Local to Global Principles for Generation Time over Commutative Rings

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January 17th, 2020

Example: Projective Dimension

M a finitely generated module of finite projective dimension.

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

- generator: R
- generation time $= n + 1 = \mathrm{pd}_R(M) + 1$

Example: Loewy length

$(R, \mathfrak{m}, k)$ a local ring and M a finite length module.

$$0 = \mathfrak{m}^n M \subsetneq \mathfrak{m}^{n-1} M \subsetneq \cdots \subsetneq \mathfrak{m} M \subsetneq M$$

- generator: k
- generation time = Loewy length = $n = \inf\{i \geq 0 | \mathfrak{m}^i M = 0\}$

Setting

$\mathcal{T}$ = triangulated category with suspension Σ

Example

$\mathcal{T} = \mathrm{D}(R)$ (or $\mathrm{D}_f(R)$) the derived category (of complexes with finitely generated bounded homology)

- Objects: Complexes of R-modules
- Morphisms: Chain maps with quasi-isomorphisms formally inverted

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- Suspension/Translation: "move X to the left":

$$X : \quad \cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots$$

$$\Sigma X : \cdots \longrightarrow X_n \xrightarrow{-\partial_n} X_{n-1} \longrightarrow \cdots$$

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Example

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- Mapping cone: $X \xrightarrow{f} Y$ a chain map

$$\mathrm{cone}(f) : \cdots \to X_n \oplus Y_{n+1} \xrightarrow{\begin{pmatrix} -\partial^X & 0 \\ f & \partial^Y \end{pmatrix}} X_{n-1} \oplus Y_n \to \cdots$$

and $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to \Sigma X$ is an exact triangle

Definition: Generation

A full subcategory $\mathcal{S}$ is thick if it is closed under:

- direct summands,
- suspensions,
- cones.

$\text{thick}(G) :=$ smallest thick subcategory containing G.

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Definition (Bondal–van den Bergh, 2003)

- $\operatorname{thick}^1(G) :=$ smallest subcategory containing G, closed under finite direct sums, suspensions and direct summands.
- $\operatorname{thick}^n(G) :=$ smallest subcategory containing all X such that there exists an exact triangle

$$Y \to X \oplus X' \to Z \to \Sigma Y$$

with $Y \in \operatorname{thick}^{n-1}(G)$ and $Z \in \operatorname{thick}^1(G)$.

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with $Y \in \mathrm{thick}^{n-1}(G)$ and $Z \in \mathrm{thick}^1(G)$.
- $\mathrm{level}^G(X) := \inf\{n \geq 0 | X \in \mathrm{thick}^n(G)\}$

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- $\text{thick}^1(R) \supseteq$ finitely generated R-modules
- $0 \to P_1 \to P_0 \to 0 = \text{cone}\,(P_1 \to P_0) \in \text{thick}^2(R)$
- $0 \to P_2 \to P_1 \to P_0 \to 0$

$$= \text{cone}\left(\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \cdots \end{array}\right) \in \text{thick}^3(R)$$

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- $\text{level}^R(M) \leq n+1$

Example: Projective Dimension, Part 2

- If M a finitely generated module then $\text{level}^R(M) = \text{pd}_R(M) + 1$.
- For complexes this need not to hold:

$$R \oplus \Sigma^n R = (0 \to R \to 0 \to \cdots \to 0 \to R \to 0) \in \text{thick}^1(R)$$

- $\text{thick}(R) = \text{Perf}(R)$

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Example: Loewy length

$(R, \mathfrak{m}, k)$ a local ring and M a complex with $\mathfrak{m}^n M \simeq 0$.

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- generator: k
- $\mathfrak{m}^{i+1} M \to \mathfrak{m}^i M \to \underbrace{\mathfrak{m}^i M / \mathfrak{m}^{i+1} M}_{\in \operatorname{thick}^1(k)} \to \Sigma \mathfrak{m}^{i+1} M$
- $M \in \operatorname{thick}^n(k)$ that is $\operatorname{level}^k(M) \leq n$
- If M is a module then $\operatorname{level}^k(M) = n$

Motivation

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If $\mathsf{f}\colon \mathcal{S} \to \mathcal{T}$ is an exact functor of triangulated categories then

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$$\operatorname{level}_{\mathcal{S}}^{G}(X) \geq \operatorname{level}_{\mathcal{T}}^{\mathsf{f}(G)}(\mathsf{f}(X))\,.$$

Question

When is this an equality?

Theorem 1

Theorem (L.)

Let $\varphi\colon R \to S$ be a faithfully flat ring map. For $X, G \in \mathrm{D}_f(R)$, one has for $\varphi^* := S \otimes^{\mathbf{L}}_R -$:

$$\operatorname{level}_R^G(X) = \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X)) .$$

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$$\operatorname{level}_R^G(X) = \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X)).$$

Corollary (L.)

Let $(R, \mathfrak{m}, k)$ be a local ring and let $\widehat{(-)}$ be the completion with respect to $\mathfrak{m}$. Then for any $X, G \in \mathrm{D}_f(R)$ one has

$$\operatorname{level}_R^G(X) = \operatorname{level}_{\widehat{R}}^{\widehat{G}}(\widehat{X}).$$

Theorem 2

Theorem (L.)

For $G, X \in \mathrm{D}_f(R)$ *one has*

$$\mathrm{level}_R^G(X) = \sup\{\mathrm{level}_{R_\mathfrak{p}}^{G_\mathfrak{p}}(X_\mathfrak{p}) | \mathfrak{p} \in \mathrm{Spec}(R)\}\,.$$

Moreover

$$\mathrm{level}_R^G(X) < \infty \iff \mathrm{level}_{R_\mathfrak{p}}^{G_\mathfrak{p}}(X_\mathfrak{p}) < \infty \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R)\,.$$

(Co)ghost maps

Definition

A map $f\colon X \to Y$ is G-ghost if

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma^i G, X) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma^i G, Y) = 0 \text{ for all } i\,.$$

The ghost index with respect to G is

$$\mathrm{gin}^{\mathcal{C}}(X) := \inf\{n \mid \text{all } n\text{-fold } \mathcal{C}\text{-ghost maps } X \to Y \text{ are zero}\}\,.$$

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A map $f\colon X \to Y$ is <u>G-coghost</u> if

$$\mathrm{Hom}_{\mathcal{T}}(Y, \Sigma^i G) \to \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i G) = 0 \text{ for all } i\,.$$

The <u>coghost index with respect to G</u> is

$$\mathrm{cogin}^{\mathcal{C}}(Y) := \inf\{n \mid \text{all } n\text{-fold } \mathcal{C}\text{-coghost maps } X \to Y \text{ are zero}\}\,.$$

Example: R-coghost Maps

$G = R = k[x, y]$

$$\begin{array}{ccccccccccccc}
R & = & Y_2 & : & & & & & 0 & \longrightarrow & R & \longrightarrow & 0\\
& & \downarrow f_2 & & & & & & \downarrow & & \| & & \\
& & Y_1 & : & & & 0 & \longrightarrow & R^2 & \xrightarrow{(x\ y)} & R & \longrightarrow & 0 \\
& & \downarrow f_1 & & & & \downarrow & & \| & & \| & & \\
k & \simeq & Y_0 & : & 0 & \longrightarrow & R & \xrightarrow{\binom{-y}{x}} & R^2 & \xrightarrow{(x\ y)} & R & \longrightarrow & 0
\end{array}$$

- $\mathrm{Hom}(Y_1, R) = 0 \implies f_2$ is R-coghost
- $\mathrm{Hom}(Y_2, \Sigma R) = 0 \implies f_1$ is R-coghost

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- $\mathrm{Hom}(Y_1, R) = 0 \implies f_2$ is R-coghost
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$$\implies \mathrm{cogin}^R(k) \geq 3$$

(Co)ghost Lemma

(Co)ghost Lemma (Kelly, 1965)

$$\operatorname{gin}^G(X) \leq \operatorname{level}^G(X) \quad \text{and} \quad \operatorname{cogin}^G(X) \leq \operatorname{level}^G(X)\,.$$

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(Co)ghost Lemma (Kelly, 1965)

$$\operatorname{gin}^G(X) \leq \operatorname{level}^G(X) \quad \text{and} \quad \operatorname{cogin}^G(X) \leq \operatorname{level}^G(X)\,.$$

Converse Coghost Lemma (Oppermann–Šťovíček, 2012)

Let R be a Noether algebra. Then for any $G, X \in \mathrm{D}_f(R)$ one has the equality

$$\operatorname{level}^G(X) = \operatorname{cogin}^G_{\mathrm{D}_f(R)}(X)\,.$$

Proof idea

Theorem (L.)

Let $\varphi\colon R \to S$ be a faithfully flat ring map. For $X, G \in \mathrm{D}_f(R)$, one has for $\varphi^* := S \otimes^{\mathbf{L}}_R -$:

$$\operatorname{level}_R^G(X) = \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X)).$$

Proof idea:

- φ^* preserves G-coghost maps
- φ^* is a faithful functor

$$\operatorname{level}_R^G(X) = \operatorname{cogin}_{\mathrm{D}_f(R)}^G(X) \leq \operatorname{cogin}_{\mathrm{D}_f(S)}^{\varphi^*(G)}(\varphi^*(X)) = \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X))$$

Equality holds, since $\operatorname{level}_R^G(X) \geq \operatorname{level}_S^{\varphi^*(G)}(\varphi^*(X))$ always holds.

Application: Hopkins–Neeman

Theorem (Hopkins, 1987; Neeman, 1992)

For perfect complexes X and Y:

$$\operatorname{level}^{Y}(X) < \infty \iff \operatorname{Supp}_R(X) \subseteq \operatorname{Supp}_R(Y)\,.$$

Application: Hopkins–Neeman

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Corollary (L.)

For complexes X and Y of finite injective dimension with finitely generated homology:

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For complexes X and Y of finite injective dimension with finitely generated homology:

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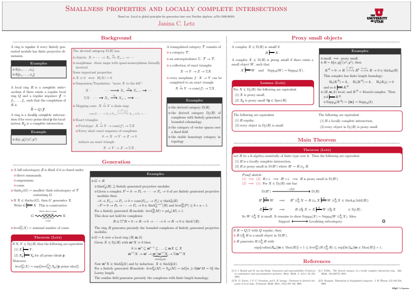
Proof idea: If the ring R has a dualizing complex ω:

$$\{\text{perfect complexes}\} \xleftrightarrow{\mathbf{R}\mathrm{Hom}_R(-,\omega)} \left\{ \begin{array}{c} \text{complexes of injective dimension} \\ \text{with finitely generated homology} \end{array} \right\}$$

Using Theorem 1 & 2: Reduce to complete local rings.

Application: Smallness Properties and Locally Complete Intersections

Come to the AWM poster session: Today 5:00–6:16 p.m. in Upper Lobby D



Thank you!

A conference for advanced undergraduate and early graduate students.

- Dates: May 20–22, 2020
- Location: University of Utah, Salt Lake City
- Deadline for Funding: January 31, 2020