Smallness properties and locally complete intersections

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Generation

- A full subcategory $\mathcal{S}$ is *thick* if it is closed under:
  - direct summands,
  - suspensions,
  - cones.
- $\operatorname{thick}_{\mathcal{T}}(X)$ = smallest thick subcategory of $\mathcal{T}$ containing X.
- If $Y \in \operatorname{thick}_{\mathcal{T}}(X)$, then X *generates* Y. Write $X \models Y$. This is constructive:

$$X \xrightarrow[\text{suspensions, cones}]{\text{finite direct sums, summands}} Y\,.$$

- $\operatorname{level}^X_{\mathcal{T}}(Y)$ = minimal number of cones

Examples

1. $\operatorname{Perf}(R)$ = perfect complexes
   Then $\operatorname{thick}_R(R) = \operatorname{Perf}(R)$.
   If M is a f.g. R-module then
   $$\operatorname{level}^R_R(M) = \operatorname{pd}_R(M) + 1\,.$$
2. $\operatorname{thick}_R(k)$ = complexes with bounded finite length homology.

Theorem

If $X, Y \in \mathrm{D}_f(R)$ then TFAE
(1) $X \models Y$,
(2) $X_{\mathfrak{p}} \models Y_{\mathfrak{p}}\ \forall \mathfrak{p} \in \operatorname{Spec}(R)$.

Theorem

Let R be a k-algebra essentially of finite type over k. Then TFAE
(1) R is a locally complete intersection,
(2) R is proxy small in $\mathrm{D}(R^e)$ where $R^e = R \otimes_k R$.

Proof sketch:
(1) $\implies$ (2): R c.i. $\implies$ R^e c.i. $\implies$ R is proxy small in $\mathrm{D}(R^e)$.
(2) $\implies$ (1): For $X \in \mathrm{D}_f(R)$ one has

$$\mathrm{D}(R^e) \xrightarrow{-\otimes^{\mathbf{L}}_R X} \mathrm{D}(R)$$

$$R^e \overset{R^e}{\models} W \implies R^e \otimes^{\mathbf{L}}_R X = R \otimes_k X \overset{R}{\models} W \otimes^{\mathbf{L}}_R X \in \operatorname{thick}_R(\operatorname{Add}(R))$$

$$R \overset{R^e}{\models} W \implies R \otimes^{\mathbf{L}}_R X = X \overset{R}{\models} W \otimes^{\mathbf{L}}_R X \qquad \in \mathrm{D}_f(R)\,.$$

So $W \otimes^{\mathbf{L}}_R X$ is small. It remains to show $\operatorname{Supp}_R(X) = \operatorname{Supp}_R(W \otimes^{\mathbf{L}}_R X)$. Idea:

Support $\longleftrightarrow$ Localizing subcategory. □

If $R = Q/I$ with Q regular, then
- $R \otimes^{\mathbf{L}}_Q R$ is a small object in $\mathrm{D}(R^e)$,
- R generates $R \otimes^{\mathbf{L}}_Q R$ with

$$\sup\{\operatorname{codim}(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}+1 \le \operatorname{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R) \le \sup\{\operatorname{ht}(I_{\mathfrak{m}}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}+1\,.$$

Characterizations

TFAE
(1) R regular,
(2) every object in $\mathrm{D}_f(R)$ is small.

TFAE
(1) R a locally complete intersection,
(2) every object in $\mathrm{D}_f(R)$ is proxy small.

Proxy small objects

A complex $X \in \mathrm{D}(R)$ is *small* if
$$R \models X\,.$$
A complex $X \in \mathrm{D}(R)$ is *proxy small* if there exists a small object W, such that
$$X \models W \quad \text{and} \quad \operatorname{Supp}_R(W) = \operatorname{Supp}_R(X)\,.$$

Examples

1. small $\implies$ proxy small.
2. $(R, \mathfrak{m}, k)$ local, and
   K^R = Koszul complex. Then
   (a) $k \models K^R$,
   (b) $\operatorname{Supp}_R(K^R) = \{\mathfrak{m}\} = \operatorname{Supp}_R(k)$.

Lemma

For $X \in \mathrm{D}_f(R)$ TFAE
(1) X is proxy small,
(2) $X_{\mathfrak{p}}$ is proxy small $\forall \mathfrak{p} \in \operatorname{Spec}(R)$.

References

[1] W. G. Dwyer, J. P. C. Greenlees, and S. B. Iyengar. Finiteness in derived categories of local rings. *Comment. Math. Helv.*, 81(2):383–432, 2006.

[2] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.

[3] J. Pollitz. The derived category of a locally complete intersection ring. *Adv. Math.*, 354:106752, 2019.