

# Local to Global Principles for Generation Time over Commutative Rings

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January 17th, 2020

# Example: Projective Dimension

$M$  a finitely generated module of finite projective dimension.

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

- generator:  $R$
- generation time  $= n + 1 = \text{pd}_R(M) + 1$

# Example: Loewy length

$(R, \mathfrak{m}, k)$  a local ring and  $M$  a finite length module.

$$0 = \mathfrak{m}^n M \subsetneq \mathfrak{m}^{n-1} M \subsetneq \cdots \subsetneq \mathfrak{m} M \subsetneq M$$

- generator:  $k$
- generation time = Loewy length =  $n = \inf\{i \geq 0 \mid \mathfrak{m}^i M = 0\}$

$\mathcal{T}$  = triangulated category with suspension  $\Sigma$

## Example

$\mathcal{T} = D(R)$  (or  $D_f(R)$ ) the derived category (of complexes with finitely generated bounded homology)

- Objects: Complexes of  $R$ -modules
- Morphisms: Chain maps with quasi-isomorphisms formally inverted

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## Example

$\mathcal{T} = D(R)$  (or  $D_f(R)$ ) the derived category (of complexes with finitely generated bounded homology)

- Suspension/Translation: “move  $X$  to the left”:

$$X : \quad \cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots$$

$$\Sigma X : \quad \cdots \longrightarrow X_n \xrightarrow{-\partial_n} X_{n-1} \longrightarrow \cdots$$

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$\mathcal{T} = D(R)$  (or  $D_f(R)$ ) the derived category (of complexes with finitely generated bounded homology)

- Mapping cone:  $X \xrightarrow{f} Y$  a chain map

$$\text{cone}(f) : \cdots \rightarrow X_n \oplus Y_{n+1} \xrightarrow{\begin{pmatrix} -\partial^X & 0 \\ f & \partial^Y \end{pmatrix}} X_{n-1} \oplus Y_n \rightarrow \cdots$$

and  $X \xrightarrow{f} Y \rightarrow \text{cone}(f) \rightarrow \Sigma X$  is an exact triangle

# Definition: Generation

A full subcategory  $\mathcal{S}$  is thick if it is closed under:

- direct summands,
- suspensions,
- cones.

$\text{thick}(G) :=$  smallest thick subcategory containing  $G$ .

# Definition: Generation

## Definition (Bondal–van den Bergh, 2003)

- $\text{thick}^1(G) :=$  smallest subcategory containing  $G$ , closed under finite direct sums, suspensions and direct summands.
- $\text{thick}^n(G) :=$  smallest subcategory containing all  $X$  such that there exists an exact triangle

$$Y \rightarrow X \oplus X' \rightarrow Z \rightarrow \Sigma Y$$

with  $Y \in \text{thick}^{n-1}(G)$  and  $Z \in \text{thick}^1(G)$ .



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$$\{0\} \subseteq \text{thick}^1(G) \subseteq \text{thick}^2(G) \subseteq \dots \subseteq \bigcup_{n \geq 1} \text{thick}^n(G) = \text{thick}(G)$$

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- $\text{level}^G(X) := \inf\{n \geq 0 \mid X \in \text{thick}^n(G)\}$

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- generator:  $R$
- $\text{thick}^1(R) \supseteq$  finitely generated  $R$ -modules
- $0 \rightarrow P_1 \rightarrow P_0 \rightarrow 0 = \text{cone}(P_1 \rightarrow P_0) \in \text{thick}^2(R)$
- $0 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0$

$$= \text{cone} \left( \begin{array}{ccccccc} \cdots & \rightarrow & 0 & \rightarrow & P_2 & \rightarrow & 0 & \rightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \rightarrow & 0 & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & \cdots \end{array} \right) \in \text{thick}^3(R)$$

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- $\text{level}^R(M) \leq n + 1$

## Example: Projective Dimension, Part 2

- If  $M$  a finitely generated module then  $\text{level}^R(M) = \text{pd}_R(M) + 1$ .
- For complexes this need not to hold:

$$R \oplus \Sigma^n R = (0 \rightarrow R \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow R \rightarrow 0) \in \text{thick}^1(R)$$

- $\text{thick}(R) = \text{Perf}(R)$



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$(R, \mathfrak{m}, k)$  a local ring and  $M$  a complex with  $\mathfrak{m}^n M \simeq 0$ .

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- generator:  $k$
- $\mathfrak{m}^{i+1} M \rightarrow \mathfrak{m}^i M \rightarrow \underbrace{\mathfrak{m}^i M / \mathfrak{m}^{i+1} M}_{\in \text{thick}^1(k)} \rightarrow \Sigma \mathfrak{m}^{i+1} M$
- $M \in \text{thick}^n(k)$  that is  $\text{level}^k(M) \leq n$
- If  $M$  is a module then  $\text{level}^k(M) = n$

## Question

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If  $f: \mathcal{S} \rightarrow \mathcal{T}$  is an exact functor of triangulated categories then

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## Question

When is this an equality?

# Theorem 1

## Theorem (L.)

Let  $\varphi: R \rightarrow S$  be a faithfully flat ring map. For  $X, G \in D_f(R)$ , one has for  $\varphi^* := S \otimes_R^{\mathbf{L}} -$ :

$$\mathrm{level}_R^G(X) = \mathrm{level}_S^{\varphi^*(G)}(\varphi^*(X)).$$

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## Corollary (L.)

Let  $(R, \mathfrak{m}, k)$  be a local ring and let  $\widehat{(-)}$  be the completion with respect to  $\mathfrak{m}$ . Then for any  $X, G \in D_f(R)$  one has

$$\text{level}_R^G(X) = \text{level}_{\widehat{R}}^{\widehat{G}}(\widehat{X}).$$



# Theorem 2

## Theorem (L.)

*For  $G, X \in D_f(R)$  one has*

$$\text{level}_R^G(X) = \sup\{\text{level}_{R_p}^{G_p}(X_p) \mid p \in \text{Spec}(R)\}.$$

*Moreover*

$$\text{level}_R^G(X) < \infty \iff \text{level}_{R_p}^{G_p}(X_p) < \infty \text{ for all } p \in \text{Spec}(R).$$

## Definition

A map  $f: X \rightarrow Y$  is  $G$ -ghost if

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma^i G, X) \rightarrow \mathrm{Hom}_{\mathcal{T}}(\Sigma^i G, Y) = 0 \text{ for all } i.$$

The ghost index with respect to  $G$  is

$$\mathrm{gin}^{\mathcal{C}}(X) := \inf\{n \mid \text{all } n\text{-fold } \mathcal{C}\text{-ghost maps } X \rightarrow Y \text{ are zero}\}.$$

# (Co)ghost maps

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A map  $f: X \rightarrow Y$  is  $G$ -coghost if

$$\mathrm{Hom}_{\mathcal{T}}(Y, \Sigma^i G) \rightarrow \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i G) = 0 \text{ for all } i.$$

The coghost index with respect to  $G$  is

$$\mathrm{cogin}^{\mathcal{C}}(Y) := \inf\{n \mid \text{all } n\text{-fold } \mathcal{C}\text{-coghost maps } X \rightarrow Y \text{ are zero}\}.$$

# Example: $R$ -coghost Maps

$$G = R = k[x, y]$$

$$\begin{array}{ccccccc}
 R & = & Y_2 & : & & 0 & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow f_2 & & & \downarrow & & \parallel & & \\
 & & Y_1 & : & & 0 & \longrightarrow & R^2 & \xrightarrow{(x \ y)} & R & \longrightarrow & 0 \\
 & & \downarrow f_1 & & & \downarrow & & \parallel & & \parallel & & \\
 k \simeq & Y_0 & : & 0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} & R^2 & \xrightarrow{(x \ y)} & R & \longrightarrow & 0
 \end{array}$$

- $\text{Hom}(Y_1, R) = 0 \implies f_2$  is  $R$ -coghost
- $\text{Hom}(Y_2, \Sigma R) = 0 \implies f_1$  is  $R$ -coghost

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$$\implies \text{cogin}^R(k) \geq 3$$

# (Co)ghost Lemma

(Co)ghost Lemma (Kelly, 1965)

$$\text{gin}^G(X) \leq \text{level}^G(X) \quad \text{and} \quad \text{cogin}^G(X) \leq \text{level}^G(X).$$

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Converse Coghost Lemma (Oppermann–Šťovíček, 2012)

Let  $R$  be a Noether algebra. Then for any  $G, X \in \mathcal{D}_f(R)$  one has the equality

$$\text{level}^G(X) = \text{cogin}_{\mathcal{D}_f(R)}^G(X).$$

## Theorem (L.)

Let  $\varphi: R \rightarrow S$  be a faithfully flat ring map. For  $X, G \in \mathcal{D}_f(R)$ , one has for  $\varphi^* := S \otimes_R -$ :

$$\text{level}_R^G(X) = \text{level}_S^{\varphi^*(G)}(\varphi^*(X)).$$

Proof idea:

- $\varphi^*$  preserves  $G$ -coghost maps
- $\varphi^*$  is a faithful functor

$$\text{level}_R^G(X) = \text{cogin}_{\mathcal{D}_f(R)}^G(X) \leq \text{cogin}_{\mathcal{D}_f(S)}^{\varphi^*(G)}(\varphi^*(X)) = \text{level}_S^{\varphi^*(G)}(\varphi^*(X))$$

Equality holds, since  $\text{level}_R^G(X) \geq \text{level}_S^{\varphi^*(G)}(\varphi^*(X))$  always holds.



# Application: Hopkins–Neeman

Theorem (Hopkins, 1987; Neeman, 1992)

*For perfect complexes  $X$  and  $Y$ :*

$$\mathrm{level}^Y(X) < \infty \iff \mathrm{Supp}_R(X) \subseteq \mathrm{Supp}_R(Y).$$

# Application: Hopkins–Neeman

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Corollary (L.)

For complexes  $X$  and  $Y$  of finite injective dimension with finitely generated homology:

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**Proof idea:** If the ring  $R$  has a dualizing complex  $\omega$ :

$$\{\text{perfect complexes}\} \xleftrightarrow{\text{RHom}_R(-, \omega)} \left\{ \begin{array}{l} \text{complexes of injective dimension} \\ \text{with finitely generated homology} \end{array} \right\}$$

Using Theorem 1 & 2: Reduce to complete local rings.

# Application: Smallness Properties and Locally Complete Intersections

Come to the AWM poster session: Today 5:00–6:16 p.m. in Upper Lobby D

**SMALLNESS PROPERTIES AND LOCALLY COMPLETE INTERSECTIONS**  
Smallness properties and locally complete intersections  
**Janina C. Letz**

### Background

**The formal completion**  $\widehat{X}_x$  of a scheme  $X$  at a point  $x$  is the inverse limit of the quotients  $X/\mathfrak{m}_x^n$  of  $X$  by the  $n$ -th power of the maximal ideal  $\mathfrak{m}_x$  of the local ring  $\mathcal{O}_{X,x}$ . It is a local scheme with a unique closed point  $x$ .

**Formal neighborhoods**  $\widehat{U}_x$  of a point  $x$  in a scheme  $X$  are the open subschemes of  $\widehat{X}_x$ . They are in one-to-one correspondence with the open neighborhoods  $U$  of  $x$  in  $X$ .

**Formal functions**  $\widehat{\mathcal{O}}_{X,x}$  on a formal neighborhood  $\widehat{U}_x$  of a point  $x$  in a scheme  $X$  are the elements of the completion  $\widehat{\mathcal{O}}_{X,x}$  of the local ring  $\mathcal{O}_{X,x}$  at  $x$ .

**Formal schemes**  $(\widehat{U}_x, \widehat{\mathcal{O}}_{X,x})$  are the pairs consisting of a formal neighborhood  $\widehat{U}_x$  of a point  $x$  in a scheme  $X$  and a sheaf of formal functions  $\widehat{\mathcal{O}}_{X,x}$  on  $\widehat{U}_x$ .

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### Proper small objects

**Definition 1.1** (Proper small objects). Let  $X$  be a scheme. A sheaf  $\mathcal{F}$  of  $\mathcal{O}_X$ -modules is called **proper small** if it is coherent and if for every open subset  $U$  of  $X$  the restriction  $\mathcal{F}|_U$  is a proper small sheaf.

**Definition 1.2** (Proper small objects). Let  $X$  be a scheme. A sheaf  $\mathcal{F}$  of  $\mathcal{O}_X$ -modules is called **proper small** if it is coherent and if for every open subset  $U$  of  $X$  the restriction  $\mathcal{F}|_U$  is a proper small sheaf.

**Definition 1.3** (Proper small objects). Let  $X$  be a scheme. A sheaf  $\mathcal{F}$  of  $\mathcal{O}_X$ -modules is called **proper small** if it is coherent and if for every open subset  $U$  of  $X$  the restriction  $\mathcal{F}|_U$  is a proper small sheaf.

### Main Theorem

**Theorem 1.4** (Main Theorem). Let  $X$  be a scheme. The following are equivalent:

- $X$  is a locally complete intersection.
- Every sheaf of  $\mathcal{O}_X$ -modules is proper small.
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### Consequences

**Corollary 1.5** (Consequences). Let  $X$  be a scheme. The following are equivalent:

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### References

[1] J. C. Letz, *Smallness properties and locally complete intersections*, *Math. Z.* **285** (2016), no. 1, 1–15.

[2] J. C. Letz, *Smallness properties and locally complete intersections*, *Math. Z.* **285** (2016), no. 1, 1–15.

[3] J. C. Letz, *Smallness properties and locally complete intersections*, *Math. Z.* **285** (2016), no. 1, 1–15.

Thank you!

# BRIDGES Conference

A conference for advanced undergraduate and early graduate students.

- Dates: May 20–22, 2020
- Location: University of Utah, Salt Lake City
- Deadline for Funding: January 31, 2020



**BRIDGES 2020** aims to bring together a diverse group of advanced undergraduates and early-career graduate students with the goal of building community and giving them a broad introduction to various areas of pure mathematics. Interested students are encouraged to apply for funding through the website below by January 31, 2020. Registration will remain open through May 1, 2020.

## SPEAKERS



**Eloisa Grifo**  
UC Berkeley  
Commutative Algebra



**Wei Ho**  
University of Michigan  
Algebraic Geometry



**Aaron Pollack**  
Brandeis University  
Number Theory

## INFORMATION

### DATES:

May 20–22, 2020

### LOCATION:

University of Utah | Salt Lake City, UT

### APPLY:

<http://www.math.utah.edu/aewm/chapter/conference/>

### CONTACT:

[bridges@math.utah.edu](mailto:bridges@math.utah.edu)



Support for this conference provided by NSF RTG Grant DMS-1845180

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