Smallness properties and locally complete intersections

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Theorem

Let k be a field and R a commutative k-algebra essentially of finite type over k. Then the following are equivalent

1. *R is a locally complete intersection*
2. *R is proxy small in $\mathrm{D}(R^e)$.*

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$$R \otimes_k R$$

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$\widehat{R_{\mathfrak{m}}} = Q/(f_1, \dots, f_c)$ with
Q regular
$f_1, \dots, f_c$ regular sequence

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1. *R is ~~a locally complete intersection~~ smooth*
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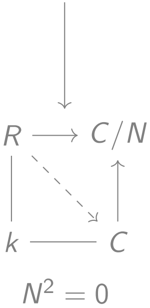

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$X \overset{\mathcal{T}}{\models\!\!=} Y$: "$X$ generates Y" if $Y \in \mathrm{thick}_{\mathcal{T}}(X)$.

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$P = R \otimes^{\mathbf{L}}_{Q} R$

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$X \overset{\mathcal{T}}{\models} Y$: "$X$ generates Y" if $Y \in \text{thick}_{\mathcal{T}}(X)$.

$$X \underset{\text{summands, suspensions}}{\overset{\text{cones, finite direct sums}}{\rightsquigarrow}} Y$$

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$\mathrm{level}^{X}_{\mathcal{T}}(Y) =$ minimal number of cones

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$R = Q/I$ with Q regular

$$R \models^{\mathrm{D}(R^e)} R \otimes^{\mathbf{L}}_{Q} R$$

$$\begin{matrix}\text{something depending} \\ \text{on } R = Q/I\end{matrix} \leq \mathrm{level}^{R}_{R^e}(R \otimes^{\mathbf{L}}_{Q} R) \leq \begin{matrix}\text{something else depending} \\ \text{on } R = Q/I\end{matrix} .$$

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Proof sketch.

Theorem

Let R be a Noether algebra with center $\mathsf{Z}(R)$. For any objects G and X in $\mathrm{D}_f(R)$, there is an equality

$$\mathsf{level}_R^G(X) = \sup\left\{\mathsf{level}_{R_\mathfrak{p}}^{G_\mathfrak{p}}(X_\mathfrak{p}) \,\middle|\, \mathfrak{p} \in \mathsf{Spec}(\mathsf{Z}(R))\right\}.$$

Moreover

$$G \models X \iff G_\mathfrak{p} \models X_\mathfrak{p} \text{ for all prime ideals } \mathfrak{p} \text{ in } \mathsf{Z}(R).$$

Theorem (Pollitz, 2018)

A local ring R is a complete intersection if and only if every nontrivial object of $\mathrm{D}_f(R)$ is virtually small.

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Finiteness in derived categories of local rings

W. Dwyer, J. P. C. Greenlees and S. Iyengar*

9.10 Question. Over a local ring R, if each homologically finite complex is virtually small, then is R complete intersection?

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Proof sketch.

$1 \implies 2$:

R complete intersection
$\implies R^e$ complete intersection
$\implies R$ proxy small in $\mathrm{D}(R^e)$

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Proof sketch.

$2 \implies 1$: $X \in \mathrm{D}_f(R)$, need to show: X proxy small in $\mathrm{D}(R)$

$- \otimes^{\mathbf{L}}_R X \colon \mathrm{D}(R^e) \to \mathrm{D}(R)$ gives

$$X \models^{\mathrm{D}(R)} P \otimes^{\mathbf{L}}_R X \text{ perfect} \qquad \mathrm{Loc}_R(X) = \mathrm{Loc}_R(P \otimes^{\mathbf{L}}_R X)$$

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