Generation in Derived Categories and Applications to Commutative Algebra

- survey
- for questions please interrupt

Motivation Hilbert's Syzygy Theorem

$R = k[x_1, \dots, x_n]$, M f.g. R-module $\Rightarrow$ $\operatorname{pd}_R M \leq n$
$\operatorname{gldim} R = n$

projective dimension = # steps to build M from R

Question Given a module G, does G build M, if so in how many steps?

I) Generation

Construction [Bondal; van den Bergh, 2003]

$\mathcal{T}$ = Δ'ed category with suspension Σ
$G \in \mathcal{T}$

$\operatorname{thick}_{\mathcal{T}}(G)$ = smallest thick subcategory containing G

$\operatorname{thick}_{\mathcal{T}}(G)$ has a filtration:

- $\operatorname{thick}^0(G) = \{0\}$

- $\operatorname{thick}^1(G)$ = smallest subcat containing G closed under suspension, finite coproducts and summands

- $\operatorname{thick}^n(G) = \left\{ X \;\middle|\; \begin{array}{l} \exists \text{ exact } \Delta: Y \to X \oplus X' \to Z \to \text{ st} \\ Y \in \operatorname{thick}^1(G),\ Z \in \operatorname{thick}^{n-1}(G) \end{array} \right\}$

$\rightsquigarrow$ $\{0\} = \operatorname{thick}^0(G) \subseteq \operatorname{thick}^1(G) \subseteq \dots \subseteq \bigcup_{n \geq 0} \operatorname{thick}^n(G) = \operatorname{thick}(G)$

If $X \in \text{thick}(G)$, then <u>G finitely builds X</u> and $G \vDash X$.

$\text{level}^G(X) = \inf\{n \mid X \in \text{thick}^n(G)\}$ "# steps"

G is a <u>strong generator</u> if $\text{thick}^n(G) = \mathcal{T}$.

<u>Rouquier dimension</u> $\dim \mathcal{T} := \inf\{n \mid \text{thick}^n(G) = \mathcal{T}\} - 1$

Question Why is the existence of a strong generator useful?

Fact [Rouquier; 2008] R = comm. noeth. ring

If $\mathcal{T}$ is an R-linear, Ext-finite, Karoubian Δ'ed cat with a strong generator, then a cohomological functor $h: \mathcal{T}^{op} \to \text{Mod}(R)$ is representable if and only if it is locally finite

$\bigoplus_{n \in \mathbb{Z}} h(\Sigma^n X)$ f.g. $/R$

II) <u>Generation in derived categories</u>

R = comm. noeth. ring
$D(R)$ = derived category of $\text{Mod}(R)$
$D^f_b(R)$ = subcategory of complexes with f.g. bdd homology

Notation $\text{thick}_{D(R)} = \text{thick}_{D^f(R)} = \text{thick}_R$

Facts [D. Christensen; 1998] [Rouquier; 2008]

① $G = R$

$\text{thick}(R)$ = perfect complexes
= quasi-isomorphic to a bdd cx of f.g. proj
= <u>compact</u> / <u>small</u> objects in $D(R)$

Explicitly: P = cx of f.g. proj.

$$P_{\leq n} \longrightarrow P_{\leq n+1} \longrightarrow \underbrace{\Sigma^{n+1} P_{n+1}}_{\text{thick}^1(R)} \longrightarrow \Sigma P_{\leq n}$$

$$\text{level}^R(P) \leq \sup P - \inf P + 1$$
$$= \text{amplitude } P + 1$$

If M f.g. R-module: $\text{level}^R(M) = \text{projdim}_R M + 1$

For complexes this need not hold:
$$\text{level}^R(R \oplus \Sigma^n R) = 1$$

$X \in D^f_b(R)$: $\text{level}^R(X) \leq \text{gldim}\, R + 1$

If R is regular with $\text{gldim}\, R < \infty$
- R is a strong generator
- $\dim D^f_b(R) \leq \text{gldim}\, R$.

(Open) Question Is $\dim D^f_b(R) = \text{gldim}\, R$ when R regular?

Known
- R = f.g. k-algebra, domain, regular
- R = DVR (regular local ring of Krull dim = 1)

(2) $G = k$ when $(R, \mathfrak{m}, k)$ local

thick(k) = complexes with bdd, finite length homology

$X \in D^f_b(R)$: $X \in \text{thick}(X) \iff \mathfrak{m}^n X \simeq 0$

If M f.g. module:

$$\text{level}^k(M) = \ell\ell_R(M) \quad \text{Loewy length}$$
$$= \inf\{n \mid \mathfrak{m}^n M = 0\}$$

$\underline{x} = x_1, \dots, x_c \in R$, $\mathrm{Kos}(\underline{x})$

If $(\underline{x})$ is $\mathfrak{m}$-primary , i.e. $\sqrt{(\underline{x})} = \mathfrak{m}$,
then $\mathrm{Kos}(\underline{x}) \in \mathrm{thick}(k)$
and also $\mathrm{Kos}(\underline{x}) \in \mathrm{thick}(R)$

If $(R, \mathfrak{m}, k)$ artinian, local, then
- k is a strong generator
- $\dim \mathcal{D}^f_b(R) \leq \ell\ell_R(R) - 1$

Facts [L.]

① R = Noether algebra

$$\mathrm{level}^G_R(X) < \infty \iff \mathrm{level}^{G_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p}) < \infty \quad \forall \mathfrak{p} \in \mathrm{Spec}(Z(R))$$

and $\mathrm{level}^G_R(X) = \sup\{ \mathrm{level}^{G_\mathfrak{p}}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(Z(R)) \}$

BUT: $G_\mathfrak{p}$ strong generator of $\mathcal{D}^f_b(R_\mathfrak{p})$ $\forall \mathfrak{p}$

$\not\Rightarrow$ G strong generator of $\mathcal{D}^f_b(R)$

② $(R, \mathfrak{m}, k)$ local and $\widehat{(-)}$ $\mathfrak{m}$-adic completion

$$\mathrm{level}^G_R(X) = \mathrm{level}^{\hat{G}}_{\hat{R}}(\hat{X})$$

III) Applications

loc(G) = smallest localizing subcategory containing G
(Δ'ed + closed under coproducts)

If $X \in \mathrm{loc}(G)$ write $G \vdash X$, G builds X

Fact $G \models X \implies G \vdash X$

Back to D(R): For $X \in D(R)$

$$\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \underbrace{k(\mathfrak{p})}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}} \otimes^L_R X \neq 0\}$$

When $X \in D^f_b(R)$: $\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \underbrace{X_\mathfrak{p} \neq 0}_{H(X)_\mathfrak{p} \neq 0}\}$

Facts [Hopkins; 1987], [Neeman; 1992]

① $\mathrm{supp}_R(X) \subseteq \mathrm{supp}_R(G) \iff G \vdash X$

② If X, G are small, then

$$\mathrm{supp}_R(X) \subseteq \mathrm{supp}_R(G) \iff G \models X$$

Facts

① R regular $\iff$ every object in $D^f_b(R)$ is small

② $\varphi: R \to S$ map of comm. noeth. rings st $\mathrm{fd}_R S < \infty$ and φ essentially of finite type. Then
(S localization of an R-algebra of finite type)

$$\varphi \text{ smooth} \iff S \text{ small in } D(S \otimes_R S)$$

"Next best rings/maps": (locally) complete intersection rings/maps

Definition

① $\underline{x} = x_1, \dots, x_c$ is a *regular sequence* if x_i a non-zero divisor in $R/(x_1,\dots,x_{i-1})$ and $R/(\underline{x}) \neq 0$.

② $(R, \mathfrak{m}, k)$ is *complete intersection* (ci) if $\hat{R} = Q/(\underline{x})$ where Q a regular local ring and $\underline{x}$ a regular sequence

③ R (not local) is *locally complete intersection* (lci) if $R_\mathfrak{p}$ ci $\forall \mathfrak{p}$

④ $\varphi : R \twoheadrightarrow S$ surjective is *complete intersection* (ci) if $\ker(\varphi)$ can be generated by a regular sequence

⑤ $\varphi : R \twoheadrightarrow S$ surjective is *locally complete intersection* (lci) if $\varphi_\mathfrak{p}$ ci for $\mathfrak{p} \in \operatorname{Spec}(S)$.

there are also dfn's of (l)ci for maps essentially of finite type

Definition

$\mathcal{T}$ = Δ'ed cat

$X \in \mathcal{T}$ is *proxy small* if $\exists$ small object K st

$$X \vDash K \quad \text{and} \quad K \vdash X$$

In $D(R)$: $\operatorname{supp}_R(K) = \operatorname{supp}_R(X)$ and $R \vDash K$

If $\operatorname{supp}_R X$ closed, then one can choose $K = \operatorname{Kos}(\underline{x})$ where $V(\underline{x}) = \operatorname{supp}_R X$.

Fact

[Dwyer, Greenlees, Iyengar; 2006], [Pollitz; 2019], [L]

R (l)ci $\iff$ every object in $D^f_b(R)$ is proxy small

Fact [Briggs, Iyengar, L, Pollitz]

$\varphi : R \twoheadrightarrow S$ surjective. TFAE

- φ lci
- δ^{φ} is proxy small in $D(S \otimes^{L}_{R} S)$ and $\mathrm{Tor}^{R}_{\gg 0}(S,S)=0$
- $\mathrm{pd}_R S < \infty$ and proxy smallness ascends along φ

If $X \in D(S)$ is proxy small in $D(R)$ then also over $D(S)$.