The spectrum of a well-generated tensor triangulated category

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FD Seminar
February 24th, 2022

Outline

1. Motivation
2. Background: Frames and classifications of thick tensor ideals
3. Radical α-localizing tensor ideals
4. Filtration of Spc($\mathcal{T}$)

Motivation: Well-generated triangulated categories

not the sum of fewer than α cardinals, all smaller than α

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and α a regular cardinal.

$X \in \mathcal{T}$ is α-small if

$$\begin{tikzcd} X \arrow[rr, "\forall"] \arrow[dr, dashed, "\exists"'] & & \coprod_{Y \in \mathcal{Y}} Y \\ & \coprod_{Y \in \mathcal{Y}'} Y \arrow[ur] & \end{tikzcd}$$

for any $\mathcal{Y} \subseteq \mathcal{T}$ $\exists\, \mathcal{Y}' \subseteq \mathcal{Y},\ |\mathcal{Y}'| < \alpha$

$\mathcal{T}^\alpha$ is the maximal α-perfect class of α-small objects of $\mathcal{T}$.
An object $X \in \mathcal{T}^\alpha$ is called α-compact.

when $\alpha = \aleph_0$: $\aleph_0$-compact $= \aleph_0$-small $=$ compact

$\mathcal{T}$ is α-compactly generated, if there exists a set $\mathcal{G}$ of α-compact objects and $\operatorname{Loc}(\mathcal{G}) = \mathcal{T}$.

$\mathcal{T}$ is well-generated, if it is α-compactly generated for some regular cardinal α.

For $\beta \geq \alpha$: $\mathcal{T}^\beta = \operatorname{loc}^\beta(\mathcal{T}^\alpha)$, $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ filtration

Motivation: Well-generated triangulated categories, part 2

- Every compactly generated category is well-generated.

  eg : $D(R)$, $StMod(kG)$

- Let $\mathcal{T}$ be a well-generated triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ a localizing subcategory generated by a set of objects. Then

  $\mathcal{S}$, $\mathcal{T}/\mathcal{S}$ are well-generated

  If $\mathcal{T}$ is α-compactly generated and $\mathcal{S}$ is generated by β-compact objects and $\beta \geq \alpha$, then $\mathcal{S}$, $\mathcal{T}/\mathcal{S}$ are β-compactly generated.

- When $\mathcal{T}$ is α-compactly generated, then $\mathcal{T}^\alpha$ is essentially small.

  $\rightsquigarrow$ α-localizing subcategories form a set

Motivation: Filtration of the spectrum

$\mathcal{T}$ = compactly generated tensor triangulated category

radical thick tensor ideals of $\mathcal{T}^c$

Balmer spectrum

$$\begin{tikzcd}
\mathrm{Rad}^c(\mathcal{T}^c) \arrow[rr, leftrightsquigarrow, "\text{classification}"] \arrow[d, tail] & & \mathrm{Spc}^c(\mathcal{T}^c) \\
\mathrm{Rad}^\alpha(\mathcal{T}^\alpha) \arrow[rr, leftrightsquigarrow] \arrow[d, tail] & & \mathrm{Spc}^\alpha(\mathcal{T}^\alpha) \arrow[u] \\
\mathrm{Rad}(\mathcal{T}) \arrow[rr, leftrightsquigarrow, "\text{classification}"] & & \mathrm{Spc}(\mathcal{T}) \arrow[u]
\end{tikzcd}$$

radical localizing tensor ideals of $\mathcal{T}$

topological space

Examples

$\mathcal{T} = D(R)$

$\mathcal{T}^c = \mathrm{Perf}(R)$

$\mathrm{Spc}^c(\mathcal{T}^c) = \mathrm{Spec}(R)$ Zariski spectrum

$\mathrm{Spc}(\mathcal{T}) = \mathrm{Spec}(R)$ with discrete topology

$\mathcal{T} = \mathrm{StMod}(kG)$

$\mathcal{T}^c = \mathrm{stmod}(kG)$

$\mathrm{Spc}^c(\mathcal{T}^c) = \mathrm{Proj}(H^*(G,k))$

$\mathrm{Spc}(\mathcal{T}) = \mathrm{Proj}(H^*(G,k))$ with discrete topology eg $\mathcal{K} = \mathcal{T}$

Let $\mathcal{K}$ be an essentially small tensor triangulated category.

$$\mathcal{I} \longmapsto \bigcup_{x \in \mathcal{I}} \operatorname{supp} X$$

$$\{X \in \mathcal{K} \mid \operatorname{supp} X \subseteq V\} \longleftarrow V$$

unions of closed subsets

$$\mathrm{Rad}^c(\mathcal{K}) = \left\{ \begin{matrix} \text{radical thick tensor} \\ \text{ideals of } \mathcal{K} \end{matrix} \right\} \xleftrightarrow{\text{[Balmer; 2005]}} \left\{ \begin{matrix} \text{Thomason subsets} \\ \text{of } \mathrm{Spc}^c(\mathcal{K}) \end{matrix} \right\}$$

↳ structure:
- inclusion ⇝ poset
- intersection, smallest radical thick tensor ideal containing the union ⇝ lattice
- infinite-join distributive law ⇝ <u>frame</u>

prime thick tensor ideals of $\mathcal{K}$ with topology given by the basis of closed sets:

$\operatorname{supp} X = \{\mathcal{P} \mid X \notin \mathcal{P}\}$

$\bigvee A$ $\bigwedge A$

$a \vee b$ $a \wedge b$

A poset $(L, \leq)$ is a lattice, if the join and the meet of any non-empty finite set exists.

1 0

A lattice is distributive, if the greatest and least element exist, and

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{for all } a, b, c \in L\,.$$

A lattice $(F, \leq)$ is a frame, if joins and finite meets exist, and

$\Rightarrow 0, 1$ exist

$$a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b) \quad \text{for all } a \in F \text{ and } B \subseteq F\,.$$

infinite join distributive law

Frm = category of frames

morphisms: maps that preserve order, joins, finite meets, greatest and least element

Background: Spatial frames

prime element
$p_x = \bigvee x^{-1}(0)$

A point of F is a frame morphism $x \colon F \to \{0, 1\}$.
$\mathrm{pt}(F)$ is the set of points of F. It is a topological space with open sets

$$U(a) := \{x \in \mathrm{pt}(F) \mid \overbrace{x(a) = 1}^{a \not\leq p_x}\} \quad \text{for any } a \in F\,.$$

$X \longmapsto$ frame of open sets

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\text{adjoint}"] & \mathrm{Frm}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{pt}"] \\
\{\text{sober spaces}\} \arrow[u, phantom, "\cup\!|"] \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\equiv"] & \mathrm{SpFrm}^{\mathrm{op}} \arrow[u, phantom, "\cup\!|"] \arrow[l, shift left]
\end{tikzcd}$$

Stone duality

A frame F is spatial, if for all $a, b \in F$ satisfying $a \not\leq b$ there is a point x such that $x(a) = 1$ and $x(b) = 0$. "F has enough points"

SpFrm = category of spatial frames

morphisms: morphisms of frames

Background: Coherent frames

$a \in F$ is compact, if $a \leq \bigvee A \Rightarrow a \leq \bigvee A'$ for $A' \subseteq A$ with $|A'| < \infty$

F^c is the set of compact elements of F.

F can be recovered from F^c

A frame F is coherent, if

1. every element is the join of compact elements,
2. the greatest element 1 is compact, and
3. the compact elements are closed under finite meets.

F^c is a distributive lattice

Coh = category of coherent frames

morphisms: morphisms of frames, that restrict to morphisms on the compact elements

Fact

coherent $\Longrightarrow$ *spatial*

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{sober}\\ \text{spaces}\end{matrix}\right\} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\equiv"] & \mathrm{SpFrm}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{pt}"] & & & & \\
\rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & & & & \\
\left\{\begin{matrix}\text{spectral}\\ \text{spaces}\end{matrix}\right\} \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\equiv"] \arrow[rrrrr, bend left, leftrightarrow, "\text{Hochster dual}"] & \mathrm{Coh}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{pt}"] \arrow[r, shift left, "(-)^c"] \arrow[r, phantom, "\equiv"] & \mathrm{DLat}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{Idl}"] \arrow[r, leftrightarrow, "(-)^{\mathrm{op}}"] & \mathrm{DLat}^{\mathrm{op}} \arrow[r, phantom, "\equiv"] & \mathrm{Coh}^{\mathrm{op}} \arrow[r, phantom, "\equiv"] & \left\{\begin{matrix}\text{spectral}\\ \text{spaces}\end{matrix}\right\} \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\
\mathrm{pt}(\mathrm{Rad}^c(\mathcal{T}^c)) \arrow[r, leftrightarrow] & \mathrm{Rad}^c(\mathcal{T}^c) \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{principle}\\ \text{tensor}\\ \text{ideals}\end{matrix}\right\} \arrow[rrr, leftrightarrow] & & & \mathrm{Spc}^c(\mathcal{T}^c)
\end{tikzcd}
$$

For the rest of the talk:
focus on this

$\mathrm{Rad}^\alpha(\mathcal{K})$: Definition

Let $\mathcal{K}$ be an essentially small tensor triangulated category with α-coproducts and α a regular cardinal.

A full subcategory $\mathcal{I} \subseteq \mathcal{K}$ is

- a tensor ideal if for $X \in \mathcal{I}$ and $Y \in \mathcal{K}$ also $X \otimes Y \in \mathcal{I}$,
- α-localizing if $\mathcal{I}$ is triangulated, closed under α-coproducts and closed under direct summands, and
- radical if whenever $X^{\otimes n} \in \mathcal{I}$ for some $n \geq 1$ also $X \in \mathcal{I}$.

$\aleph_0$-loc.
$\shortparallel$
thick

$\mathrm{rad}^\alpha(\mathcal{X}) =$ smallest radical α-localizing tensor ideal containing $\mathcal{X}$

$\mathrm{Rad}^\alpha(\mathcal{K}) =$ set of all radical α-localizing tensor ideals of $\mathcal{K}$

$\mathrm{Rad}^{\alpha}(\mathcal{K})$: Properties

$\mathrm{Rad}^{\alpha}(\mathcal{K})$

1. has a partial order — *inclusion*
2. has a greatest element — $1 = \mathcal{K}$
3. has a least element — $0 = \mathrm{rad}^{\alpha}(0)$
4. has α-joins
5. has finite meets
6. satisfies the infinite-join distributive law
7. every element is a join of elements $\mathrm{rad}^{\alpha}(X)$ for $X \in \mathcal{K}$

frame

next slide

$$I = \bigvee_{x \in I} \mathrm{rad}^{\alpha}(X)$$

$\mathrm{Rad}^\alpha(\mathcal{K})$: Infinite-join-distributive law

Lemma

$\mathrm{Rad}^\alpha(\mathcal{K})$ *is a frame.*

Proof.

Need to show:

$$\mathcal{I} \wedge \bigvee A = \bigvee_{\mathcal{J} \in A} (\mathcal{I} \wedge \mathcal{J}) \quad \text{for any } \mathcal{I} \in \mathrm{Rad}^\alpha(\mathcal{K}) \text{ and } A \subseteq \mathrm{Rad}^\alpha(\mathcal{K})\,.$$

($\geq$) always

($\leq$) $X \in \mathcal{I} \wedge \bigvee A$

$\mathcal{C}_X := \{ Y \in \bigvee A \mid X \otimes Y \in \bigvee_{\mathcal{J} \in A} (\mathcal{I} \wedge \mathcal{J}) \}$ radical α-localizing tensor ideal

$\mathcal{J} \in \mathcal{C}_X \quad \forall \mathcal{J} \in A \implies \bigvee A \in \mathcal{C}_X \implies X \otimes X \in \mathcal{C}_X \implies X \in \mathcal{C}_X$

$\square$

$\mathrm{Rad}^{\alpha}(\mathcal{K})$: α-coherent frames

$a \in F$ is compact, if for every set $A \subseteq F$ with $a \leq \bigvee A$ there exists $B \subseteq A$ with

$$a \leq \bigvee B \quad \text{and} \quad |B| < \infty\,.$$

F^c is the set of compact elements of F.

A frame F is coherent, if

1. every element is the join of compact elements,
2. the greatest element 1 is compact, and
3. the compact elements are closed under finite meets.

Fact

coherent $\implies$ *spatial*

$\mathrm{Rad}^{\alpha}(\mathcal{K})$: α-coherent frames

$a \in F$ is α-compact, if for every set $A \subseteq F$ with $a \leq \bigvee A$ there exists $B \subseteq A$ with

$$a \leq \bigvee B \quad \text{and} \quad |B| < \alpha\,.$$

F^{α} is the set of α-compact elements of F.

A frame F is α-coherent, if

1. every element is the join of α-compact elements,
2. the greatest element 1 is α-compact, and
3. the α-compact elements are closed under finite meets.

Fact

α-coherent $\not\Longrightarrow$ *spatial*

$\mathrm{Rad}^\alpha(\mathcal{K})$: α-compact elements

Lemma

The α-compact elements of the frame $\mathrm{Rad}^\alpha(\mathcal{K})$ are precisely those of the form $\mathrm{rad}^\alpha(X)$ for some $X \in \mathcal{K}$.

Proof.

Let $\mathcal{I} \in \mathrm{Rad}^\alpha(\mathcal{K})$ be an α-compact element.

$$\mathcal{I} = \bigvee_{X \in \mathcal{I}} \mathrm{rad}^\alpha(X) = \bigvee_{X \in \mathcal{X}} \mathrm{rad}^\alpha(X) = \mathrm{rad}^\alpha\Big(\coprod_{X \in \mathcal{X}} X\Big)$$

for $\mathcal{X} \subseteq \mathcal{I}$ with $|\mathcal{X}| < \alpha$ $\qquad$ $\mathcal{K}$ closed under α-coproducts

Let $\mathcal{I} = \mathrm{rad}^\alpha(X)$ with

$$\mathrm{rad}^\alpha(X) \leq \bigvee_{J \in A} J \overset{\text{wlog}}{=} \bigvee_{X \in \mathcal{X}} \mathrm{rad}^\alpha(X) \qquad \text{for some } \mathcal{X} \subseteq \mathcal{J}.$$

$$= \mathrm{rad}^\alpha(\mathcal{X}) = \bigcup_{\substack{\mathcal{X}' \subseteq \mathcal{X} \\ |\mathcal{X}'| < \alpha}} \mathrm{rad}^\alpha(\mathcal{X}') \quad \Rightarrow \quad \mathrm{rad}^\alpha(X) \leq \mathrm{rad}^\alpha(\mathcal{X}')$$

an α-join because $|\mathcal{X}'| < \alpha$ $\square$

$\mathrm{Rad}^\alpha(\mathcal{K})$

Generalizes [Kock, Pitsch; 2017]

Theorem (Krause, L)

Let $\mathcal{K}$ be an essentially small tensor triangulated category with α-coproducts and α a regular cardinal. Suppose that

$$\mathrm{rad}^\alpha(X \otimes Y) = \mathrm{rad}^\alpha(X) \wedge \mathrm{rad}^\alpha(Y) \rightsquigarrow$$

α-compact elements are closed under meets

for any $X, Y \in \mathcal{K}$.
Then $\mathrm{Rad}^\alpha(\mathcal{K})$ is α-coherent and the map

"support of X"

$$\mathrm{Obj}(\mathcal{K}) \longrightarrow \mathrm{Rad}^\alpha(\mathcal{K}), \quad X \mapsto \mathrm{rad}^\alpha(X)$$

is an α-support and it is initial among all α-supports on $\mathcal{K}$.

$\rightsquigarrow$ $\mathrm{Rad}^\alpha(\mathcal{K})$ is the best possible frame to classify the objects respecting the triangulated structure

$\mathrm{Spc}(\mathcal{T})$: Direct system of $\mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha})$

Let α be a regular cardinal and $\mathcal{T}$ an α-compactly generated tensor triangulated category.

$$\mathcal{T} = \bigcup_{\alpha} \mathcal{T}^{\alpha} \qquad\qquad \mathcal{T}^{\beta} = \mathrm{loc}^{\beta}(\mathcal{T}^{\alpha})$$

For a regular cardinal $\beta \geq \alpha$ there is a pair of maps

$$(-)\restriction^{\beta}_{\alpha}\colon\ \mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha}) \rightleftarrows \mathrm{Rad}^{\beta}(\mathcal{T}^{\beta})\colon\ (-)\downharpoonleft^{\beta}_{\alpha}$$

$$\mathcal{I} \longmapsto \mathrm{rad}^{\beta}(\mathcal{I})$$

$$\mathcal{J} \cap \mathcal{T}^{\alpha} \longleftarrow \mathcal{J}$$

Warning: these need not be morphisms of frames

Fact (Thomason's localization theorem)

$$(\mathcal{I}\restriction^{\beta}_{\alpha})\downharpoonleft^{\beta}_{\alpha} = \mathcal{I} \quad \textit{and} \quad (\mathcal{J}\downharpoonleft^{\beta}_{\alpha})\restriction^{\beta}_{\alpha} \leq \mathcal{J}$$

Spc($\mathcal{T}$): Spatial frames

Lemma

Let $\alpha \leq \beta$ be regular cardinals and $\mathcal{T}$ α-compactly generated. Suppose that

$$\mathrm{rad}^{\beta}(X \otimes Y) = \mathrm{rad}^{\beta}(X) \wedge \mathrm{rad}^{\beta}(Y)$$

for any $X, Y \in \mathcal{T}^{\alpha}$.
Then

1. *$\mathcal{T}^{\alpha}$ satisfies the α-radical tensor property,*
2. *the extension map $(-)\restriction_{\alpha}^{\beta}$ is a morphism of frames, and*
3. *if $\mathrm{Rad}^{\beta}(\mathcal{T}^{\beta})$ is spatial, then $\mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha})$ is spatial.*

Spc($\mathcal{T}$): Inverse System of spaces

Suppose that $\mathcal{T}$ is α-compactly generated, $\mathsf{Rad}(\mathcal{T})$ is spatial and

$$\mathrm{rad}(X \otimes Y) = \mathrm{rad}(X) \wedge \mathrm{rad}(Y)$$

for any $X, Y \in \mathcal{T}$.

spatial

$$\mathsf{Rad}^\alpha(\mathcal{T}^\alpha) \longrightarrow \mathsf{Rad}^\beta(\mathcal{T}^\beta) \longrightarrow \dots \longrightarrow \mathsf{Rad}(\mathcal{T})$$

in examples: bijection of sets

$$\begin{tikzcd} \mathrm{pt}(\mathsf{Rad}^\alpha(\mathcal{T}^\alpha)) & \mathrm{pt}(\mathsf{Rad}^\beta(\mathcal{T}^\beta)) \arrow[l] & \dots \arrow[l] & \mathrm{pt}(\mathsf{Rad}(\mathcal{T})) \arrow[l] \arrow[lll, bend right] \end{tikzcd}$$

Spc($\mathcal{T}$): α-compactly stratified

Definition

Let $\mathcal{T}$ be tensor triangulated category, and assume the localizing tensor ideals that are generated by sets of objects form a set. We call $\mathcal{T}$ α-compactly stratified if

1. the triangulated category $\mathcal{T}$ is α-compactly generated,
2. the frame $\mathrm{Rad}(\mathcal{T})$ is spatial, ⇝ descends to $\mathrm{Rad}^\alpha(\mathcal{T}^\alpha)$
3. for any $X, Y \in \mathcal{T}$

$$\mathrm{rad}(X \otimes Y) = \mathrm{rad}(X) \wedge \mathrm{rad}(Y),$$

⇝ extension is a morphism of frames

4. the induced map $\mathrm{pt}(\mathrm{Rad}(\mathcal{T})) \to \mathrm{pt}(\mathrm{Rad}^\alpha(\mathcal{T}^\alpha))$ is a bijection, and
5. every point of $\mathrm{Rad}^\alpha(\mathcal{T}^\alpha)$ is locally closed → $\{x\} = \underset{\text{open}}{U} \cap \underset{\text{closed}}{V}$

Spc($\mathcal{T}$)

Theorem (Krause, L)

Let $\mathcal{T}$ be a tensor triangulated category and $\alpha \leq \beta$ regular cardinals. If $\mathcal{T}$ is α-compactly stratified, then $\mathcal{T}$ is β-compactly stratified.

$$\begin{tikzcd}
\mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha}) \arrow[r, tail] & \mathrm{Rad}^{\beta}(\mathcal{T}^{\beta}) \arrow[r, tail] & \mathrm{Rad}(\mathcal{T}) \\
\mathrm{pt}(\mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha})) & \mathrm{pt}(\mathrm{Rad}^{\beta}(\mathcal{T}^{\beta})) \arrow[l] & \mathrm{pt}(\mathrm{Rad}(\mathcal{T})) \arrow[l] \arrow[ll, bend left]
\end{tikzcd}$$

α-compactly stratified: bijection of sets

$\Rightarrow$ get a filtration of $\mathrm{pt}(\mathrm{Rad}(\mathcal{T}))$ where the underlying set is the same and the topology is refined

$\mathrm{Spc}(\mathcal{T})$

Theorem (Krause, L)

Let $\mathcal{T}$ be a tensor triangulated category and $\alpha \leq \beta$ regular cardinals. If $\mathcal{T}$ is α-compactly stratified, then $\mathcal{T}$ is β-compactly stratified.

Lemma

Suppose

$$F \xrightarrow{\varphi} G \xrightarrow{\psi} H$$

are injective maps of spatial frames and every point of F is locally closed. Then

$$\mathrm{pt}(\varphi) \text{ and } \mathrm{pt}(\psi) \text{ bijective} \iff \mathrm{pt}(\psi \circ \varphi) \text{ bijective}.$$

$\mathrm{Spc}(\mathcal{T})$: Verdier quotients

Corollary

Let $\mathcal{T}$ be a tensor triangulated category. Suppose that $\mathcal{T}$ is α-compactly stratified and $\mathcal{S} \subseteq \mathcal{T}$ a radical localizing tensor ideal generated by β-compact objects.
Then the quotient $\mathcal{T}/\mathcal{S}$ is β-compactly stratified.

Spc($\mathcal{T}$): Next steps and open questions

We have a filtration:

$$\mathrm{pt}(\mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha})) \longleftarrow \mathrm{pt}(\mathrm{Rad}^{\beta}(\mathcal{T}^{\beta})) \longleftarrow \dots \longleftarrow \mathrm{pt}(\mathrm{Rad}(\mathcal{T}))$$

Q: What is the topology on $\mathrm{pt}(\mathrm{Rad}^{\alpha}(\mathcal{T}^{\alpha}))$?

Q: In $D(R)$: $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ residue field at prime ideal $\mathfrak{p}$

Is the point $\mathfrak{p}$ closed in $\mathrm{pt}(\mathrm{Rad}^{\alpha}(D(R)^{\alpha}))$ if and only if $\kappa(\mathfrak{p}) \in D(R)^{\alpha}$?

Thank you

Paul Balmer, The spectrum of prime ideals in tensor triangulated categories, J. Reine Angew. Math. **588** (2005), 149–168.

Joachim Kock and Wolfgang Pitsch, Hochster duality in derived categories and point-free reconstruction of schemes, Trans. Amer. Math. Soc. **369** (2017), no. 1, 223–261.

Amnon Neeman, Triangulated categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.