Computations for thick subcategories in Macaulay2

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Outline

1. Definitions: What do we want to compute?
2. Strategies: How can those be computed?
3. Implementation: What does the package compute?

Setting

- R = commutative ring
- $\mathrm{D}(R)$ = derived category of R-modules
  - objects: complexes of R-modules
  - morphisms: chain complexes with quasi-isomorphisms inverted
    - e.g. $X \cong 0$ in $\mathrm{D}(R) \iff X$ is acyclic

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  - Structure as a triangulated category:

Suspension

$\Sigma\colon \mathrm{D}(R) \to \mathrm{D}(R)$ given by

$$X: \quad \xrightarrow{\partial} X_n \xrightarrow{\partial} X_{n-1} \to$$
$$\Sigma X\colon \to X_n \xrightarrow{-\partial} X_{n-1} \xrightarrow{-\partial}$$

Exact triangles

Replacement for SESs:

$$X \to Y \to Z \to \Sigma X$$

satisfying some axioms.

Mapping cone

For $X \xrightarrow{f} Y$ there is an object $\mathrm{Cone}(f): \to X_n \oplus Y_{n+1} \xrightarrow{\begin{pmatrix} -\partial^X & 0 \\ f & \partial^Y \end{pmatrix}} \overbrace{X_{n-1} \oplus Y_n}^{\text{degree } n} \to$

and an exact triangle $X \xrightarrow{f} Y \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y$

Thick subcategories

A full subcategory T of D(R) is thick if it is closed under:

- retracts,
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Example

thick(R) = {complexes quasi-isomorphic to bounded complexes of f.g. projective R-modules}

Example

thick(k) = {complexes with finite length homology}

Building and level

A filtration

$$\{0\} \subseteq \operatorname{thick}^1(G) \subseteq \operatorname{thick}^2(G) \subseteq \cdots \subseteq \bigcup_{n \geq 0} \operatorname{thick}^n(G) = \operatorname{thick}(G)$$

is given by:

- $\operatorname{thick}^0(G) \coloneqq \{0\}$
- $\operatorname{thick}^1(G) \coloneqq$ smallest subcategory containing G, closed under finite direct sums, suspensions and retracts.
- $\operatorname{thick}^n(G) \coloneqq$ smallest subcategory containing all X such that there exists an exact triangle

  $$Y \to X \oplus X' \to Z \to \Sigma Y$$

  with $Y \in \operatorname{thick}^{n-1}(G)$ and $Z \in \operatorname{thick}^1(G)$.

$$\operatorname{level}^G(X) \coloneqq \inf\{n \geq 0 | X \in \operatorname{thick}^n(G)\}$$

Building and level

Example

- P a f.g. projective R-modules: $\operatorname{level}^R(\Sigma^n P) = 1$
- $X = 0 \to P \xrightarrow{f} Q \to 0 = \operatorname{Cone}(P \xrightarrow{f} Q)$ with P, Q f.g. projective R-modules: $\operatorname{level}^R(X) \leq 2$
- X a bounded complex of f.g. projective R-modules: $\operatorname{level}^R(X) \leq \max(X) - \min(X) + 1$
- M a f.g. R-module: $\operatorname{level}^R(M) = \operatorname{proj\,dim}_R(M) + 1$

Example

$(R, \mathfrak{m}, k)$ a local ring.

- X a complex with f.g. bounded homology and $\mathfrak{m}^n X \simeq 0$: $\operatorname{level}^k(X) \leq n$
- M a f.g. R-module: $\operatorname{level}^k(M) = \inf \{n \geq 0 \mid \mathfrak{m}^n M = 0\} = \text{Loewy length}$

Goal

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2. Compute $\text{level}^G(X)$.

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Idea

$X \in \text{thick}(G)$ implies containment of supports for different notions of support.

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Support

1. $\operatorname{supp}(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid X_{\mathfrak{p}} \not\simeq 0\}$
   For an exact triangle $X \to Y \to Z \to \Sigma X$:
   $$\operatorname{supp}(Y) \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Z)\,.$$
   Thus $\quad X \in \operatorname{thick}(G) \implies \operatorname{supp}(X) \subseteq \operatorname{supp}(G)$
   $$\overset{k=\bar{k}}{\iff} \sqrt{\operatorname{ann}(\mathrm{H}(X))} \supseteq \operatorname{ann}(\mathrm{H}(G))$$

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2. $R = Q/(\boldsymbol{f})$ with $\boldsymbol{f} = f_1, \ldots, f_c$.
   $$E = \text{Kos}^Q(\boldsymbol{f})\,, \quad \mathcal{S} = Q[\chi_1, \ldots, \chi_c]$$
   $$V_E(X, Y) = \text{supp}^+_\mathcal{S}(\text{Ext}_E(X, Y))$$
   Then
   $$\begin{aligned} X \in \text{thick}_R(G) &\Longrightarrow X \in \text{thick}_E(G) \\ &\Longleftrightarrow V_E(X, X) \subseteq V_E(G, G) \\ &\Longrightarrow V_E(X, Y) \subseteq V_E(G, Y) \text{ for any } Y \end{aligned}$$

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   $$\Longleftrightarrow V_E(X, X) \subseteq V_E(G, G)$$
   $$\underset{\Longrightarrow}{\Longleftarrow} V_E(X, k) \subseteq V_E(G, k) \qquad {\color{red} \text{length}\, \mathrm{H}(X) < \infty}$$

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2. Compute $\text{level}^G(X)$.

Idea

$$\text{level}^G(X) = \inf \left\{ n \geq 0 \;\middle|\; \begin{array}{c} X = X_0 \xrightarrow{f(X_0)} X_1 \xrightarrow{f(X_1)} X_2 \to \ldots \to X_n \\ \text{is zero in } \mathrm{D}(R) \end{array} \right\}$$

with $f(X_i)\colon X_i \to X_{i+1}$ constructed from a right G-approximation of X_i.

Right approximations

A right G-approximation of X is a map $G' \to X$ with

$$G' = \bigoplus \Sigma^n G^{d_n} \quad \text{and} \quad \begin{array}{ccc} \Sigma^d G & \longrightarrow & X \,. \\ \downarrow \exists & \nearrow & \\ G' & & \end{array}$$

Completing this to an exact triangle

$$G' \to X \xrightarrow{f(X)} Y \to \Sigma G'$$

yields

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Package

1. `isBuilt(X,G)` checks if $X \in \mathrm{thick}(G)$
2. `level(G,X)` returns $\mathrm{level}^G(X)$