level}_{R}^{G}(X) &= \sup\left\{ \mathrm{level}_{R_{\mathfrak{m}}}^{G_{\mathfrak{m}}}(X_{\mathfrak{m}}) \;\middle|\; \mathfrak{m} \in \mathrm{Max}(\mathrm{Z}(R)) \right\} \\ &\leq \sup\left\{ \circleddash_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{m}})}(G_{\mathfrak{m}}) \;\middle|\; \mathfrak{m} \in \mathrm{Max}(\mathrm{Z}(R)) \right\} + 1 \leq N + 1 \,, \end{aligned}$$

and so G is a strong generator of $\mathrm{D}^{\mathrm{f}}(R)$ with $\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(G) \leq N$. □

This statement does not hold without a uniform bound on local generation time.

Example 5.4.8. In [Nag62, Appendix A1], Nagata constructed a commutative noetherian ring R of infinite Krull dimension, such that $R_{\mathfrak{m}}$ is regular and of finite Krull dimension for all maximal ideals $\mathfrak{m}$. So

$$\circleddash_{\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{m}})}(R_{\mathfrak{m}}) = \mathrm{gldim}(R_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}}) < \infty$$

for any maximal ideal $\mathfrak{m}$. But

$$\circleddash_{\mathrm{D}^{\mathrm{f}}(R)}(R) = \mathrm{gldim}(R) = \dim(R) = \infty \,.$$

So just because $G_{\mathfrak{m}}$ is a strong generator of $\mathrm{D}^{\mathrm{f}}(R_{\mathfrak{m}})$ for all $\mathfrak{m}$ does not mean G is a strong generator of $\mathrm{D}^{\mathrm{f}}(R)$.

Chapter 6

Applications

In this chapter, I present some applications of generation in the triangulated category, and of the local to global principles in Chapter 5. If a triangulated category has finite Rouquier dimension, or equivalently a strong generator, then one can weaken some of the conditions for the Brown representability theorems. More specifically, Rouquier proved [Rou08, 4.3] that an R-linear triangulated category need not have all coproducts. Instead, it has to fulfill some finiteness conditions. I extend this result to the graded case, when a graded ring acts on the triangulated category.

The local to global principles of Chapter 5 allow reducing questions about finite generation to the case of a complete local ring. Since these rings have a dualizing complex, I conclude from Hopkins and Neeman's theorem 4.7.5 a converse of 4.7.3 for complexes of finite injective dimension with finitely generated homology.

Last I extend the homotopical characterization of complete intersection by [Pol19] to a characterization of locally complete intersections.

6.1 Representable Functors

An object X in $\mathcal{T}$ induces a cohomological functor

$$\mathsf{h}_X := \operatorname{Hom}_{\mathcal{T}}(-, X)\colon \mathcal{T}^{op} \to \mathcal{A}b\,;$$

see 3.1.2. If a cohomological functor $\mathcal{T}^{op} \to \mathcal{A}b$ is naturally isomorphic to such a functor h_X for some $X \in \mathcal{T}$, then it is called *representable*.

There are a number of results in various settings when every 'reasonable' functor is representable. The first such result is due to Brown; see [Bro62]. In [Nee96, Theorem 3.1], the result for the triangulated category was established:

Theorem 6.1.1. *Let $\mathcal{T}$ be a triangulated category with a set $\mathcal{C}$ of compact objects such that every object X with*

$$\operatorname{Hom}_{\mathcal{T}}(\Sigma^i C, X) = 0 \quad \textit{for all } i \in \mathbb{Z} \textit{ and } C \in \mathcal{C}$$

is zero. Additionally, assume the coproduct of every set of objects exists in $\mathcal{T}$. If $\mathsf{h}\colon \mathcal{T}^{op} \to \mathcal{A}b$ is a cohomological functor such that for every set of objects $\mathcal{X}$ the natural map

$$\mathsf{h}(\coprod_{X \in \mathcal{X}} X) \to \prod_{X \in \mathcal{X}} \mathsf{h}(X)$$

is an isomorphism, then h is representable. □

The condition on the functor h is necessary, since it holds for any functor h_X; compare to 2.5.1.

The assumptions of this theorem are rather strong; it requires that every coproduct exists. For example, this does not hold for the derived category of complexes with bounded finitely generated homology $\mathrm{D}^{\mathrm{f}}(R)$. When the triangulated category has a strong generator, it is possible to weaken this assumption, by [Rou08, 4.3].

Let R be a commutative noetherian ring and $\mathcal{T}$ an R-linear triangulated category. That means for X and Y in $\mathcal{T}$, the set of morphisms $\operatorname{Hom}_{\mathcal{T}}(X, Y)$ is an R-module and the composition is R-bilinear. Then the representable functors h_X map to the category of R-modules, denoted R-Mod.

Definition 6.1.2. An R-linear triangulated category $\mathcal{T}$ is *Ext-finite* if for all $X, Y \in \mathcal{T}$, the R-module $\operatorname{Ext}_{\mathcal{T}}(X, Y)$ is finitely generated.

A triangulated category $\mathcal{T}$ is *Karoubian* if for every object X in $\mathcal{T}$ and every idempotent $e \in \operatorname{End}_{\mathcal{T}}(X)$, that means $e^2 = e$, there exists an object Y and maps

$$i\colon Y \to X \quad \text{and} \quad p\colon X \to Y$$

such that $p \circ i = \mathrm{id}_Y$ and $i \circ p = e$.

A cohomological functor $\mathsf{f}\colon \mathcal{T}^{op} \to R$-Mod is *locally finite* if the R-module $\coprod_{d \in \mathbb{Z}} \mathsf{f}(\Sigma^d Y)$ is finitely generated for all $Y \in \mathcal{T}$.

Note, in an Ext-finite category, locally finite is a necessary property for a representable functor.

Then Rouquier's result is [Rou08, Corollary 4.18]:

Theorem 6.1.3. *Let $\mathcal{T}$ be an Ext-finite, Karoubian triangulated category with a strong generator. Then a cohomological functor $\mathcal{T}^{op} \to R$-Mod is representable if and only if it is locally finite.* □

It is necessary to introduce the Ext-finite property to make sure that in the construction, all coproducts are finite. The assumption that $\mathcal{T}$ is Karoubian is weaker than the existence of all coproducts; see [Nee01, 1.6.8].

6.2 Representable Functors in the Graded Setting

Let R be a $\mathbb{Z}$-graded commutative noetherian ring. Say a triangulated category $\mathcal{T}$ is *graded R-linear* if

1. for any objects X and Y in $\mathcal{T}$, the abelian group $\operatorname{Ext}_{\mathcal{T}}(X, Y)$ is a graded R-module with the grading given by the direct sum in 3.1.4, and
2. the composition map is R-bilinear.

In this context, I say a functor $\mathcal{T}^{op} \to R\text{-}\operatorname{grMod}$ is *representable* if it is naturally isomorphic to a functor

$$\mathsf{g}_X := \operatorname{Ext}_{\mathcal{T}}(-, X) \colon \mathcal{T}^{op} \to R\text{-}\operatorname{grMod} .$$

Here R- grMod denotes the category of graded R-modules.

The nth shift $M[n]$ of a graded R-module M is given by $(M[n])_d = M_{n+d}$. Note that the suspension of $\mathcal{T}$ in the first component of $\operatorname{Ext}_{\mathcal{T}}(-,-)$ corresponds to the negative shift in R- Mod:

$$\mathsf{g}_Y(\Sigma^n X) = \operatorname{Ext}_{\mathcal{T}}(\Sigma^n X, Y) \cong \operatorname{Ext}_{\mathcal{T}}(X, Y)[-n] = \mathsf{g}_Y(X)[-n] . \tag{6.2.1}$$

I adapt the techniques used to prove 6.1.3, to obtain a similar statement in the graded case. The first step is to find a graded analog of Yoneda's lemma.

6.2.2. Yoneda's lemma states that in a category $\mathcal{C}$, the map

$$\operatorname{Nat}(\mathsf{h}_X, \mathsf{f}) \to \mathsf{f}(X) \quad \text{given by} \quad \eta \mapsto \eta(X)(\mathrm{id}_X)$$

is an isomorphism of sets for any functor $\mathsf{f} \colon \mathcal{C}^{op} \to \mathcal{S}et$ and any object X in $\mathcal{C}$.

For the rest of this section, let $\mathcal{T}$ be a graded R-linear category.

Lemma 6.2.3 (Graded version of Yoneda's lemma)**.** *Let* $\mathsf{f} \colon \mathcal{T}^{op} \to R\text{-}\operatorname{grMod}$ *be a functor that respects suspension and shift as in* (6.2.1)*, and* X *an object in* $\mathcal{T}$*. Then the map*

$$\operatorname{Nat}(\mathsf{g}_X, \mathsf{f}) \to \mathsf{f}(X)_0 \quad \textit{given by} \quad \eta \mapsto \eta(X)(\mathrm{id}_X)$$

is an isomorphism of abelian groups. The codomain is the degree 0 *component of the graded* R*-module* $\mathsf{f}(X)$*.*

Proof. First note that the given map is a map of abelian groups. It is enough to construct an inverse map. For $u \in \mathsf{f}(X)_0$, define a natural transformation

$$\eta_u \colon \mathsf{g}_X \to \mathsf{f} \quad \text{as} \quad \eta_u(Y)(f) := \mathsf{f}(f)(u)$$

where $Y \in \mathcal{T}$ and $f \in \operatorname{Ext}_{\mathcal{T}}(Y,X)$. Note $\eta_u(Y)$ is a homogeneous map of graded R-modules. Given a map $g \colon Y \to Z$ in $\mathcal{T}$, the diagram

$$\begin{array}{ccc} \mathsf{g}_X(Z) & \xrightarrow{g^*(X)} & \mathsf{g}_X(Y) \\ {\scriptstyle \eta_u(Z)}\downarrow & & \downarrow{\scriptstyle \eta_u(Y)} \\ \mathsf{f}(Z) & \xrightarrow{\mathsf{f}(g)} & \mathsf{f}(Y) \end{array}$$

commutes, and so η_u is a natural transformation. It remains to show the maps are inverse to each other. For $u \in \mathsf{f}(X)_0$, one has

$$\eta_u(X)(\mathrm{id}_X) = \mathsf{f}(\mathrm{id}_X)(u) = \mathrm{id}_{\mathsf{f}(X)}(u) = u\,.$$

For a natural transformation $\eta \colon \mathsf{g}_X \to \mathsf{f}$, set $u := \eta(X)(\mathrm{id}_X)$. Then

$$\eta_u(Y)(f) = \mathsf{f}(f)(u) = (\mathsf{f}(f) \circ \eta(X))(\mathrm{id}_X) = (\eta(Y) \circ f^*(X))(\mathrm{id}_X) = \eta(Y)(f)$$

for any map $f \in \operatorname{Ext}_{\mathcal{T}}(Y,X)$. So $\eta = \eta_u$. □

6.2.4. It is straightforward to check that for a map $f \colon X \to Y$, the following diagram commutes

$$\begin{array}{ccccc} \mathsf{f}(Y)_0 & \longrightarrow & \mathrm{Nat}(\mathsf{g}_Y, \mathsf{f}) & \longrightarrow & \mathsf{f}(Y)_0 \\ \downarrow{\scriptstyle \mathsf{f}(f)_0} & & \downarrow{\scriptstyle -\circ f_*} & & \downarrow{\scriptstyle \mathsf{f}(f)_0} \\ \mathsf{f}(X)_0 & \longrightarrow & \mathrm{Nat}(\mathsf{g}_X, \mathsf{f}) & \longrightarrow & \mathsf{f}(X)_0\,. \end{array}$$

So the two maps in 6.2.3 between $\mathrm{Nat}(\mathsf{g}_X, \mathsf{f})$ and $\mathsf{f}(X)_0$ are functorial in X.

The following definition for a resolution of a functor was introduced in [BvdB03, 2.3]. Let $(G_i, d_i)_{i>0}$ be a directed system in a category $\mathcal{C}$, that is

$$G_1 \xrightarrow{d_1} G_2 \to \cdots \to G_i \xrightarrow{d_i} G_{i+1} \to \cdots\,.$$

Definition 6.2.5. A directed system $(G_i, d_i)_{i>0}$ in $\mathcal{C}$ is of *order* n if any composition of n consecutive transition maps is zero. That is

$$d_{i+n-1} \circ \ldots \circ d_i = 0 \quad \text{for all } i > 0\,.$$

6.2.6. A map between directed systems $f\colon (F_i, d_i^F)_{i>0} \to (G_i, d_i^G)_{i>0}$ consists of morphisms $f_i\colon F_i \to G_i$ that commute with the transition maps.

In an abelian category, a sequence of directed systems

$$(F_i, d_i^F)_{i>0} \to (G_i, d_i^G)_{i>0} \to (H_i, d_i^H)_{i>0}$$

is exact if the maps are exact in each degree. In this situation, if $(F_i, d_i^F)_{i>0}$ is of order m and $(H_i, d_i^H)_{i>0}$ of order n, then $(G_i, d_i^G)_{i>0}$ is of order $m+n$.

For the rest of this section, let $\mathsf{f}\colon \mathcal{T}^{op} \to R\text{-}\operatorname{grMod}$ be a cohomological functor that commutes suspension and shift as in (6.2.1).

Definition 6.2.7. Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$ closed under suspension. An *n-resolution of f with respect to $\mathcal{C}$* is a directed system $(X_i, d_i)_{i>0}$ in $\mathcal{T}$ together with compatible natural transformations $\zeta_i\colon \mathsf{g}_{X_i} \to \mathsf{f}$, such that for any $Z \in \mathcal{C}$, one has

1. $\zeta_i(Z)$ is surjective for all $i > 0$, and

2. the direct system $(\ker(\zeta_i(Z)), a_i)_{i>0}$ is of order n where a_i are the maps induced by the commutative diagram

$$\begin{array}{ccccc}
\ker(\zeta_i(Z)) & \longrightarrow & \mathsf{g}_{X_i}(Z) & \xrightarrow{\zeta_i(Z)} & \mathsf{f}(Z) \\
\downarrow{\scriptstyle a_i} & & \downarrow{\scriptstyle (d_i)_*(Z)} & & \| \\
\ker(\zeta_{i+1}(Z)) & \longrightarrow & \mathsf{g}_{X_{i+1}}(Z) & \xrightarrow{\zeta_{i+1}(Z)} & \mathsf{f}(Z)\,.
\end{array}$$

Note, a direct system is an n-resolution of f with respect to $\mathcal{C}$ if and only if it is an n-resolution of f with respect to $\operatorname{thick}^1(\mathcal{C})$.

Lemma 6.2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{T}$ closed under suspension. If the direct system $(X_i, d_i)_{i>0}$ is an m-resolution of f with respect to $\mathcal{C}$ and an n-resolution with respect to $\mathcal{D}$, both compatible with the same natural transformations $\zeta_i\colon \mathsf{g}_{X_i} \to \mathsf{f}$, then $(X_i, d_i)_{i>n}$ is an $(m+n)$-resolution of f with respect to $\mathcal{C} \diamond \mathcal{D}$.*

Proof. By 2.2.7 (2), it is enough to show the claim for $\mathcal{C} \star \mathcal{D}$. For $Z \in \mathcal{C} \star \mathcal{D}$, one has to show that $\zeta_i(Z)$ is surjective, that is $\operatorname{coker}(\zeta_i(Z)) = 0$, for $i > n$ and that $(\ker(\zeta_i(Z)), a_i)_{i>n}$ is a direct system of order $m+n$.

For Z, there exists an exact triangle $U \to Z \to V \to \Sigma U$ with $U \in \mathcal{C}$ and $V \in \mathcal{D}$. This gives a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \ker(\zeta_i(\Sigma U)) & \longrightarrow & \mathsf{g}_{X_i}(\Sigma U) & \longrightarrow & \mathsf{f}(\Sigma U) & \longrightarrow & 0 & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \ker(\zeta_i(V)) & \longrightarrow & \mathsf{g}_{X_i}(V) & \longrightarrow & \mathsf{f}(V) & \longrightarrow & 0 & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \ker(\zeta_i(Z)) & \longrightarrow & \mathsf{g}_{X_i}(Z) & \longrightarrow & \mathsf{f}(Z) & \longrightarrow & \operatorname{coker}(\zeta_i(Z)) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \ker(\zeta_i(U)) & \longrightarrow & \mathsf{g}_{X_i}(U) & \longrightarrow & \mathsf{f}(U) & \longrightarrow & 0 & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \ker(\zeta_i(\Sigma^{-1}V)) & \to & \mathsf{g}_{X_i}(\Sigma^{-1}V) & \to & \mathsf{f}(\Sigma^{-1}V) & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

where the third and fourth column, and all rows, are exact.

By a straightforward diagram chase, one gets that the sequence

$$\ker(\zeta_i(V)) \to \ker(\zeta_i(Z)) \to \ker(\zeta_i(U))$$

is exact, and that there exists a surjective map

$$f\colon \operatorname{coker}(\zeta_i(Z)) \twoheadrightarrow \ker(\ker(\zeta_i(\Sigma^{-1}V)) \to \ker(\zeta_i(\Sigma^{-1}Z))).$$

By 6.2.6, the exact sequence implies that $(\ker(\zeta_i(Z)), a_i)_{i>0}$ is of order $m+n$.

Similar as for the kernel, the commutative diagram

$$\begin{array}{ccccccc}
\mathsf{g}_{X_i}(Z) & \longrightarrow & \mathsf{f}(Z) & \longrightarrow & \operatorname{coker}(\zeta_i(Z)) & \longrightarrow & 0 \\
\downarrow {\scriptstyle (d_i)_*(Z)} & & \| & & \downarrow {\scriptstyle b_i} & & \\
\mathsf{g}_{X_{i+1}}(Z) & \longrightarrow & \mathsf{f}(Z) & \longrightarrow & \operatorname{coker}(\zeta_{i+1}(Z)) & \longrightarrow & 0
\end{array}$$

induces a map b_i for every positive integer i. Note, these maps are surjective. Then applying 6.2.6 twice to the surjective map f, one has that the direct system $(\operatorname{coker}(\zeta_i(Z)), b_i)_{i>0}$ is of order n. Then the surjectivity of b_i implies that $\operatorname{coker}(\zeta_i(Z)) = 0$ for all integers $i > n$. □

Corollary 6.2.9. *Let $(X_i, d_i)_{i>0}$ be an m-resolution of f with respect to $\mathcal{C}$. Then $(X_i, d_i)_{i>nm}$ is an (nm)-resolution with respect to* $\operatorname{thick}^n(\mathcal{C})$ *for any positive integer n.* □

Lemma 6.2.10. *Let $(X_i, d_i)_{i>0}$ be an m-resolution of f with respect to $\mathcal{C}$. Then for any $n > m$, the functor f is a direct summand of g_{X_n} when restricted to* $\operatorname{thick}^1(\mathcal{C})$.

Proof. One may assume $\mathcal{C} = \operatorname{thick}^1(\mathcal{C})$. For $Z \in \mathcal{C}$, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\zeta_{n-m}(Z)) & \longrightarrow & \mathsf{g}_{X_{n-m}}(Z) & \xrightarrow{\zeta_{n-m}(Z)} & \mathsf{f}(Z) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle 0} & & \big\downarrow & \swarrow & \big\| & & \\ 0 & \longrightarrow & \ker(\zeta_n(Z)) & \longrightarrow & \mathsf{g}_{X_n}(Z) & \twoheadrightarrow & \mathsf{f}(Z) & \longrightarrow & 0\,. \end{array}$$

The left vertical map is zero, since $(\ker(\zeta_i(Z)), a_i)_{i>0}$ is of order m. Then there exists a map $\mathsf{f}(Z) \to \mathsf{g}_{X_n}(Z)$ such that the upper triangle commutes. Since $\zeta_{n-m}(Z)$ is surjective, the lower triangle also commutes. So $\mathsf{f}(Z)$ is a direct summand of $\mathsf{g}_{X_n}(Z)$. Since the induced map is natural in Z, the functor f is a direct summand of g_{X_n} on $\operatorname{thick}^1(\mathcal{C})$. □

The property of a functor $\mathcal{T}^{op} \to R\text{-}\operatorname{grMod}$ that correspond to the locally finiteness of a functor $\mathcal{T}^{op} \to R\text{-}\operatorname{mod}$ is that it maps to finitely generated graded R-modules. Let $R\text{-}\operatorname{grmod}$ be the category of finitely generated graded R-modules. For such a functor $\mathcal{T}^{op} \to R\text{-}\operatorname{grmod}$, I construct a 1-resolution with respect to $\operatorname{thick}^1(Z)$ for any object Z.

Lemma 6.2.11. *Let Z be an object in $\mathcal{T}$ and $\mathsf{f}\colon \mathcal{T}^{op} \to R\text{-}\operatorname{grmod}$ a functor. Then there exists an object X in $\mathcal{T}$ and a natural transformation $\zeta\colon \mathsf{g}_X \to \mathsf{f}$ such that ζ is surjective on* $\operatorname{thick}^1(Z)$.

Proof. Let $z_1, \dots, z_n$ be a set of homogeneous generators of $\mathsf{f}(Z)$ in degrees $d_1, \dots, d_n$. Since f maps to finitely generated graded R-modules, it is possible to choose a finite set. Define

$$X := \bigoplus_{j=1}^{n} \Sigma^{d_j} Z\,.$$

For every element z_j of the generating set, there exist canonical maps

$$\Sigma^{d_j} Z \xrightarrow{i_j} X \xrightarrow{p_j} \Sigma^{d_j} Z$$

whose composition is the identity map on $\Sigma^{d_j} Z$. Let $x \in \mathsf{f}(X)$ be the canonical element, for which

$$z_j = \mathsf{f}(i_j)(x) \quad \text{for } 1 \le j \le n\,.$$

Because of the suspensions introduced in the definition of X, the element x is homogeneous of degree 0. By Yoneda's lemma 6.2.3, the element x corresponds to the natural transformation $\zeta\colon \mathsf{g}_X \to \mathsf{f}$ with $\zeta(X)(\mathrm{id}_X) = x$. Then $\zeta(Z)(i_j) = z_j$, and so $\zeta(Z)$ is surjective. □

The finite generation of the image of f is required so that the direct sum in the definition of X is finite. Next, one builds a resolution of f inductively. Here it is important that the representable functors g_X map to R-grmod so that Lemma 6.2.11 can be applied to the kernel of a natural transformation $\mathsf{g}_X \to \mathsf{f}$.

Lemma 6.2.12. *Let $\mathcal{T}$ be an Ext-finite triangulated category, Z an object in $\mathcal{T}$, and $\mathsf{f}\colon \mathcal{T}^{op} \to R\text{-}\operatorname{grmod}$ a functor. Then f has a 1-resolution with respect to* $\operatorname{thick}^1(Z)$.

Proof. By the previous lemma, there exists $X_1 \in \mathcal{T}$ and a natural transformation $\zeta_1\colon \mathsf{g}_{X_1} \to \mathsf{f}$, such that ζ_1 is surjective on $\operatorname{thick}^1(Z)$.

I construct a direct system $(X_i, d_i)_{i>0}$ as follows: Assume $(X_i, d_i)_{0<i<n}$ satisfies the conditions for a 1-resolution. I will construct an object X_n, a map d_{n-1}, and a natural transformation ζ_n. Set $\mathsf{f}'(-) := \ker(\zeta_{n-1}(-))$. Then f' is a functor $\mathcal{T}^{op} \to R\text{-}\operatorname{grmod}$, since R is noetherian and $\mathcal{T}$ is Ext-finite. This functor commutes suspension and shift the same as f and g_X. So by the previous lemma, there exists $Y \in \mathcal{T}$ and a natural transformation $\eta\colon \mathsf{g}_Y \to \mathsf{f}'$ that is surjective on $\operatorname{thick}^1(Z)$. By Yoneda's lemma 6.2.3, the composition of natural transformations $\mathsf{g}_Y \to \mathsf{f}' \to \mathsf{g}_{X_{n-1}}$ corresponds to an element $f \in \mathsf{g}_{X_{n-1}}(Y)_0 = \operatorname{Hom}_{\mathcal{T}}(Y, X_{n-1})$. Then the composition is the natural transformation f_* and the sequence of functors

$$\mathsf{g}_Y \xrightarrow{f_*} \mathsf{g}_{X_{n-1}} \xrightarrow{\zeta_{n-1}} \mathsf{f} \to 0$$

is exact on $\operatorname{thick}^1(Z)$. The morphism f fits into an exact triangle

$$Y \xrightarrow{f} X_{n-1} \xrightarrow{d_{n-1}} X_n \to \Sigma Y\,.$$

Applying the functor f to this exact triangle and using the functoriality of the correspondence in Yoneda's lemma 6.2.4 gives the exact sequence

$$\operatorname{Nat}(\mathsf{g}_{X_n}, \mathsf{f}) \to \operatorname{Nat}(\mathsf{g}_{X_{n-1}}, \mathsf{f}) \to \operatorname{Nat}(\mathsf{g}_Y, \mathsf{f})$$

of abelian groups. By construction, the natural transformation ζ_{n-1} is mapped to 0. So there exists a natural transformation $\zeta_n\colon \mathsf{g}_{X_n} \to \mathsf{f}$, such that $\zeta_{n-1} = \zeta_n \circ (d_{n-1})_*$. In particular, the natural transformation ζ_n is surjective on $\operatorname{thick}^1(Z)$.

It remains to show that the maps d_{n-1} induce the zero map on the kernel of the natural transformations ζ_i. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathsf{g}_Y & \longrightarrow & \mathsf{f}' = \ker(\zeta_{n-1}(-)) & \hookrightarrow & \mathsf{g}_{X_{n-1}} & \xrightarrow{\zeta_{n-1}} & \mathsf{f} \\ & & \downarrow & & \downarrow{\scriptstyle (d_{n-1})_*} & & \| \\ & & \ker(\zeta_n(-)) & \hookrightarrow & \mathsf{g}_{X_n} & \xrightarrow{\zeta_n} & \mathsf{f}\,. \end{array}$$

By construction, the natural transformation $\mathsf{g}_Y \to \mathsf{g}_{X_n}$ is zero, and $\mathsf{g}_Y \to \mathsf{f}'$ is surjective on $\operatorname{thick}^1(Z)$. So the induced map between the kernel is zero. □

Now combining 6.2.9, 6.2.10, and 6.2.12, I can prove a graded version of Theorem 6.1.3.

Theorem 6.2.13. *Let $\mathcal{T}$ be an Ext-finite, Karoubian triangulated category with a strong generator. Then any cohomological functor $\mathsf{f}\colon \mathcal{T}^{op} \to R\text{-}\operatorname{grmod}$ that commutes suspension and shift as* (6.2.1) *is representable.*

Proof. Let Z be a strong generator of $\mathcal{T}$. By Lemma 6.2.12, there exists a 1-resolution $(X_i, d_i)_{i>0}$ of f with respect to $\operatorname{thick}^1(Z)$. Since Z is a strong generator, there exists n, such that $\mathcal{T} = \operatorname{thick}^n(Z)$. So by 6.2.9, the direct system $(X_i, d_i)_{i>n}$ is an n-resolution of f with respect to $\mathcal{T}$. Then by Lemma 6.2.10, the functor f is a direct summand of $\mathsf{g}_{X_{n+1}}$. Let π and ι be the natural projection and injection of f as a direct summand of $\mathsf{g}_{X_{n+1}}$. Let the endomorphism $e\colon X_{n+1} \to X_{n+1}$ be the image of $\operatorname{id}_{X_{n+1}}$ under the map

$$\mathsf{g}_{X_{n+1}}(X_{n+1}) \xrightarrow{\pi(X_{n+1})} \mathsf{f}(X_{n+1}) \xrightarrow{\iota(X_{n+1})} \mathsf{g}_{X_{n+1}}(X_{n+1})\,.$$

Then $e^2 = e$ and e is idempotent. Since $\mathcal{T}$ is Karoubian, there exists an object Y in $\mathcal{T}$ and maps

$$i\colon Y \to X_{n+1} \quad \text{and} \quad p\colon X_{n+1} \to Y \quad \text{with} \quad p \circ i = \operatorname{id}_Y \text{ and } i \circ p = e\,.$$

Then the natural transformations

$$\mathsf{f} \xrightarrow{\iota} \mathsf{g}_{X_{n+1}} \xrightarrow{p_*} \mathsf{g}_Y \quad \text{and} \quad \mathsf{g}_Y \xrightarrow{i_*} \mathsf{g}_{X_{n+1}} \xrightarrow{\pi} \mathsf{f}$$

are inverse to each other. In particular, the functors f and g_Y are naturally isomorphic. □

6.3 Theorem of Hopkins and Neeman for Complexes of Finite Injective Dimension

I prove a converse of Lemma 4.7.3 for complexes of finite injective dimension with finitely generated homology. Using the dualizing complex introduced in Section 5.2, one gets a connection between the complexes of finite projective dimension and the complexes of finite injective dimension; see [Rob80, Chapter 3].

Lemma 6.3.1. *Assume a commutative noetherian ring R has a dualizing complex ω. Then there is an equivalence*

$$\operatorname{Perf}(R) \underset{\mathbf{R}\operatorname{Hom}_R(-,\omega)}{\overset{\mathbf{R}\operatorname{Hom}_R(-,\omega)}{\rightleftarrows}} \mathrm{K}_{b,f}(R\text{-}\operatorname{Inj})$$

where $\mathrm{K}_{b,f}(R\text{-}\mathrm{Inj})$ *is the homotopy category of all bounded complexes of injective R-modules with finitely generated homology.* □

A commutative noetherian ring R need not have a dualizing complex, but every complete local ring has a dualizing complex. With the results of Chapter 5, it is possible to reduce to complete local rings so that I can conclude from the theorem of Hopkins and Neeman 4.7.5:

Theorem 6.3.2. *Let R be a commutative noetherian ring, and let X and Y be complexes of finite injective dimension with finitely generated homology. If* $\operatorname{Supp}_R(X) \subseteq \operatorname{Supp}_R(Y)$, *then* $\operatorname{level}^Y(X) < \infty$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be any prime ideal. Then $X_\mathfrak{p}$ and $Y_\mathfrak{p}$ are complexes of finite injective dimension with finitely generated homology. Also localization preserves the inclusion of their support and, by 5.4.5, $\operatorname{level}^Y_R(X) < \infty$ if and only if $\operatorname{level}^{Y_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p}) < \infty$ for all prime ideals $\mathfrak{p}$. So one may assume R is local.

Let $\widehat{(-)}$ denote the completion with respect to the maximal ideal and k the residue field of R. By [AF91, 5.5(I)], a complex X lies in $\mathrm{K}_{b,f}(R\text{-}\mathrm{Inj})$ if and only if $\mathbf{R}\mathrm{Hom}_R(k, X)$ is a bounded above complex. Since X lies in $\mathrm{K}_{b,f}(R\text{-}\mathrm{Inj})$, in particular, it lies in $\mathrm{D}^{\mathrm{f}}(R)$, so that $\widehat{X} = X \otimes^{\mathbf{L}}_R \widehat{R}$. Then

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(k, \widehat{X}) \cong \mathbf{R}\mathrm{Hom}_R(k, \widehat{X}) \cong \mathbf{R}\mathrm{Hom}_R(k, X) \otimes^{\mathbf{L}}_R \widehat{R}$$

and thus $X \in \mathrm{K}_{b,f}(R\text{-}\mathrm{Inj})$ if and only if $\widehat{X} \in \mathrm{K}_{b,f}(\widehat{R}\text{-}\mathrm{Inj})$.

It is well-known that

$$({}^a\varphi)^{-1}(\operatorname{Supp}_R(X)) = ({}^a\varphi)^{-1}(\operatorname{Supp}_R(\mathrm{H}(X))) = \operatorname{Supp}_{\widehat{R}}(\widehat{R} \otimes_R \mathrm{H}(X)) = \operatorname{Supp}_{\hat{R}}(\widehat{X})$$

where $\varphi\colon R \to \widehat{R}$ is the canonical ring homomorphism and ${}^a\varphi\colon \operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ the induced map. So completion preserves the inclusion of the support.

Last we have $\operatorname{level}^Y_R(X) < \infty$ if and only if $\operatorname{level}^{\widehat{Y}}_{\widehat{R}}(\widehat{X}) < \infty$ by 5.3.12. So without loss of generality, I assume R is a complete local ring.

Now R has a dualizing complex ω. Set $(-)^\dagger := \mathbf{R}\mathrm{Hom}_R(-, \omega)$. Then the complexes $X^\dagger$ and $Y^\dagger$ are perfect by 6.3.1. Since X has finitely generated homology, one has $(X^\dagger)_\mathfrak{p} = (X_\mathfrak{p})^\dagger$ and thus $\operatorname{Supp}_R(X^\dagger) \subseteq \operatorname{Supp}_R(X)$. Since $(-)^\dagger$ is an auto-equivalence, the supports are equal. The same holds for Y, so $\operatorname{Supp}_R(X^\dagger) \subseteq \operatorname{Supp}_R(Y^\dagger)$. By 4.7.5, one has $\operatorname{level}^{Y^\dagger}_R(X^\dagger) < \infty$, and thus $\operatorname{level}^Y_R(X) < \infty$. □

6.4 Virtual and Proxy Smallness

In homotopy theory, the compact objects are known as the *small* objects. By 4.3.1, the small objects in $\mathrm{D}(R)$ are the perfect complexes. There are two approaches on how to weaken the notion of smallness; see [DGI06].

Definition 6.4.1. A complex X in $\mathrm{D}(R)$ is *virtually small*, if $X \simeq 0$ or there exists $W \not\simeq 0$ in $\mathrm{D}(R)$, such that

$$\operatorname{level}^R(W) < \infty \quad \text{and} \quad \operatorname{level}^X(W) < \infty \,.$$

If additionally $\operatorname{Supp}_R(X) = \operatorname{Supp}_R(W)$, then X is *proxy small*. The complex W is the *witness* of X.

6.4.2. By the theorem of Hopkins and Neeman 4.7.5, for proxy smallness the witness is only unique up to its support. By [DGI06, 4.4], a complex $X \not\simeq 0$ is proxy small if and only if it is proxy small with witness the Koszul complex $\operatorname{Kos}(I)$ on the ideal I, where $V(I) = \operatorname{Supp}_R(X)$. For virtual smallness, one can choose a witness with smaller, though non-empty, support. By [DGI06, 4.5], a complex $X \not\simeq 0$ is virtually small if and only if there exists a maximal ideal $\mathfrak{m} \in \operatorname{Supp}_R(X)$, such that X is virtually small with witness the Koszul complex $\operatorname{Kos}(\mathfrak{m})$ on $\mathfrak{m}$.

By 5.4.5, a complex is small if and only if it is small locally. Similarly, I track the behavior of proxy smallness under localization.

Proposition 6.4.3. *Let R be a Noether algebra and X in $\mathrm{D}^{\mathrm{f}}(R)$. Then X is proxy small if and only if $X_{\mathfrak{p}}$ is proxy small for all $\mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))$.*

Proof. Let I be an ideal with $V(I) = \operatorname{Supp}_R(X)$. For any prime ideal $\mathfrak{p}$, one has $\operatorname{Kos}(I)_{\mathfrak{p}} \simeq \operatorname{Kos}(I_{\mathfrak{p}})$. By 6.4.2, it is enough to show

$$X \models \operatorname{Kos}(I) \iff X_{\mathfrak{p}} \models \operatorname{Kos}(I_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \operatorname{Spec}(\mathrm{Z}(R))\,.$$

This holds by 5.4.5. □

Virtual smallness does not behave in the same way. If W is a perfect complex, that is built by X, and it does not have the same support as X, then there exists a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(X)$ with $W_{\mathfrak{p}} \simeq 0$. Thus if the complex X is virtually small, the localizations $X_{\mathfrak{p}}$ need not be.

Proposition 6.4.4. *Let R be a Noether algebra and $X \not\simeq 0$ a complex over R. Then $X_{\mathfrak{m}} \not\simeq 0$ is virtually small for some maximal ideal $\mathfrak{m}$ of $\mathrm{Z}(R)$ if and only if X is virtually small.*

Proof. First assume X is virtually small and let $W \not\simeq 0$ be a perfect complex, such that $\operatorname{level}_R^X(W) < \infty$. Since W lies in $\mathrm{D}^{\mathrm{f}}(R)$, its support is closed and there exists a maximal ideal $\mathfrak{m} \in \operatorname{Supp}_R(W)$. In particular, $X_{\mathfrak{m}} \not\simeq 0$. Then $W_{\mathfrak{m}} \not\simeq 0$ is a perfect complex and $\operatorname{level}_{R_{\mathfrak{m}}}^{X_{\mathfrak{m}}}(W_{\mathfrak{m}}) < \infty$. Thus $X_{\mathfrak{m}}$ is virtually small.

By 6.4.2, there exists $\mathfrak{m} \in \operatorname{Supp}_R(X)$, such that $X_{\mathfrak{m}}$ is virtually small and thus

$$\operatorname{level}_{R_{\mathfrak{m}}}^{X_{\mathfrak{m}}}(\operatorname{Kos}(\mathfrak{m})_{\mathfrak{m}}) < \infty\,.$$

For any prime ideal $\mathfrak{p} \neq \mathfrak{m}$, one has $\operatorname{Kos}(\mathfrak{m})_{\mathfrak{p}} = 0$. So by Theorem 5.4.5

$$\operatorname{level}_R^X(\operatorname{Kos}(\mathfrak{m})) < \infty\,.$$

Since the Koszul complex $\operatorname{Kos}(\mathfrak{m})$ is perfect, X is virtually small. □

One can also track the behavior of virtually and proxy small under a faithfully flat ring map.

Proposition 6.4.5. *Let $\varphi\colon R \to S$ be a faithfully flat ring map of commutative noetherian rings and $X \in \mathrm{D}^{\mathrm{f}}(R)$.*

1. *X is proxy small if and only if $\varphi^*(X) := X \otimes_R^{\mathbf{L}} S$ is proxy small in $\mathrm{D}(R)$.*

2. *If X is virtually small, then $\varphi^*(X)$ is virtually small in $\mathrm{D}(R)$.*

Proof. By 5.3.10, the functor φ^* is faithful. So $X \simeq 0$ if and only if $\varphi^*(X) \simeq 0$, and I may assume $X \not\simeq 0$. Let I be an ideal in R, such that $V(I) = \operatorname{Supp}_R(X)$. Given that S is faithfully flat over R, it is well-known that

$$\operatorname{Kos}(I) \otimes_R^{\mathbf{L}} S = \operatorname{Kos}(I \otimes_R S) \quad \text{and} \quad \operatorname{Supp}_S(\varphi^*(X)) = V(I \otimes_R S)\,.$$

Then by 5.3.11, one has

$$\operatorname{level}_R^X(\operatorname{Kos}(I)) = \operatorname{level}_S^{\varphi^*(X)}(\operatorname{Kos}(I \otimes_R S))\,.$$

Now X is proxy small if and only if $\operatorname{level}_R^X(\operatorname{Kos}(I)) < \infty$ and $\varphi^*(X)$ is proxy small if and only if $\operatorname{level}_S^{\varphi^*(X)}(\operatorname{Kos}(I \otimes_R S)) < \infty$. This shows the claim of (1).

For (2), let $W \not\simeq 0$ be a perfect complex, such that $\operatorname{level}_R^X(W) < \infty$. By 5.3.10, the functor φ^* is faithful, so $\varphi^*(W) \not\simeq 0$ and by 2.3.3 (3), one has

$$\operatorname{level}_S^{\varphi^*(X)}(\varphi^*(W)) \leq \operatorname{level}_R^X(W) < \infty\,.$$

So $\varphi^*(X)$ is virtually small. □

The properties virtually and proxy smallness can be used to give a categorical description of a complete intersection. A local ring $(R, \mathfrak{m}, k)$ is a *complete intersection* if its $\mathfrak{m}$-adic completion $\widehat{R}$ is of the form $\widehat{R} = Q/(f_1, \ldots, f_c)$ where Q is a regular local ring and $f_1, \ldots, f_c$ a regular sequence in Q.

A commutative noetherian ring R is a locally complete intersection if for any prime ideal $\mathfrak{p}$, the ring $R_\mathfrak{p}$ is a complete intersection. Using [Pol19, Theorem 5.4] and 6.4.3, I get a characterization of locally complete intersections.

Theorem 6.4.6. *For a commutative noetherian ring R, the following are equivalent:*

1. *R is a locally complete intersection, and*
2. *every object in $\mathrm{D}^{\mathrm{f}}(R)$ is proxy small.*

Proof. Assume R is a locally complete intersection. That is $\widehat{R_\mathfrak{p}}$ is a quotient of a regular local ring by a regular sequence for every prime ideal $\mathfrak{p}$. By [DGI06, Theorem 9.4], every object in $\mathrm{D}^{\mathrm{f}}(\widehat{R_\mathfrak{p}})$ is proxy small. Then by 6.4.3 and 6.4.5, every object in $\mathrm{D}^{\mathrm{f}}(R)$ is proxy small.

For the opposite direction, by 5.4.6, the functor $\mathrm{D}^{\mathrm{f}}(R) \to \mathrm{D}^{\mathrm{f}}(R_\mathfrak{p})$ is essentially surjective and thus since every object in $\mathrm{D}^{\mathrm{f}}(R)$ is proxy small, so is every object in $\mathrm{D}^{\mathrm{f}}(R_\mathfrak{p})$. Then by [Pol19, Theorem 5.2], the localization $R_\mathfrak{p}$ is a complete intersection. □

In [Pol19, Theorem 5.4], Pollitz proved that (1) holds if and only if every object in $\mathrm{D}^{\mathrm{f}}(R)$ is virtually small.

Over a local ring $(R, \mathfrak{m}, k)$ a complex $X \in \mathrm{D}^{\mathrm{f}}(R)$ has *finite CI-dimension*, if there exist local homomorphisms $R \to R' \leftarrow Q$, such that

- $R \to R'$ is faithfully flat,
- $Q \to R'$ is surjective and the kernel is generated by a regular sequence, and
- $\operatorname{fd}_Q(R' \otimes^{\mathbf{L}}_R X) < \infty$.

This was first introduced by [AGP97] and extended to complexes by [SW04].

Theorem 5.3.11 answers the question raised in [DGI06, 9.6 Remarks]. Therefore, we can complete the proof that a complex of finite CI-dimension is virtually small. This fact has been proven via a different method in [Ber09]. Using 6.4.5, I strengthen the result to the following.

Proposition 6.4.7. *Every complex in* $\mathrm{D}^{\mathrm{f}}(R)$ *of finite CI-dimension is proxy small.*

Proof. Let X be a complex in $\mathrm{D}^{\mathrm{f}}(R)$ of finite CI-dimension and let $R \to R' \leftarrow Q$ be a diagram of local homomorphisms satsifying the required conditions. Then $R' \otimes^{\mathbf{L}}_R X$ has finite homology over R' and, in particular, over Q. So $R' \otimes^{\mathbf{L}}_R X$ is a perfect complex over Q. Then by [DGI06, Theorem 9.1], the complex $R' \otimes^{\mathbf{L}}_R X$ is proxy small in $\mathrm{D}(R')$ and by 6.4.5 (1), X also in $\mathrm{D}(R)$. □

The converse does not hold. In a local ring, the residue field is proxy small, but it has finite CI-dimension if and only if the ring is a complete intersection.

The condition given in Theorem 6.4.6 to characterize a locally complete intersection by its derived category is broad. In some settings, there exists an object that encodes the information when every object in $\mathrm{D}^{\mathrm{f}}(R)$ is proxy small.

Given a commutative k-algebra R, the enveloping algebra of R is $R^e := R \otimes_k R$. Then R^e acts on R diagonally.

Theorem 6.4.8. *Let k be a field and R a commutative k-algebra essentially of finite type over k. Then the following are equivalent:*

1. *R is a locally complete intersection, and*
2. *R is proxy small in* $\mathrm{D}(R^e)$.

Proof. Both conditions are local conditions. So it is enough to show a local ring R of finite type over k is a complete intersection if and only if R is proxy small in $\mathrm{D}(R^e)$.

Since R is a complete intersection, so is R^e by [Avr99, 5.11]. Then by 6.4.6, every object in $\mathrm{D}^{\mathrm{f}}(R^e)$ is proxy small and thus R is proxy small in $\mathrm{D}(R^e)$.

For the converse direction, assume R is proxy small in $\mathrm{D}(R^e)$. That is there exists a non-zero complex W in $\mathrm{D}(R^e)$, such that

$$\operatorname{level}^{R^e}_{R^e}(W) < \infty \quad \text{and} \quad \operatorname{level}^{R}_{R^e}(W) < \infty \quad \text{and} \quad \operatorname{Supp}_{R^e}(W) = \operatorname{Supp}_{R^e}(R)\,.$$

Let $X \in \mathrm{D}^{\mathrm{f}}(R)$. By 6.4.6, it is enough to show X is proxy small in $\mathrm{D}(R)$. Any complex Y in $\mathrm{D}(R^e)$ has a left and a right R-action. Thus $Y \otimes^{\mathbf{L}}_R X$ has a left R-action through the left R-action of Y. This induces the exact functor

$$- \otimes^{\mathbf{L}}_R X \colon \mathrm{D}(R^e) \to \mathrm{D}(R)$$

and by 2.3.3 (3), one has

$$\operatorname{level}^{R \otimes_k X}_R (W \otimes^{\mathbf{L}}_R X) < \infty \quad \text{and} \quad \operatorname{level}^{X}_R (W \otimes^{\mathbf{L}}_R X) < \infty\,.$$

The object $R \otimes_k X$ is a direct sum of suspensions of R. Then

$$R \otimes_k X \in \operatorname{Add}(R) \quad \text{and so} \quad \operatorname{level}^{\operatorname{Add}(R)}_R (W \otimes^{\mathbf{L}}_R X) < \infty\,.$$

In particular, the complex $W \otimes^{\mathbf{L}}_R X$ has a finite resolution by projective modules. Since $W \otimes^{\mathbf{L}}_R X$ is built by X, it has finite homology. Thus $W \otimes^{\mathbf{L}}_R X$ has a finite resolution of finitely generated projective modules, that is it is perfect.

It remains to show $W \otimes^{\mathbf{L}}_R X$ has the same support as X. The *localizing* subcategory generated by an object X in $\mathrm{D}(R)$ is the smallest triangulated subcategory of $\mathrm{D}(R)$, which is closed under coproducts and contains X. By [Nee92, Theorem 2.8], two complexes with finitely generated homology have the same support if and only if they generate the same localizing subcategory. Now since W and R have the same support over R^e, they have the same localizing subcategories in $\mathrm{D}(R^e)$. So $W \otimes^{\mathbf{L}}_R X$ and $R \otimes^{\mathbf{L}}_R X = X$ have the same localizing subcategories in $\mathrm{D}(R)$, and thus the same support over R. □

This characterization is similar to the characterization for a smooth ring: If k is a field and R a k-algebra essentially of finite type over k, then R is smooth if and only if R is small in $\mathrm{D}(R^e)$.

It is possible to specify how R is proxy small in $\mathrm{D}(R^e)$. I will give a witness for R and bounds for its level with respect to R.

The following is partially proved in [DGI06, Theorem 9.1], before the invariant level was developed.

Lemma 6.4.9. *Let $Q \twoheadrightarrow R$ be a surjective map of commutative noetherian rings with kernel I. If I is generated by a regular sequence $\boldsymbol{f} = f_1, \ldots, f_c$, then*

$$\operatorname{level}^{Q}_{Q}(R) = c + 1 \quad \textit{and} \quad \operatorname{level}^{R}_{R \otimes^{\mathbf{L}}_Q R}(R \otimes^{\mathbf{L}}_Q R) \leq c + 1\,.$$

Proof. Let $Q\langle \boldsymbol{x} | \partial(x_i) = f_i \rangle$ be the Koszul complex on $\boldsymbol{f}$. Since $\boldsymbol{f}$ is a regular sequence, the natural map $Q\langle \boldsymbol{x} \rangle \to R$ is a quasi-isomorphism. This is a minimal projective resolution of R, which proves the first equality.

For the second equality, consider

$$R \otimes^{\mathbf{L}}_Q R \simeq Q\langle \boldsymbol{x} \rangle \otimes^{\mathbf{L}}_Q Q\langle \boldsymbol{x} \rangle = Q\langle \boldsymbol{y}, \boldsymbol{z} \rangle$$

where the variables $\boldsymbol{y}$ correspond to $\boldsymbol{x} \otimes 1$ and $\boldsymbol{z}$ to $1 \otimes \boldsymbol{x}$. Given a set $I \subseteq \{1, \ldots, c\}$, I show by induction on the cardinality of I the inequality

$$\operatorname{level}^{Q\langle x \rangle}_{Q\langle y,z \rangle}(Q\langle \boldsymbol{y}, \boldsymbol{z} \rangle / (y_i - z_i | i \notin I)) \leq |I| + 1 \,.$$

Note that one has the identification $Q\langle \boldsymbol{x} \rangle \cong Q\langle \boldsymbol{y}, \boldsymbol{z} \rangle / (y_i - z_i | 1 \leq i \leq c)$. So there is nothing to show for $|I| = 0$. For I not empty, let $\boldsymbol{x}'$, $\boldsymbol{y}'$, and $\boldsymbol{z}'$ be the variables of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ with index in I, and $\boldsymbol{u}$ the ones of $\boldsymbol{x}$ not in $\boldsymbol{x}'$. Set $A := Q\langle \boldsymbol{x}' \rangle$. Then there exists a short exact sequence

$$0 \to \bigoplus_{j \in I} \Sigma A\langle \boldsymbol{y}', \boldsymbol{z}' \rangle / (y'_j - z'_j) \xrightarrow{1 \mapsto y'_j - z'_j} A\langle \boldsymbol{y}', \boldsymbol{z}' \rangle \xrightarrow{y'_i \mapsto x'_i, z'_i \mapsto x'_i} A\langle \boldsymbol{x}' \rangle \to 0 \,.$$

This induced an exact triangle and by induction, one has

$$\operatorname{level}^{Q\langle x \rangle}_{Q\langle y,z \rangle}(A\langle \boldsymbol{y}', \boldsymbol{z}' \rangle / (y'_j - z'_j)) \leq |I \setminus \{j\}| + 1 = |I| \,,$$

and so the claim holds by 2.3.2 (2). □

Proposition 6.4.10. *Let k be a field and R a k-algebra essentially of finite type over k, that is a locally complete intersection. If $R = Q/I$ is the quotient of a regular ring Q, then $R \otimes^{\mathbf{L}}_Q R$ is a small object in $\mathrm{D}(R^e)$ and $\operatorname{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R)$ is bounded below by*

$$\sup \left\{ \operatorname{codim}(\widehat{R_{\mathfrak{m}}}) \,\middle|\, \mathfrak{m} \in \operatorname{Max}(R) \right\} + 1$$

and above by

$$\sup \left\{ \operatorname{height}(\widehat{I_{\mathfrak{m}}}) \,\middle|\, \mathfrak{m} \in \operatorname{Max}(R) \right\} + 1 \,.$$

Proof. To see $R \otimes^{\mathbf{L}}_Q R$ is small in $\mathrm{D}(R^e)$, consider the functor

$$\mathsf{f} \colon \mathrm{D}(Q) \to \mathrm{D}(R^e) \quad \text{with} \quad X \mapsto (R \otimes^{\mathbf{L}}_Q X) \otimes_k R \,.$$

Since Q is regular, it generates k in $\mathrm{D}(Q)$ and thus

$$R^e = \mathsf{f}(Q) \models\!\!= \mathsf{f}(k) = R \otimes^{\mathbf{L}}_Q R \,.$$

First assume R is a complete local ring. Then the ideal I is generated by a regular sequence of length $\operatorname{height}(I)$ and there exists a map $R^e \to R \otimes^{\mathbf{L}}_Q R$. Thus by 6.4.9

$$\operatorname{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R) \leq \operatorname{level}^R_{R \otimes^{\mathbf{L}}_Q R}(R \otimes^{\mathbf{L}}_Q R) = \operatorname{height}(I) + 1\,.$$

For the lower bound, consider the functor

$$\mathtt{t} := - \otimes^{\mathbf{L}}_R k \colon \mathrm{D}(R^e) \to \mathrm{D}(R)\,.$$

Since $\mathtt{t}(R^e) = R$, the functor $\mathtt{t}$ sends perfect complexes to perfect complexes. Then the inequality

$$\operatorname{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R) \geq \operatorname{level}^k_R(\mathtt{t}(R \otimes^{\mathbf{L}}_Q R)) \geq \operatorname{codim}(R) + 1$$

holds by [ABIM10, Theorem 11.3] . This proves the claim if R is a complete local ring.

If R is locally a complete intersection, then by 5.3.12 and 5.4.4

$$\operatorname{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R) = \sup\left\{ \operatorname{level}^{\widehat{R_{\mathfrak{m}}}}_{\widehat{(R^e)_{\mathfrak{m}}}}(\widehat{(R \otimes^{\mathbf{L}}_Q R)_{\mathfrak{m}}}) \,\middle|\, \mathfrak{m} \in \operatorname{Max}(R^e) \right\}.$$

The multiplication map $\mu\colon R^e \to R$ induces an injective map $\mu^a\colon \operatorname{Spec}(R) \to \operatorname{Spec}(R^e)$, which restricts to a map on maximal ideals $\operatorname{Max}(R) \to \operatorname{Max}(R^e)$. For any maximal ideal $\mathfrak{m}$ of R^e that lies not in the image of this map, the localization $R_{\mathfrak{m}}$ is zero, and thus irrelevant for the calculation of the supremum. That is

$$\operatorname{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R) = \sup\left\{ \operatorname{level}^{\widehat{R_{\mathfrak{m}}}}_{(\widehat{R_{\mathfrak{m}}})^e}(\widehat{R_{\mathfrak{m}}} \otimes^{\mathbf{L}}_{\widehat{Q_{\varphi^a(\mathfrak{m})}}} \widehat{R_{\mathfrak{m}}}) \,\middle|\, \mathfrak{m} \in \operatorname{Max}(R) \right\}$$

where $\varphi\colon Q \to R$ and φ^a the induced map on the set of maximal ideals. Now the claim follows from the complete local case. □

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