Smallness properties and locally complete intersections

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Background

A ring is *regular* if every finitely generated module has finite projective dimension.

Examples

- $k[x_1, \dots, x_n]$
- $k[\![x_1, \dots, x_n]\!]$

A local ring R is a *complete intersection* if there exists a regular local ring Q and a regular sequence $\boldsymbol{f} = f_1, \dots, f_c$, such that the completion of R is

$$\widehat{R} = Q/\boldsymbol{f}\,.$$

A ring is a *locally complete intersection* if for every prime ideal $\mathfrak{p}$ the localization $R_\mathfrak{p}$ is a complete intersection.

Example

- $k[x,y]/(x^2,y^2)$

The *derived category* $\mathrm{D}(R)$ has:

- objects: $X = \cdots \to X_n \xrightarrow{\partial_n} X_{n-1} \to \cdots$
- morphisms: chain maps with quasi-isomorphisms formally inverted.

Some important properties:

- $X \simeq 0 \iff \mathrm{H}(X) = 0$
- Suspension/Translation: "move X to the left":

$$X\ :\qquad \cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \to \cdots$$
$$\Sigma X:\ \cdots \longrightarrow X_n \xrightarrow{-\partial_n} X_{n-1} \longrightarrow \cdots$$

- Mapping cone: $X \xrightarrow{f} Y$ a chain map

$$\mathrm{cone}(f) : \cdots \to X_n \oplus Y_{n+1} \xrightarrow{\left(\begin{smallmatrix} -\partial^X & 0 \\ f & \partial^Y \end{smallmatrix}\right)} X_{n-1} \oplus Y_n \to \cdots$$

- Exact triangles:
  - Prototype: $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to \Sigma X$
  - Every short exact sequence of complexes
  $$0 \to X \to Y \to Z \to 0$$
  induces an exact triangle
  $$X \to Y \to Z \to \Sigma X\,.$$

A triangulated category $\mathcal{T}$ consists of

- a category $\mathcal{T}$,
- an autoequivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$,
- a collection of exact triangles
$$X \to Y \to Z \to \Sigma X$$
- every morphism $f\colon X \to Y$ can be completed to an exact triangle
$$X \xrightarrow{f} Y \to \mathrm{cone}(f) \to \Sigma X\,.$$

Examples

- the derived category $\mathrm{D}(R)$
- the derived category $\mathrm{D}_f(R)$ of complexes with finitely generated, bounded cohomology
- the category of vector spaces over a fixed field
- the stable homotopy category in topology

Generation

- A full subcategory $\mathcal{S}$ is *thick* if it is closed under:
  - direct summands,
  - suspensions,
  - cones.
- $\mathrm{thick}_\mathcal{T}(G)$ = smallest thick subcategory of $\mathcal{T}$ containing G.
- If $X \in \mathrm{thick}_\mathcal{T}(G)$, then G *generates* X. Write $G \models X$. This is constructive:

$$G \xrightarrow[\text{suspensions, cones}]{\text{finite direct sums, summands}} X\,.$$

- $\mathrm{level}^G_\mathcal{T}(X)$ = minimal number of cones

Theorem (Letz)

If $X, Y \in \mathrm{D}_f(R)$ then the following are equivalent

(1) $X \models Y$,

(2) $X_\mathfrak{p} \models Y_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$.

Moreover,

$$\mathrm{level}^G_R(X) = \sup\{\mathrm{level}^{G_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p}) | \mathfrak{p} \text{ prime ideal}\}\,.$$

Examples

- $G = R$
  - $\mathrm{thick}^1_R(R) \supseteq$ finitely generated projective modules
  - Given a complex $P = 0 \to P_b \to \cdots \to P_a \to 0$ of are finitely generated projective modules then:
    - $0 \to P_{a+1} \to P_a \to 0 = \mathrm{cone}(P_{a+1} \to P_a) \in \mathrm{thick}^2_R(R)$
    - $P = 0 \to P_b \to \cdots \to P_a \to 0 \in \mathrm{thick}^{b-a+1}_R(R)$ and $\mathrm{level}^R_R(P) \le b + a - 1$.

  For a finitely generated R-module: $\mathrm{level}^R_R(M) = \mathrm{pd}_R(M) + 1$.

  This does not hold for complexes:
  $$R \oplus \Sigma^n R = 0 \to R \to 0 \to \cdots \to 0 \to R \to 0 \in \mathrm{thick}^1(R)\,.$$
  The ring R generates precisely the bounded complexes of finitely generated projective modules.
- $G = k$ over a local ring $(R, \mathfrak{m}, k)$

  Given $X \in \mathrm{D}_f(R)$ with $\mathfrak{m}^n X \simeq 0$ then
  $$0 \simeq \mathfrak{m}^n X \subseteq \mathfrak{m}^{n-1} X \subseteq \ldots \subseteq \mathfrak{m} X \subseteq X$$
  $$\mathfrak{m}^{i+1} X \to \mathfrak{m}^i X \to \underbrace{\mathfrak{m}^i X / \mathfrak{m}^{i+1} X}_{\in \mathrm{thick}^1_R(k)} \to \Sigma \mathfrak{m}^{i+1} X$$
  Now $\mathfrak{m}^n X \in \mathrm{thick}^0_R(k)$ and by induction: $X \in \mathrm{thick}^n_R(k)$.

  For a finitely generated R-module: $\mathrm{level}^k_R(M) = \mathrm{ll}_R(M) = \inf\{n \ge 0 | \mathfrak{m}^n M = 0\}$ the Loewy length.

  The residue field generates precisely the complexes with finite length homology.

Proxy small objects

A complex $X \in \mathrm{D}(R)$ is *small* if
$$R \models X\,.$$

A complex $X \in \mathrm{D}(R)$ is *proxy small* if there exists a small object W, such that
$$X \models W \quad \text{and} \quad \mathrm{Supp}_R(W) = \mathrm{Supp}_R(X)\,.$$

Lemma (Letz)

For $X \in \mathrm{D}_f(R)$ the following are equivalent

(1) X is proxy small,

(2) $X_\mathfrak{p}$ is proxy small $\forall \mathfrak{p} \in \mathrm{Spec}(R)$.

Examples

- small $\implies$ proxy small.
- $R = k[\![x,y]\!]/(x^2,y^2)$, then
$$K^R = 0 \to R \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} R^2 \xrightarrow{(x\ y)} R \to 0 \in \mathrm{thick}_R(R)$$
This complex has finite length homology:
$$\mathrm{H}_0(K^R) = k\,, \quad \mathrm{H}_1(K^R) = k\,, \quad \mathrm{H}_2(K_R) = 0$$
and so $k \models K^R$.
- $(R, \mathfrak{m}, k)$ local, and K^R = Koszul complex. Then
  - $k \models K^R$,
  - $\mathrm{Supp}_R(K^R) = \{\mathfrak{m}\} = \mathrm{Supp}_R(k)$.

The following are equivalent

(1) R regular,

(2) every object in $\mathrm{D}_f(R)$ is small.

The following are equivalent

(1) R a locally complete intersection,

(2) every object in $\mathrm{D}_f(R)$ is proxy small.

Main Theorem

Theorem (Letz)

Let R be a k-algebra essentially of finite type over k. Then the following are equivalent

(1) R is a locally complete intersection,

(2) R is proxy small in $\mathrm{D}(R^e)$ where $R^e = R \otimes_k R$.

Proof sketch:

(1) $\implies$ (2): R c.i. $\implies$ R^e c.i. $\implies$ R is proxy small in $\mathrm{D}(R^e)$.

(2) $\implies$ (1): For $X \in \mathrm{D}_f(R)$ one has

$$\mathrm{D}(R^e) \xrightarrow{\quad -\otimes^{\mathbf{L}}_R X \quad} \mathrm{D}(R)$$

$$R^e \overset{R^e}{\models} W \implies R^e \otimes^{\mathbf{L}}_R X = R \otimes_k X \overset{R}{\models} W \otimes^{\mathbf{L}}_R X \in \mathrm{thick}_R(\mathrm{Add}(R))$$

$$R \overset{R^e}{\models} W \implies R \otimes^{\mathbf{L}}_R X = X \overset{R}{\models} W \otimes^{\mathbf{L}}_R X \quad \in \mathrm{D}_f(R)\,.$$

So $W \otimes^{\mathbf{L}}_R X$ is small. It remains to show $\mathrm{Supp}_R(X) = \mathrm{Supp}_R(W \otimes^{\mathbf{L}}_R X)$. Idea:

Support $\longleftrightarrow$ Localizing subcategory. □

If $R = Q/I$ with Q regular, then

- $R \otimes^{\mathbf{L}}_Q R$ is a small object in $\mathrm{D}(R^e)$,
- R generates $R \otimes^{\mathbf{L}}_Q R$ with
$$\sup\{\mathrm{codim}(R_\mathfrak{m}) | \mathfrak{m} \in \mathrm{Max}(R)\} + 1 \le \mathrm{level}^R_{R^e}(R \otimes^{\mathbf{L}}_Q R) \le \sup\{\mathrm{ht}(I_\mathfrak{m}) | \mathfrak{m} \in \mathrm{Max}(R)\} + 1\,.$$

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